Дубна, 26/11/2015

The Geometry of Supermanifolds and Applications

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in collaboration with G. Policastro, M. Marescotti, D. Matessi, M. Debernardi, R. Catenacci, L. Andrianopoli, M. Trigiante, R. D'Auria, L. Castellani, and recently with C. Maccaferri

- Picture Changing Operators Friedan-Martinec-Shenker
- Some math work by Schwartz, Voronov, Zorich, Bernstein, Leites, Manin
- Two interesting papers (unpublished) by Belopolsky
- Multiloop amplitudes in Pure Spinor String Theory by Berkovits
- Geometry of supermanifolds and target space PCO PAG-Policastro
- Integration of superforms and super-Thom class PAG-Marescotti
- Balanced super varieties and Integral form cohomology Catenacci-Debernardi-Matessi-PAG
- Superstring perturbation theory revised Witten
- Poincare' dual, Hodge operator, Hodge theory and dualities Castellani-Catenacci-PAG
- Super-entropy current and fluid-dynamics Andrianopoli-D'Auria-Trigiante-PAG

BRST quartets vs PCO

Let us consider the following BRST quartet (θ, η, t, b) where the Greek letters denote anticommuting quantities and the Latin letters the commuting ones.
We assign the following transformation rules:

$$Q\theta = t\,, \qquad Qt = 0$$

$$Qb = \eta\,, \qquad Q\eta = 0$$

and we consider the following expression

$$Q(b\theta) = (Qb)\theta + bQ\theta = \eta\theta + bt$$

Now, we define the following integral

$$Y = \int db d\eta e^{i(\eta\theta + bt)} = \int db d\eta (1 + i\eta\theta) e^{ibt}$$
$$= i \int db \theta e^{ibt} = i\theta\delta(t)$$

This quantity has the following two properties: 1) it is BRST closed and 2) it is not exact.

$$Q(\theta\delta(t)) = t\delta(t) = 0 \qquad \theta\delta(t) \neq Q\Omega$$

where the distributional rules have been applied.

This quantity can be integrated w.r.t. to the variables (θ, t)

$$\int d\theta dt Y = i$$

Y is known in the literature as Picture Changing Operator where the variables (θ, η, t, b) are interpreted as the ghost fields of RNS/Pure Spinor/Topological String theory.

Let us generalise it: instead of a single quartet, we consider a multiplet

$$(\theta_i, \eta_i, t_i, b_i)$$

with i = 1, ..,m and we can construct a PCO as a product of m PCO's as

$$Y_m = \prod_{i=1}^{m} \theta_i \delta(t_i) = \theta_1 \dots \theta_m \delta(t_1) \dots \delta(t_m)$$

which is invariant under any transformation of GL(m,C) of the variables $\theta_i \to \Lambda_i{}^j \theta_j \,, \qquad t_i \to \Lambda_i{}^j t_j \,,$

since we have

$$\theta_1 \dots \theta_m \to \det(\Lambda) \theta_1 \dots \theta_m \qquad\qquad \delta(t_1) \dots \delta(t_m) \to \frac{1}{\det(\Lambda)} \delta(t_1) \dots \delta(t_m)$$

Again the integral of Y_m gives 1 (with the correct normalisation).

Integration on supermanifolds

Let us now move to supermanifolds. We denote by $\mathcal{M}$ a (n|m)-dimensional supermanifold parametrised by the local coordinates (x_μ, θ_i).

We introduce also the corresponding 1-forms $(dx_\mu, d\theta_i)$ with the properties

$$dx_\mu \wedge dx_\nu = -dx_\nu \wedge dx_\mu \qquad dx_\mu \wedge d\theta_i = d\theta_i \wedge dx_\mu \qquad d\theta_i \wedge d\theta_j = d\theta_j \wedge d\theta_i$$

Then a generic (super) form has the form

$$\omega = \sum_{k=1,l=1}^{k=p,l=q} \omega_{[\mu_1 \dots \mu_k](i_1 \dots i_l)} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \wedge d\theta^{i_1} \wedge \dots \wedge d\theta^{i_l}$$

where the components $\omega_{[\mu_1 \dots \mu_k](i_1 \dots i_l)}(x, \theta)$ are functions of the manifold coordinates. The indices $[\mu_1 \dots \mu_k]$ are antisymmetrized while $(i_1 \dots i_l)$ are symmetrized. The total form degree is fixed by the p + q..

This implies that there is no upper bound to the form degree and there is no top form.

A. How to define a sensible theory of integration of differential forms on supermanifolds?

B. How to define de Rham/Čech cohomology in the space of superforms?

C. How to construct the Poincaré duality in the space of superforms?

D. How to define a Hodge operator?

E. How to construct the Hodge theory for a supermanifold?

We answer to all of these questions and apply the results to

1. Constructing new supersymmetric actions with higher-derivative terms
2. Constructing the Hodge dualities for superfields

A. Integration of Forms on Supermanifolds

Let us begin with a conventional manifold $\mathcal{M}$ with dimension = n, given a generic differential form

$$\omega \in \Omega^{\bullet}(\mathcal{M})$$

This is a section of the exterior bundle and it can be decomposed as

$$\omega = \omega^0 + \omega^1 + \omega^2 + \cdots + \omega^n$$

where the last term is the top form element. Locally it can be written as

$$\omega(x,\psi) = \sum_{p=0}^{n} \omega_{[\mu_1 \ldots \mu_p]}(x) dx^{\mu_1} \ldots dx^{\mu_p}$$

and its integral on the manifold is

$$\int_{\mathcal{M}} \omega = \int f(x) d^n x \,, \quad f(x) = \sqrt{g}\, \omega_{[\mu_1 \ldots \mu_n]}(x) \epsilon^{\mu_1 \cdots \mu_n}$$

where the second member is a Lebesgue/Riemann integral of the function built in terms of the components of the degree n term of the differential form. This integral can be view from a more algebraic way by introducing a set of anti-commuting variables ψ_μ rewriting the diff. form as function in a superspace

$$\hat{\omega}(x,\psi) = \sum_{p=0}^{n} \omega_{[\mu_1 \ldots \mu_p]}(x) \psi^{\mu_1} \ldots \psi^{\mu_p}$$

and its integral is defined as

$$\int_{\mathcal{M}} \omega = \int_{\hat{\mathcal{M}}} \hat{\omega}$$

where the integral over the anticommuting coordinates is performed by usual Berezin integral. The measure for the r.h.s. is sometimes written as $[d^n x d^n \psi]$ (where I neglect details for curved spaces).

Let us move to supermanifolds. A **superform** for a supermanifold is written as

$$\omega = \sum_{p,q} \omega_{[\mu_1 \dots \mu_p](i_1 \dots i_q)}(x,\theta) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge d\theta^{i_1} \wedge \dots \wedge d\theta^{i_q}$$

and we can apply the same strategy:

$$dx^\mu \to \psi^\mu \,, \qquad d\theta^i \to t^i$$

and finally we would like to use the same definition of the integral as above by considering the superform ω as a function on this new superspace $\omega(x, \theta, \psi, t)$

The integrals over the fermionic coordinates are Berezin integrals, over the x-coordinates are the usual Lebesgue/Riemann integrals, but the integral over t is not well defined on the superforms.

One way to solve this issue is to construct the new form

$$\omega(x,\theta,\psi,t) \to \omega(x,\theta,\psi,t)\prod_{i=1}^{m}\theta^i\delta(t^i)$$

by multiplying it by the PCO! This leads to a new quantity: an **INTEGRAL FORM** as

$$\boxed{\omega^{(n|m)} = \omega_n(x,\theta)\psi^n\prod_{i=1}^{m}\theta^i\delta(t^i)}$$

where **n = form degree** and **m = picture number**

For the Dirac delta functions we assume the following properties (distributional properties)

$$\delta(t^i)\wedge\delta(t^j) = -\delta(t^j)\wedge\delta(t^i)$$

$$t^i\delta(t^i) = 0$$

$$t^i\delta^{(n)}(t^i) = -n\delta^{(n-1)}(t^i)$$

Now a generic **(pseudo)-form** can be written as

$$\omega = \sum_{p,r,s} \omega_{[\mu_1 \dots \mu_p](i_1 \dots i_r)[i_{r+1} \dots i_s]}(x,\theta) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge d\theta^{i_1} \wedge \dots \wedge d\theta^{i_r} \wedge \delta(d\theta^{i_{r+1}}) \wedge \dots \wedge \delta(d\theta^{i_s})$$

each pieces are differential forms with fixed **form degree = p + r** and **picture number = s - r**

There is an additional detail that we can add: we should also admit derivatives of the Dirac delta functions

$$\boxed{\delta^{(p)}(d\theta^i)}$$

and they have **picture number = +1** but **form degree = - p**. They have negative form numbers!!

A generic (p|q) form is written in terms of

$$dx^\mu \wedge \dots \wedge d\theta^i \dots \wedge \delta^{(p)}(d\theta^j)$$

and we denote by $\Omega^{(p|q)}(\mathcal{M})$ the space of pseudo-forms. For q=0, we have the well-known **superforms**, for q=m we have the **integral forms** and for 0< q<m, we have the space of **pseudo-forms**.

B. Cohomology

Now we have the following complexes

$$0 \to \Omega^{(0|0)} \to \Omega^{(0|0)} \to \dots \to \Omega^{(n|0)} \to \Omega^{(n+1|0)} \dots$$

where all spaces are finite dimensional. The complex is not bounded from above. The differential **d** acts along the arrows.

$$\dots \to \Omega^{(-2|m)} \to \Omega^{(-1|m)} \to \dots \to \boxed{\Omega^{(n|m)}} \to 0$$

this is the complex of integral forms. It is unbounded from below, but it is bounded from above. The last space is the space of top forms. Notice that since we have the maximum number of delta's, there is no room for $d\theta' s$

Then finally we have

$$\int_{\hat{\mathcal{M}}} \hat{\omega} = \int_{\mathcal{M}} \omega_{[\mu_1 \dots \mu_n][i_1 \dots i_m]}(x, \theta) \epsilon^{\mu_1 \dots \mu_n} \epsilon^{i_1 \dots i_m}$$

There are additional complexes of the form:

$$\dots \to \Omega^{(-2|q)} \to \Omega^{(-1|q)} \to \dots \to \Omega^{(n|q)} \to \dots$$

which is not bounded from above nor from below. In addition, each single space is **infinite dimensional** space and their geometry is completely unknown.

In summary, we have

$$\begin{array}{ccccccccccccc}
 & 0 & \xrightarrow{\mathrm{d}} & \Omega^{(0|0)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|0)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|0)} & \xrightarrow{\mathrm{d}} & \Omega^{(m+1|0)} & \dots \\
 & {\scriptstyle Z}\big\uparrow & & {\scriptstyle Z}\big\uparrow\big\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\big\uparrow\big\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\big\uparrow\big\downarrow{\scriptstyle Y} & & \big\downarrow{\scriptstyle Y} & \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots & \\
\dots & \Omega^{(-1|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|s)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|s)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(m+1|s)} & \dots \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots & \\
 & {\scriptstyle Z}\big\uparrow & & {\scriptstyle Z}\big\uparrow\big\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\big\uparrow\big\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\big\uparrow\big\downarrow{\scriptstyle Y} & & \big\downarrow{\scriptstyle Y} & \\
\dots & \Omega^{(-1|n)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|n)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|n)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|n)} & \xrightarrow{\mathrm{d}} & 0 &
\end{array}$$

The operators Y and Z are known as Picture Changing Operators and act vertically in the complexes.

These two operators (the expression of **Z** is rather complicate and it is not displayed) are d-closed and they are not d-exact. The **Y** operators are elements of the cohomology

$$H^{(0|m)}(\mathcal{M})$$

This implies that given a pseudo form (p|q) and multiplying it by a PCO $Y_i = \theta_i \delta(d\theta_i)$ we have

$$Y_i : H^{(p|q)}(\mathcal{M}) \to H^{(p|q+1)}(\mathcal{M})$$

This observation implies that if there were cohomology in a given space, this can be mapped into a space with another picture. Since the two complexes $\Omega^{(p|0)}(\mathcal{M})$ and $\Omega^{(p|m)}(\mathcal{M})$ are either bounded from below or from above, this means that there is no cohomology below and above.

So, the cohomology is entirely contained into the square bounded by the 0-forms with 0 pictures and from the integral forms with n-form degree and m-picture.

For the finite dimensional spaces $\Omega^{(p|0)}(\mathcal{M})$ $\Omega^{(p|m)}(\mathcal{M})$

we can establish an isomorphism, namely we have the following property

$$\dim \Omega^{(p|q)}(\mathcal{M}) = \dim \Omega^{(n-p|m-q)}(\mathcal{M})$$

which is the Poincaré duality for the pseudo-forms. For finite dimensional spaces it is easy to count the generators of these spaces, but it is also possible for infinite dimensional spaces since they have countable infinite number of generators.

As in the conventional manifolds, we have

$$\Omega^{(0|0)} \ni 1 \to \boxed{f(x) d^n x \theta^m \delta^m(d\theta) \in \Omega^{(n|m)}}$$

namely, 1 is mapped into a top form.

D. Hodge operator

To define a Hodge operator we recall the equation discussed in the beginning using the BRST quartets

Introduce a set of dual variables $p^\mu \leftrightarrow x^\mu \,, \qquad \eta^i \leftrightarrow \theta^i$

$$\Omega^{(p|q)} \ni \omega(x, \theta, dx, d\theta) \to \omega(x, \theta, p, \eta)$$

Then we define the Hodge dual as

$$\star\omega(x, \theta, dx, d\theta) = \mathcal{N} \int_{\tilde{\mathcal{M}}} \omega(x, \theta, p, \eta) e^{i(p^\mu G_{\mu\nu} dx^n + p^\mu G_{\mu j} d\theta^j + \eta^i G_{i\nu} dx^\nu + \eta^i G_{ij} d\theta^j)}$$

with the properties

$$\star : \Omega^{(p|q)} \to \Omega^{(n-p|m-q)} \qquad\qquad \star^2 = \pm 1$$

The matrix

$$\begin{pmatrix} G_{\mu\nu} & G_{\mu j} \\ G_{i\nu} & G_{ij} \end{pmatrix}$$

defines a **supermetric** for the supermanifold and the entries are either bosonic or fermonic in order to respect the parity of the exponential in the definition the Hodge dual.

In terms of it, we have the remarkable equation for a curved supermanifold

$$\star 1 = \mathrm{Sdet}(G) d^n x \delta^m(d\theta)$$

As an application, given a superfield (namely a (0|0)-superform) $\phi(x,\theta)$

$$\phi(x,\theta) \to d\phi(x,\theta) \to \star d\phi(x,\theta) \in \Omega^{(n-1|m)}$$

And finally, we have

$$\boxed{S_\sigma = \int_{\mathcal{M}} d\phi \wedge \star d\phi}$$

Wess-Zumino action plus high derivative terms

Let us compute the WZ action explicitly

$$d\phi = \partial_\mu \phi (dx^\mu + \theta\gamma^\mu d\theta) + D_i \phi d\theta^i$$

then we can compute its hodge dual (with an invertible metric)

$$\star d\phi = \partial_\mu \phi \star (dx^\mu + \theta\gamma^\mu d\theta) + D_i \phi \star d\theta^i$$

where

$$\star(dx^\mu + \theta\gamma^\mu d\theta) = \epsilon^{\mu\nu\rho}(dx^\nu + \theta\gamma^\nu d\theta)(dx^\rho + \theta\gamma^\rho d\theta)\delta^2(d\theta)$$
$$\star d\theta^i = \Lambda\epsilon^{ij}(dx + \theta\gamma d\theta)^3 \iota_i \delta^2(d\theta)$$

where we used the supermetric

$$\left(\begin{array}{cc} \eta_{\mu\nu} & 0 \\ 0 & \Lambda\epsilon_{ij} \end{array} \right)$$

where Λ is a dimensionfull constant.

Using the integration rules, we finally find

$$S_\sigma = \int_{\mathcal{M}} d\phi \wedge \star d\phi = \fint \Big[\partial_\mu \phi \partial^\mu \phi + \Lambda D_i \phi D^i \phi\Big]$$

which corresponds to the Wess-Zumino action (in 3d) with high-derivative terms .

In 4d the action of pure supergravity N=1 can be written as

$$S_{\mathrm{sugraN=1}} = \int_{\mathcal{M}} \star 1$$

SYM theory can also be written as

$$S_{SYM} = \int_{\mathcal{M}} F^{(2|0)} \wedge \star F^{(2|0)}$$

For SYM, there are some constraints to select the physical degrees of freedom, and they can be implemented by some differential equations.

E. HODGE THEORY

As in the conventional formalism, we set

$$\delta = \star d \star : \Omega^{(p|q)} \rightarrow \Omega^{(p-1|q)}$$

what is relevant here that the operation involves passing through negative-form degree pseudo-forms.

In addition, we can define a Laplace-Beltrami differential

$$\Delta = \delta\, d + d\, \delta$$

- We have checked that for a supermetric, this produces a well-defined operator.
- It reproduces the quadratic Casimir operator for supergroups such as PSU(n|n)
- We have not yet analysed the Hodge theory and the relation between cohomology and harmonic forms

F. DUALITY FOR SUPERFIELDS

By considering YM in (3|2) dimensions, we start from a (1|0)-form

$$A^{(1|0)} \to F^{(2|0)} = dA^{(1|0)} \to \star F^{(2|0)} = F^{(1|2)} \in \Omega^{(1|2)}$$

As usual, the closure of the last field strength implies that there is a (0|2)-form for the dual gauge field.

$$dF^{(1|2)} = 0 \Rightarrow F^{(1|2)} = dA^{(0|2)}$$

$$dF^{(1|2)} \Rightarrow d \star F^{(2|0)} = 0$$

$$dF^{(2|0)} \Rightarrow d \star F^{(1|2)} = 0$$

so the closure of the field strength implies the equations of motion on the dual fields.

Self-duality

The condition for self-duality

$$\star\Omega^{(p|q)} = \Omega^{(n-p|m-q)}$$
$$p = n/2\,, \qquad q = m/2$$

which implies that self-dual superfields live in the pseudo-integral form spaces. But they have an infinite number of components (this is rather similar to what happen in Open String Field Theory).

What about Chern-Simons theory on supermanifolds.
Let us consider (3|2)-dimensional supermanifold?

$$S_{SCS} = \int_{\mathcal{M}} A^{(1|1)} \wedge dA^{(1|1)}$$

work in collaboration with C.Maccaferri

Conclusions

Recently E. Witten wrote a series of papers on the subject revisiting the perturbation theory for superstrings, and there are several applications where this formalism can be used.

This new part of mathematics has a lot of interesting applications:
from string theory (RNS, GS, Pure Spinors, Topological Strings…)
to quantum field theory (Supersymmetric models, Chern-Simons on supermanifolds) and it is largely unexplored.

Čech cohomology of $\mathbb{P}^{1|1}$

$\mathbf{CP}^{(1|1)}$ is the simplest non-trivial example of super projective space. It is defined as usual as an algebraic variety by quotient the complex superspace $\mathbf{C}^{2|1}$ with respect to a complex number different from zero. It can be covered by two patches.

Transition functions from one patch to another

$$U_0 = \{[z_0; z_1] \in \mathbb{P}^1 : z_0 \neq 0\},$$
$$U_1 = \{[z_0; z_1] \in \mathbb{P}^1 : z_1 \neq 0\}.$$

$$\Phi^*(\gamma) = \frac{1}{\widetilde{\gamma}}, \qquad \Phi^*(\psi) = \frac{\widetilde{\psi}}{\widetilde{\gamma}}.$$

The change of patch reflects upon the following transformation

$$\Phi^* \delta^n(d\widetilde{\psi}) = \gamma^{n+1}\delta^n\,(d\psi) - \gamma^n \psi\, d\gamma\, \delta^{n+1}(d\psi).$$

Results for cohomology

For non-negative integer

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong \mathbb{C}^{4n+4},$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong 0.$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \mathbb{C}^{4n}$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong 0,$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}.$$

Notice that $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0})$ and $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1})$ have the same dimension.

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0}) \times \check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \to \check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}$$

Super de Rham cohomology

1. d behaves as a differential on functions;
2. $d^2 = 0$;
3. d commutes with δ and its derivatives, and so $d(\delta^{(k)}(d\psi)) = 0$.

For $n \geq 0$, the holomorphic de Rham cohomology groups of $\mathbb{P}^{1|1}$ are as follows:

$$H_{\mathrm{DR}}^{n|0}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H_{\mathrm{DR}}^{-n|1}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H_{\mathrm{DR}}^{1|1}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong 0.$$

In this computation $H_{\mathrm{DR}}^{0|1}(\mathbb{P}^{1|1}, \mathrm{hol})$ is generated by the constant sheaf

$$\boxed{\psi\,\delta(d\psi)}$$