# Supergroup geometry, supergravity and noncommutative extensions

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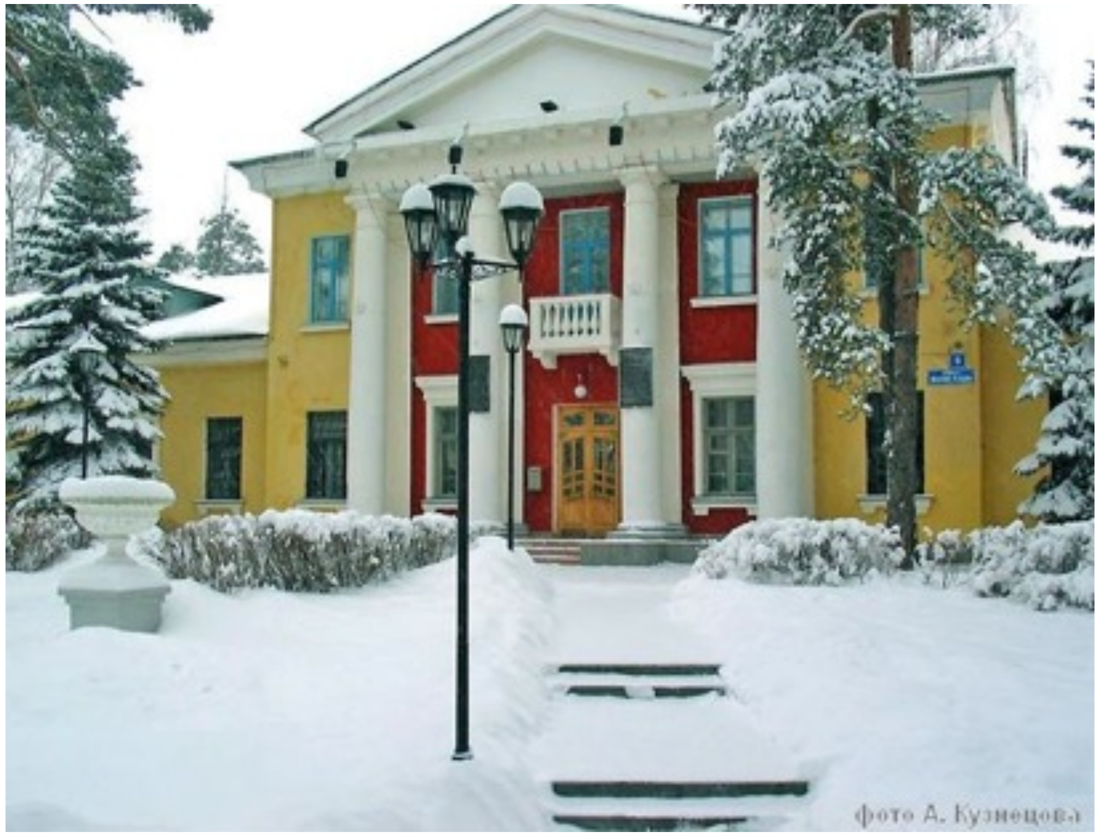


Dubna
Nov. 2015

1. Integration on supermanifolds revisited

2. Actions and invariances

3. Integral representation of the Hodge dual

4. Noncommutative extensions

LC, R.Catenacci, P.A.Grassi 1409.0192 , NPB889 (2014) 419
1503.07886, NPB899 (2015)112
1507.01421, NPB899 (2015) 570
1511.05105

# 40 years of supergravity

Component, Noether method

Superspace

(super) group manifold

## 1. Why supermanifolds

To interpret (local) supersymmetry variations of the fields as the effect of a Grassmann coordinate transformation

This idea can be extended to gauge transformations as well —> supergroup manifolds

Thus diff.s, supersymmetry, gauge transformations are all diffeomorphisms in the supergroup manifold G

They are invariances of an action invariant under group manifold diff.s

Dynamical fields: vielbeins on G

# Example: G = superPoincaré

| generators | vielbein | G-coordinates |
|---|---|---|
| $P_a$ | $\longleftrightarrow V^a$ | $x$ |
| $M_{ab}$ | $\longleftrightarrow \omega^{ab}$ | $y^{ab}$ |
| $Q_\alpha$ | $\longleftrightarrow \psi^\alpha$ | $\theta^\alpha$ |

The group structure is encoded in the Cartan-Maurer equations $d\sigma^A + \frac{1}{2}C^A{}_{BC}\,\sigma^B \wedge \sigma^C = 0$

$$dV^a - \omega^a{}_b\, V^b + \frac{i}{2}\bar{\psi}\gamma^a\psi = 0$$

$$d\omega^{ab} - \omega^a{}_c\,\omega^{cb} = 0$$

$$d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi = 0$$

## 2. Why integration on supermanifolds

To obtain a diff-invariant action on the supergroup manifold: integral of a $n$-form on a $n$-dim supergroup manifold $G$

But $n > d$, where $d$ is the dimension of spacetime.

How can we obtain a $d$-dim field theory where the fields depend only on $d$ space-time coordinates (for ex. $d=4$) ?

→ Start with a Lagrangian $d$-form and integrate it on a $d$-dim submanifold $S$ of $G$

Group geometric construction of supergravity theories, Torino group 80's
Review: LC, D'Auria, Fré

# Action in *d* dim. ⟷ Action in *n* dim.

$$i : S \to G$$

$$\int_S i^* L^{(d)} \quad = \quad \int_G L^{(d)} \wedge \eta_S$$

where $L^{(d)}$ is a *d*-form Lagrangian (on *G*)
$S$ is a (bosonic*) d-dim surface embedded in *G*,
$\eta_S$ is the Poincaré dual of *S*

- If S described locally by the vanishing of *n-d* coordinates *t*

$$\eta_S = \delta(t^1) \cdots \delta(t^{n-d}) \; dt^1 \wedge \cdots \wedge dt^{n-d}$$

a singular closed localization (*n-d*) - form. Projects on the submanifold *S* (*t=0*) and orthogonally to $dt^1 \wedge \cdots \wedge dt^{n-d}$ .

*diffeomorphic to *d*-dim Minkowski spacetime

This action, being the integral of a *n*-form on the *n*-dim supergroup manifold *G*, is invariant under *G*-diffeomorphisms

$$0 = \delta_\epsilon \int_G L^{(d)} \wedge \eta_S = \int_G \ell_\epsilon L^{(d)} \wedge \eta_S + \int_G L^{(d)} \wedge \ell_\epsilon \eta_S$$

→ a change of *S*, generated by the *Lie derivative* $\ell_\epsilon$ along a tangent vector $\epsilon$, can be compensated by a diffeomorphism applied to the fields in ***L***

# Action principle

- The action

$$I[\phi, S] = \int_G L(\phi) \wedge \eta_S$$

depends on fields $\phi$ (contained in *L*) and on the submanifold *S*

- must vary both $\phi$ and *S*
- since variation of embedding of *S* is equivalent to variation of fields, just vary $\phi$ with *S* fixed and arbitrary (variational principle does not determine S)

→ Field equations: $\frac{\delta L}{\delta \phi} = 0$

*d-1* form equations holding on *G*. *"Inner components"* along *S* directions

Finally, dependence of the fields on the extra coordinates:

- disappears for “gauge coordinates”
- for “supersymmetry coordinates” $\theta$ , the fields at $\theta$ are related to the fields at $\theta = 0$

An output of the field equations in “outer” (gauge,susy) directions:

- horizontality of the curvatures: no legs in gauge directions
- rheonomy of curvatures: legs in $\theta$ directions related to legs in x directions

# Invariances

- A kind of holography: invariances of the bulk induce invariances on the boundary (submanifold S)

- Diff.s on *G* are invariances of the action:

$$\delta_\epsilon \int_G L(\phi) \wedge \eta_S = \int_G \ell_\epsilon L(\phi) \wedge \eta_S + \boxed{\int_G L(\phi) \wedge \ell_\epsilon \eta_S}$$

- If second term vanishes, diff.s applied only to the fields in *L* are also invariances of the action

- This happens when $\epsilon$ is orthogonal to *S* (then $\ell_\epsilon \eta_S = 0$) $\longrightarrow$ *spacetime diff.s* , or more generally when

$$\boxed{i_\epsilon dL = 0}$$

(use $\ell_\epsilon = i_\epsilon d + d i_\epsilon$ , $d\eta_S = 0$ and integration by parts )

- Thus spacetime diff.s are always (off-shell) invariances of the restricted action.

**Constructive procedure** ensuring that

$$i_\epsilon dL = 0$$

- is satisfied for $\epsilon$ in "gauge directions", by horizontality
  $\longrightarrow$ restricted action is gauge invariant
- is satisfied for $\epsilon$ in "susy directions", by rheonomy
  $\longrightarrow$ restricted action is locally supersymmetric

on the "partial shell" of outer field equations

- closure of susy algebra: off-shell with auxiliary fields, otherwise only on the shell of inner field eq.s:

# Example: N=1 supergravity in d=4

## Action

$$I_{SG} = \int_{M^4} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4\ \bar{\psi} \wedge \gamma_5 \gamma_d \rho \wedge V^d$$

with $R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}$

$\rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi$

## Invariances (diff.s on superPoincaré group manifold*)

- ordinary x-diff.s
- local Lorentz rotations
- local supersymmetry

$$\delta_\epsilon V^a = i\ \bar{\epsilon}\gamma^a\psi$$

$$\delta_\epsilon \psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon$$

*soft group manifold

- Diff invariance relies on existence of a top form

$$\delta_\epsilon \int (top\ form) = \int (d i_\epsilon + i_\epsilon d)(top\ form) = \int d(i_\epsilon\ top\ form)$$

since *d(top form)* = 0

- Are there top forms also on supermanifolds ?

- Can we integrate them ?

NOTE:

We know how to integrate functions on a supermanifold (Berezin integration).

# Integration of functions on supermanifolds

- Example: real superspace $\mathbb{R}^{n|m}$

$n$ bosonic coordinates $x^i$
$m$ fermionic coordinates $\theta^\alpha$

- Integration of functions

$$f(x,\theta) = f_0(x) + \cdots + f_m(x)\ \theta^1 \cdots \theta^m$$

If the real function $f_m(x)$ is integrable in $\mathbb{R}^n$,

the Berezin integral of $f(x,\theta)$ is defined as

$$\boxed{\int_{\mathbb{R}^{n|m}} f(x,\theta)[d^n x d^m \theta] = \int_{\mathbb{R}^n} f_m(x) d^n x}$$

# Integration of forms on supermanifolds

NB: integration of usual (bosonic) forms

$$\omega = \omega_{[i_1 \cdots i_p]}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

is defined via Riemann-Lebesgue integration of functions

$$\int_{M^p} \omega = \int \omega_{[i_1 \cdots i_p]}(x) \epsilon^{i_1 \cdots i_p} d^p x$$

# Berezin for bosonic forms

Usual integration theory of differential forms for bosonic manifolds can be rephrased in terms of Berezin integration.

The idea is to interpret the differentials *dx* as anticommuting variables $\xi = dx$, similar to the Grassmann coordinates $\theta$

Then the *p*-form

$$\omega = \omega_{[i_1 \cdots i_p]}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

is reinterpreted as a function on a supermanifold $M^{p|p}$ with coordinates *x* and $\xi$

$$\omega(x, \xi) = \omega_{[i_1 \cdots i_p]}(x) \xi^{i_1} \cdots \xi^{i_p}$$

➡ The Berezin integral of this function is

$$\int_{M^{p|p}} \omega(x,\xi)[d^p x d^p \xi] = \int \omega_{[i_1 \cdots i_p]}(x) \epsilon^{i_1 \cdots i_p} d^p x$$

and reproduces $\int_{M^p} \omega$

# Top forms for supermanifolds ?

- There seems to be a problem:
  forms on a supermanifold can be of arbitrarily high order,
  since the $d\theta$ commute !

$$\omega = \omega_{[i_1 \cdots i_r](\alpha_1 \cdots \alpha_s)}(x, \theta)\ dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_s}$$

$d\omega \neq 0$ —-> forms of this type **cannot be top forms**

- Then how can we define integration of forms on a supermanifold ?

- Answer: consider $\omega$ as a function of the differentials

$$\omega = \omega(x, \theta, dx, d\theta)$$

  with *n+m* bosonic variables $x, d\theta$
  and *m+n* fermionic variables $\theta, dx$

- Use then Berezin integration on the function $\omega$ on the “double” supermanifold $M^{n+m|n+m}$

- The only functions of $x, \theta, dx, d\theta$ that can be integrated on $M^{n+m|n+m}$ are the "integral top forms" containing all the $dx$ differentials, and all the $d\theta$ differentials inside delta functions:

$$\omega = \omega_{[i_1 \cdots i_n][\alpha_1 \cdots \alpha_m]}(x,\theta)\ dx^{i_1} \cdots dx^{i_n} \delta(d\theta^{\alpha_1}) \cdots \delta(d\theta^{\alpha_m})$$

NB $\omega$ has compact support as a function of the even variables $d\theta$ : it is in fact a *distribution* with support at the origin, so that the integral over those variables makes sense.

- Note that $\delta(d\theta^\alpha)\,\delta(d\theta^\beta)$ = - $\delta(d\theta^\beta)\,\delta(d\theta^\alpha)$

to be consistent with $\int \delta(d\theta)\delta(d\theta')d(d\theta)d(d\theta') = 1$

- In analogy with the Berezin integral for bosonic forms:

$$\int_{M^{n|m}} \omega = \int_{M^{n+m|n+m}} \omega(x,\theta,dx,d\theta)[d^n x d^m \theta d^n(dx) d^m(d\theta)]$$

$$\equiv \int_{M^{n|m}} \omega_{[i_1 \cdots i_n][\alpha_1 \cdots \alpha_m]}(x,\theta)\epsilon^{i_1 \cdots i_n}\epsilon^{\alpha_1 \cdots \alpha_m}[d^n x d^m \theta]$$

→ consistent theory of integration on supermanifolds

books: Berezin, Manin, DeWitt, Rogers
review articles: Kac, Leites, Voronov, Nelson, Deligne and Morgan, Witten
theory of integral forms initiated in Bernstein and Leites (1977)

including integration on a (bosonic) submanifold of a supergroup manifold,
necessary to give a sound mathematical basis to the group-geometric method outlined above.

# 3. Hodge dual for supermanifolds

LC,Catenacci,Grassi

Based on Fourier transform of superforms.

Again, superforms can be seen as functions of $x, \theta, dx, d\theta$

Then we just need to define Fourier transform of functions of $x, \theta, dx, d\theta$ . Introducing the dual variables $y, \psi, \eta, b$ :

$$\mathcal{F}(\omega)(x, \theta, dx, d\theta) \equiv \int_{\mathbb{R}^{n+m|n+m}} \omega(y, \psi, \eta, b) e^{i(xy+\theta\psi+dx\eta+d\theta b)} [d^n y d^m \psi d^n \eta d^m b]$$

defines the Fourier transform of a superform $\omega$ in $\mathbb{R}^{n|m}$

# Integral representation of the Hodge dual

A **partial Fourier transform** only on the “differential variables”:

$$(\star\omega)(x,\theta,dx,d\theta) \equiv i^{p^2-n^2}\frac{\sqrt{|SdetG|}}{SdetG}\int_{\mathbb{R}^{m|n}}\omega(x,\theta,\eta,b)e^{i(dxA\eta+d\theta Bb)}[d^n\eta d^m b]$$

for a form $\omega$ in $\mathbb{R}^{n|m}$ with bosonic degree = $p$

and supermetric $G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

$\longrightarrow$ $SdetG = \frac{detA}{detB}$

for example in $\mathbb{R}^{3|2}$

$$\star 1 = \sqrt{|\tfrac{detA}{detB}|}\epsilon_{mnp}dx^m dx^n dx^p \delta(d\theta^1)\delta(d\theta^2) \quad \in \quad \Omega^{(3|2)}$$

$$dx^m = \sqrt{|\tfrac{detA}{detB}|}A^{mn}\epsilon_{npq}dx^p dx^q \delta(d\theta^1)\delta(d\theta^2) \quad \in \quad \Omega^{(2|2)}$$

$$d\theta^\alpha = \sqrt{|\tfrac{detA}{detB}|}B^{\alpha\beta}\epsilon_{mnp}dx^m dx^n dx^p i_\beta \delta(d\theta^1)\delta(d\theta^2) \quad \in \quad \Omega^{(2|2)}$$

$$dx^m dx^n = \sqrt{|\tfrac{detA}{detB}|}A^{mp}A^{nq}\epsilon_{pqr}dx^r \delta(d\theta^1)\delta(d\theta^2) \quad \in \quad \Omega^{(1|2)}$$

$$dx^m d\theta^\alpha = \sqrt{|\tfrac{detA}{detB}|}A^{mp}B^{\alpha\beta}\epsilon_{pqr}dx^q dx^r i_\beta \delta(d\theta^1)\delta(d\theta^2) \quad \in \quad \Omega^{(1|2)}$$

$$d\theta^\alpha d\theta^\beta = \sqrt{|\tfrac{detA}{detB}|}B^{\alpha\gamma}B^{\beta\delta}\epsilon_{pqr}dx^p dx^q dx^r i_\gamma i_\delta \delta(d\theta^1)\delta(d\theta^2) \quad \in \quad \Omega^{(1|2)}$$

Properties:

$$\star\star = (-1)^{p(p-n)} \qquad \text{on } p\text{-superforms}$$

Isomorphism

$$\star : \Omega^{(p|0)} \longleftrightarrow \Omega^{(n-p|m)}$$

between *0*-picture *p*-forms (superforms) and *m*-picture (*n*-*p*)-integral forms

⟶ finite dimensional spaces, generalizes Poincaré duality

# NB: new integral representation of Hodge dual also for usual (bosonic) p-forms

$$(\star\omega)(x,dx) \equiv i^{p^2-n^2}\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|n}}\omega(x,\eta)e^{idx\cdot g\cdot\eta}[d^n\eta]$$

Example: in $\mathbb{R}^2$

$$\star 1 = \frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}} e^{idx\cdot g\cdot\eta}[d^2\eta] = \sqrt{|g|}dx^1dx^2$$

$$\star dx^1dx^2 = \frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}}\eta^1\eta^2 e^{idx\cdot g\cdot\eta}[d^2\eta] = \frac{\sqrt{|g|}}{g}$$

$$\star dx^1 = i^{1^2-2^2}\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}}\eta^1 e^{idx\cdot g\cdot\eta}[d^2\eta] = -g^{12}\sqrt{|g|}dx^1 + g^{11}\sqrt{|g|}dx^2$$

$$\star dx^2 = i^{1^2-2^2}\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}}\eta^2 e^{idx\cdot g\cdot\eta}[d^2\eta] = -g^{22}\sqrt{|g|}dx^1 + g^{21}\sqrt{|g|}dx^2$$

# Convolution product of forms in $\mathbb{R}^n$

$\alpha$ p-form

$\beta$ q-form

$$\alpha \bullet \beta(x, dx) = (-1)^{n+pn+pq} \int_{\mathbb{R}^{0|n}} \alpha(x, \xi)\beta(x, dx - \xi)[d^n\xi]$$

defined using Berezin integration on the anticommuting variables

**Properties:** has a unit, the volume form $\star 1$

$$\alpha \bullet \beta = (-1)^{(n-p)(n-q)} \beta \bullet \alpha$$

→ A simple formula for the Hodge dual of a product:

$$\star(\alpha\beta) = (\star\alpha) \bullet (\star\beta)$$

# Some applications

- new superspace actions
- coupling of gauge fields to gravity in group manifold approach
- Hodge operator for NC spaces —> NC gauge theories

# NC spacetime

- motivations from string theory:

  - cannot resolve arbitrarily small structures with finite size objects —> generalized uncertainty principle

  - low energy limit of open strings in a background B - field —> coordinates of end points do not commute

- our perspective: non commutativity as a guide to extended gravity theories

# NC field theories, $\star$ product

- Field theories on NC spaces become especially tractable when non commutativity is encoded in a twisted $\star$product (noncommutative, associative) between ordinary fields

- Example: Moyal-Groenewold $\star$ product:

$$f(x) \star g(x) \equiv f(x) \exp[\frac{i}{2}\theta^{\mu\nu} \overleftarrow{\partial_\mu} \overrightarrow{\partial_\nu}] \, g(x)$$

$$= fg + \frac{i}{2}\theta^{\mu\nu} \, \partial_\mu f \, \partial_\nu g + \frac{1}{2!}\left(\frac{i}{2}\right)^2 \theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\mu\partial_\rho f)\,(\partial_\nu\partial_\sigma g) + \cdots$$

- generalization: abelian twist

$$\partial_\mu \to X_A = X_A^\mu(x)\partial_\mu \quad \text{with} \quad [X_A, X_B] = 0$$

- extension to p-forms: $\wedge_\star$ - product

$$X_A \to \ell_{X_A} \quad \text{(Lie derivative)}$$

$$\tau \wedge_\star \tau' \equiv \tau \wedge \tau' + \frac{i}{2}\theta^{AB}\ \ell_{X_A}\tau \wedge \ell_{X_B}\tau' + \cdots$$

- NC theories are obtained by replacing products between fields in classical actions by $\star$ - products

➡ Nonlocal actions, higher derivatives

➡ Invariant under NC $\star$ symmetries

- Examples:
  - NC Yang-Mills in flat space
  - NC metric gravity (eq.s of motion) - Münich group

- NC vierbein gravity , coupling to fermions (action), NC supergravity - P. Aschieri, LC
- NC Chern-Simons supergravity - LC

after Seiberg-Witten map —> LOCALLY LORENTZ INVARIANT

**P. Aschieri, L.C. :**

*Noncommutative D=4 gravity coupled to fermions*
JHEP 0906(2009)086

*Noncommutative supergravity in D=3 and D=4,*
JHEP 0906(2009)087

*Noncommutative gravity coupled to fermions:  second order expansion via the Seiberg-Witten map,*
JHEP 1207(2012)184

*Noncommutative gauge fields coupled to noncommutative gravity*
Gen.Rel.Grav. 45 (2013) 581-598

*Extended gravity theories from dynamical noncommutativity*
Gen.Rel.Grav. 45 (2013) 411-426

*Noncommutative gravity at second order via the SW map,*
Phys.Rev. D87 (2013) 2, 024017

*Noncommutative Chern-Simons gauge and gravity theories and their geometric Seiberg-Witten map*
JHEP 1411 (2014) 103

**P. Aschieri, L.C., M. Dimitrijevic:**

*Noncommutative gravity at second order via SW map*
Phys.Rev. D87 (2013) 2, 024017

**L.C.:**

*OSp(1|4) supergravity and its noncommutative extension,*
Phys.Rev. D88 (2013) 2, 025022

*Chern-Simons supergravities, with a twist*
JHEP 1307 (2013) 133

# NC Hodge dual, integral representation

$$(\star\omega)(x,dx)_{\star} \equiv i^{p^2-n^2} \int_{\mathbb{R}^{0|n}} \omega(x,\eta) \star e^{idx\star\eta}[d^n\eta]$$

- for constant metric g
- "good" properties, for ex. $\star(f \star \omega) = f \star (\star\, \omega)$

  (f = function 0-form)

P. Aschieri, LC, R. Catenacci, P.A. Grassi

- For Hopf algebras (e.g. quantum groups, quantum plane)

  cf. S. Majid

*Thank you !*

# NC vierbein gravity

- Classical action

$$S = \int R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} = -4 \int R \sqrt{-g}\, d^4x$$

with $V^a = V^a_\mu \, dx^\mu, \quad \omega^{ab} = \omega^{ab}_\mu \, dx^\mu$

$$R^{ab} = R^{ab}{}_{\mu\nu} \, dx^\mu \wedge dx^\nu = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b$$

- Index-free

$$S = \int Tr \, (i\gamma_5 \, R \wedge V \wedge V)$$

with $V = V^a \gamma_a, \quad \Omega = \frac{1}{4}\omega^{ab}\gamma_{ab}, \quad R = d\Omega - \Omega \wedge \Omega$

$$S = \int Tr\ (i\gamma_5\ R \wedge V \wedge V)$$

- Invariances:

  - General coordinate transformations

  - Local Lorentz rotations

$$\delta_\varepsilon V = -V\varepsilon + \varepsilon V, \quad \delta_\varepsilon \Omega = d\varepsilon - \Omega\varepsilon + \varepsilon\Omega$$

with $\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab}$

➡ $$\delta_\varepsilon R = -R\varepsilon + \varepsilon R$$

➡ $$\delta_\varepsilon \int Tr\ (i\gamma_5\ R \wedge V \wedge V) = 0$$

by cyclicity of $Tr$ and $[\gamma_5, \varepsilon] = 0$

- ⋆ vierbein gravity:

$$S = \int Tr\ (i\gamma_5\ R \wedge_\star V \wedge_\star V)$$

with $R = d\Omega - \Omega \wedge_\star \Omega$

- Invariances:

1) GCT: the action is an integral of a 4-form on a 4-manifold

2) ⋆ - gauge invariance under:

$$\delta_\varepsilon V = -V \star \varepsilon + \varepsilon \star V, \quad \delta_\varepsilon \Omega = d\varepsilon - \Omega \star \varepsilon + \varepsilon \star \Omega$$

→ $$\delta_\varepsilon R = -R \star \varepsilon + \varepsilon \star R$$

→ $$\delta_\varepsilon \int Tr\ (i\gamma_5\ R \wedge_\star V \wedge_\star V) = 0$$

by cyclicity of $Tr$ and $\int$ , and if $[\gamma_5, \varepsilon] = 0$

## Note:

$\Omega \wedge_\star \Omega$ contains $[\gamma^{ab}, \gamma^{cd}] \to \gamma^{ef}$ and

$$\{\gamma^{ab}, \gamma^{cd}\} \to 1, \gamma_5$$

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\ \omega 1 + \tilde{\omega}\gamma_5$$

$$\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\ \varepsilon 1 + \tilde{\varepsilon}\gamma_5 \qquad (\delta_\varepsilon \Omega = d\varepsilon + \cdots)$$

$$V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5$$

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + i\ r 1 + \tilde{r}\gamma_5$$

New fields: $\omega, \quad \tilde{\omega}, \quad \tilde{V}^a$

gauge invariance: $SL(2,C) \to \star GL(2,C)$

# Geometrical SW map for abelian twists

- relates NC gauge field $\widehat{\Omega}$ to ordinary (classical) $\Omega$ and $\widehat{\varepsilon}$ to $\varepsilon$ and $\Omega$ so as to satisfy:

$$\boxed{\widehat{\Omega}(\Omega) + \widehat{\delta}_{\hat{\varepsilon}}\widehat{\Omega}(\Omega) = \widehat{\Omega}(\Omega + \delta_{\varepsilon}\Omega)}$$

where

$$\delta_{\varepsilon}\Omega = d\varepsilon - \Omega\varepsilon + \varepsilon\Omega$$

$$\widehat{\delta}_{\hat{\varepsilon}}\widehat{\Omega} = d\widehat{\varepsilon} - \widehat{\Omega} \star \widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{\Omega}$$

- can be solved order by order in $\theta$

$$\widehat{\Omega} = \Omega + \Omega^1(\Omega) + \Omega^2(\Omega) + \cdots$$

$$\widehat{\varepsilon} = \varepsilon + \varepsilon^1(\varepsilon, \Omega) + \varepsilon^2(\varepsilon, \Omega) + \cdots$$

with

$$\Omega^{n+1} = \frac{i}{4(n+1)} \theta^{AB} \{\widehat{\Omega}_A, \ell_B \widehat{\Omega} + \widehat{R}_B\}_\star^n$$

$$\varepsilon^{n+1} = \frac{i}{4(n+1)} \theta^{AB} \{\widehat{\Omega}_A, \ell_B \widehat{\varepsilon}\}_\star^n$$

$$R^{n+1} = \frac{i}{4(n+1)} \theta^{AB} (\{\widehat{\Omega}_A, (\ell_B + L_B) \widehat{R}\}_\star^n - [\widehat{R}_A, \widehat{R}_B]_\star^n)$$

**recursive relations**: generalize Ulker (2008)

- for example:

$$V^{1a} = 0$$

$$\tilde{V}^{1a} = \frac{1}{4}\theta^{AB} X^{\rho}_{A} \omega^{bc}_{\rho} \varepsilon^{a}{}_{bcd} (\ell_B V^d - \frac{1}{2} X^{\sigma}_{B} \omega_{\sigma}{}^{d}{}_{e} V^e)$$

$$\omega^{1ab} = 0$$

$$\omega^1 = -\frac{1}{16}\theta^{AB} X^{\rho}_{A} \omega_{\rho,ab} (\ell_B \omega^{ab} + i_B R^{ab})$$

$$\tilde{\omega}^1 = -\frac{1}{16}\theta^{AB} X^{\rho}_{A} \omega^{ab}_{\rho} (\ell_B \omega^{cd} + i_B R^{cd}) \varepsilon_{abcd}$$

P. Aschieri, L.C. 2011

- Applying the SW map to the fields in the NC gravity action yields a higher derivative action involving only $V^a, \omega^{ab}$ and the background $X_A$ vector fields defining the $\star$product

$$S = S^0 + S^1 + S^2 + \cdots$$

with

$$S^0 = \text{classical Einstein} - \text{Hilbert action}$$

$$S^1 = 0$$

$$S^2 \neq 0$$

Note:

the expanded action, after using the SW map, is gauge invariant under usual gauge transf. order by order in $\theta$

$$S = S^0 + S^1 + S^2 + \cdots$$

Indeed:

- usual gauge transf. induce $\star$ gauge transf. in the NC fields, under which $S$ is invariant
- usual gauge transf. do not contain $\theta$

- **Index-free formalism,** applied to
  $\star$ **gravity with complex vierbein**

  does not reduce to ordinary gravity in the commutative limit,
  Chamseddine 2003

- $\star$ **gravity**, reducing to ordinary gravity in the commutative limit.
  Coupling with spin 1/2 and spin 3/2 fermions, P. Aschieri, L.C. 2009

- $\theta^2$ - expansion of fields and action via SW map,
  P. Aschieri, L.C. 2011

- gauge invariant $\theta^2$ expansion,
  P. Aschieri, M. Dimitrijevic, L.C. 2012

study Haar measure for supergroups