Space-Time Supersymmetry in Ten-dimensional String Theory

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based on paper:
A. Belavin,L. Spodyneiko. Gepner approach to Space-Time Supersymmetry in Ten-dimensional String Theory.

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Preface

- Superstring theory plays an important role in the physics and mathematics. Its essential feature is the space-time supersymmetry, proposed to solve the hierarchy problem.
- Gliozzi, Sherk and Olive discovered thet Space-Time supersymmetry in the fermionic NSR string appears after the special projection of the space of physical states . Also the numbers of physical states in bosonic and fermionic sectors are equal .
- Super-Poincare operators in the covariant approach were built by Friedan, Shenker , Martinec and by Knizhnik. They used the spin field in the matter sector and bosonization of ghost sector to construct them.

Preface

- Gepner and Banks,Dixon,Friedan,Martinec showed that the condition for Space-Time Supersymmetry after compactification of ten-dimensional strings to four-dimensional Minkowski space is the $N = 2$ superconformal symmetry in six compact dimensions.
- Gepner explained how the operator of the so-called Spectral Flow which maps NS-sector in R-sector and vice versa, can be used to construct the space-time SUSY generator.
- Meanwhile, $d = 10$ NSR string itself has a hidden $N = 2$ superconformal symmetry on the world-sheet.
- We use the operator of the corresponding Spectral Flow U , restricting the space of physical fields to ensure their locality with respect to the operator U, to determine the action of Space-Time supersymmetry on this subspace.

NSR String

A. Neveu and J.H. Schwarz. Factorizable dual model of pions. *Nucl.Phys.*, B31:86–112, 1971.

P. Ramond. Dual Theory for Free Fermions. *Phys.Rev.*, D3:2415–2418, 1971.

Space-Time supersymmetry in NSR string

F. Gliozzi, J. Scherk, and D. I. Olive. Supergravity and the Spinor Dual Model. *Phys.Lett.*, B65:282, 1976.

D. Friedan, S. H. Shenker, and E. J. Martinec. Covariant Quantization of Superstrings.*Phys.Lett.*, B160:55–61, 1985.

V.G. Knizhnik. Covariant Fermionic Vertex in Superstrings. *Phys.Lett.*, B160:403–407, 1985.

Connection between N=2 SCA and space-time SUSY

D. Gepner. Space-Time Supersymmetry in Compactified String Theory and Superconformal Models. *Nucl.Phys.*, B296:757, 1988.

T. Banks, L. J. Dixon, D. Friedan, and E. J. Martinec. Phenomenology and Conformal Field Theory Or Can String Theory Predict the Weak Mixing Angle?*Nucl.Phys.*, B299:613–626, 1988.

Main Idea

- NSR string has a hidden $N = 2$ superconformal algebra.
- $N = 2$ superconformal algebra possesses an isomorphism U called spectral flow.
- U interchange R-sector (fermions) and NS-sectors (boson)
- Therefore U is a natural candidate for space-time SUSY.
- U doesn't act on full space of states, but it acts on a reduced subspace.
- This subspace is standard GSO-projected physical states.

Conformal field theory is a field theory with vanishing trace of energy-momentum tensor

$$T_a^a = 0. \tag{1}$$

Important characteristic of field Φ is its conformal dimensions $\Delta, \overline{\Delta}$

$$\Phi(z, \bar{z}) = \lambda^{\Delta} \lambda^{\overline{\Delta}} \Phi(\lambda z, \lambda \bar{z}). \tag{2}$$

Operator product expansion

$$\Phi_1(z)\Phi_2(w) = \sum_n C_{12}^n (z - w)^{\Delta_n - \Delta_1 - \Delta_2} \Phi_n(w) \tag{3}$$

is a common way to do computations.

Variation of a field under transformation $z \to z + \varepsilon(z)$

$$\delta_\varepsilon \Phi = \int dz\, \varepsilon(z) T(z) \Phi(0,0). \tag{4}$$

If one chooses $\varepsilon = z^{n+1}$ then

$$L_n = \int dz\, z^{n+1} T(z), \tag{5}$$

Commutator of L_n

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{6}$$

can be derived from OPE

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w)$$
$$+\text{regular terms.}$$

NSR String

Matter part of NSR string have action

$$S_m = \int d^2z \left[\partial X^\mu \bar{\partial} X_\mu + \psi_\mu \bar{\partial} \psi^\mu + \widetilde{\psi}_\mu \partial \widetilde{\psi}^\mu \right], \qquad (7)$$

where $\mu = 0, \ldots, 9$. These field have OPEs

$$\begin{aligned} X_\mu(z) X_\nu(0) &\sim -\eta_{\mu\nu} \ln z, \\ \psi_\mu(z)\psi_\nu(0) &\sim \frac{\eta_{\mu\nu}}{z}, \end{aligned} \qquad (8)$$

Sectors

ψ_μ can have different monodromy around vertex operators

$$\psi_\mu(e^{2\pi i} z) V(0) = e^{2\pi i \nu} \psi_\mu(z) V(0), \tag{9}$$

$\nu = 1/2$ – R-sector, $\nu = 0$ – NS-sector

The NSR theory has an $N = 1$ SUSY generated by the currents

$$T^m = -\frac{1}{2}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu, \qquad (10)$$
$$G^m = i\psi^\mu \partial X_\mu,$$

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0), \qquad (11)$$
$$T(z)G(0) \sim \frac{3}{2z^2}G(0) + \frac{1}{z}\partial G(0),$$
$$G(z)G(0) \sim \frac{2c}{3z^3} + \frac{2}{z}T(0),$$

or, in modes , commutaters $[L_m, L_n]$ are usual and other are

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 1), \delta_{r,-s} \qquad (12)$$
$$[L_m, G_r] = \frac{m - 2r}{2} G_{m+r},$$

The modes $G(z)$ are integer in R-sector and half-integer in NS-sector.

These commutation relations lead to an important inequality.

$$\left|G_0|\Phi\rangle\right|^2 = \frac{1}{2}\langle\Phi|\{G_0, G_0\}|\Phi\rangle = \langle\Phi|\left(L_0 - \frac{c}{24}\right)|\Phi\rangle = \left(\Delta - \frac{c}{24}\right)\langle\Phi|\Phi\rangle \tag{13}$$

It follows from unitarity that dimensions of all fields in the R-sector satisfy $\Delta \geq \frac{c}{24}$. The field with conformal dimension $\Delta = \frac{c}{24}$ satisfies $G_0|\Phi\rangle = 0$ and is called Ramond vacuum.
In the matter sector of NSR string Ramond vacua $S_\alpha(z)$, with $\alpha = 1, \ldots, 32$ is 32-component spinor, it satisfyes

$$\psi^\mu(z)S_\alpha(0) \sim \frac{1}{\sqrt{2z}}\Gamma^\mu_{\alpha\beta}S_\beta(0), \tag{14}$$

Γ^μ are $10d$ 32×32 gamma matrices . The square root of z leads to the minus sign after translation of the field $\psi_\mu(z)$ around zero. It means that $S_\alpha(z)$ belongs to R-sector and is called the Spin field.

Ghost Sector

Currents $T^m(z), G^m(z)$ are used to impose the constraints in string theory. Using standard procedure, one have to introduce fermionic ghost fields b, c for constraints from $T(z)$ and bosonic ghost field β, γ for $G(z)$. BRST-charge is

$$Q_B = \int dz \left[cT^m + \gamma G^m + \frac{1}{2}\left(cT^{gh} + \gamma G^{gh} \right) \right]. \qquad (15)$$

Physical states are defined as cohomologies of BRST-charge

$$\begin{aligned} Q_B \Phi &= 0, \\ \Phi &\simeq \Phi + Q_B \Psi, \end{aligned} \qquad (16)$$

The so defined physical states have a positive norm .

Ghost action

Ghost part has action

$$S_{gh} = \int d^2 z \left[b\bar{\partial} c + \beta \bar{\partial} \gamma + h.c. \right]. \tag{17}$$

This action has $N = 1$ superconformal symmetry generated by the currents

$$\begin{aligned} T^{gh} &= -\partial b c - 2 b \partial c - \frac{1}{2} \partial \beta \gamma - \frac{3}{2} \beta \partial \gamma, \\ G^{gh} &= \partial \beta c + \frac{3}{2} \beta \partial c - 2 b \gamma. \end{aligned} \tag{18}$$

In order to have well-defined BRST-charge

$$Q_B = \int dz \left[cT^m + \gamma G^m + \frac{1}{2} \left(cT^{gh} + \gamma G^{gh} \right) \right]. \qquad (19)$$

β, γ must have the same monodromy around vertex operators

$$\begin{aligned} \psi_\mu(e^{2\pi i} z) &= e^{2\pi i \nu} \psi_\mu(z), \\ \beta(e^{2\pi i} z) &= e^{2\pi i \nu} \beta(z), \\ \gamma(e^{2\pi i} z) &= e^{2\pi i \nu} \gamma(z), \end{aligned} \qquad (20)$$

Pictures

$\beta - \gamma$ system has many vacuums V_q parameterised by number q, called the number of the picture .
The space of states generated by $\psi_\mu, \partial X_\mu, \beta, \gamma, b, c$ out of the vacuum V_q, is called the picture.
V_q is defined by

$$\begin{aligned} \beta(z)V_q(0) &\sim O(z^q), \\ \gamma(z)V_q(0) &\sim O(z^{-q}). \end{aligned} \tag{21}$$

q is half-integer in R-sector and integer in NS-sector.
The physical states (BRST-cohomologies) in the different pictures which q differ by an integer are isomorphic to each other.
The isomorphism is given by the action of the so-called picture changing operator. It is convenient to choose the canonical pictures $q = -1/2$ in the R-sector and $q = -1$ in the NS-sector.

N=2 Super Conformal Algebra

$N = 2$ superconformal algebra consists of currents $T(z), G^{\pm}(z), J(z)$. They have OPE

$$
\begin{aligned}
T(z)G^{\pm}(0) &\sim \frac{3}{2z^2}G^{\pm}(0) + \frac{1}{z}\partial G^{\pm}(0), \\
T(z)J(0) &\sim \frac{1}{z^2}J(0) + \frac{1}{z}\partial J(0), \\
G^{+}(z)G^{-}(0) &\sim \frac{2c}{3z^3} + \frac{2}{z^2}J(0) + \frac{2}{z}T(0) + \frac{1}{z}\partial J(0), \\
G^{\pm}(z)G^{\pm}(0) &\sim 0, \quad J(z)G^{\pm}(0) \sim \pm\frac{1}{z}G^{\pm}(0), \quad J(z)J(0) \sim \frac{c}{3z^2}.
\end{aligned}
$$

It has $N = 1$ subalgebra $T(z), G(z) = (G^{+} + G^{-})/\sqrt{2}$

N=2 Super Conformal Algebra

$$
\begin{aligned}
G^{\pm}(z) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}\pm\nu} G^{\pm}_{r} z^{-r-3/2}, \\
T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \\
J(z) &= \sum_{n\in\mathbb{Z}} J_n z^{-n-1},
\end{aligned}
\tag{22}
$$

$N = 2$ SCA has relations

$$
\begin{aligned}
&[L_m, G^{\pm}_r] = \left(\frac{m}{2} - r\right) G^{\pm}_{m+r}, \\
&\{G^{+}_r, G^{-}_s\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \\
&\{G^{\pm}_r, G^{\pm}_s\} = 0, \quad [L_m, J_n] = -nJ_{m+n}, \\
&[J_n, G^{\pm}_r] = \pm G^{\pm}_{r+n}, \quad [J_m, J_n] = \frac{c}{3} m\delta_{m+n,0}.
\end{aligned}
\tag{23}
$$

where $n, m \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2} \pm \nu$.

Spectral flow automorphism

$N = 2$ SCA has an isomorphism relating different values of ν

$$\begin{aligned} L'_n &= L_n + \eta J_n + \frac{1}{6}\eta^2 c\delta_{n,0} \\ J'_n &= J_n + \frac{1}{3}\eta\delta_{n,0} \\ (G_r^{\pm})' &= G^{\pm}_{r\pm\eta} \end{aligned} \tag{24}$$

This isomorphism act on a field with dimension Δ and $U(1)$-charge q as

$$\begin{aligned} \Delta' &= \Delta + \eta q + \frac{1}{6}\eta^2 c \\ q' &= q + \frac{1}{3}\eta c \end{aligned} \tag{25}$$

$U(1)$ realization of Spectral flow

Spectral flow can be realized (Gepner) in terms of the bosonic scalar field $\varphi(z)$. Let us bosonise $U(1)$-current

$$\begin{aligned} J(z) &= \partial\varphi(z) \\ \varphi(z)\varphi(0) &\sim \frac{c}{3}\ln z \end{aligned} \tag{26}$$

Define its action on field V as

$$V_\eta = Ve^{\eta\phi} \tag{27}$$

The boson $\varphi(z)$ depends on the realization of the generators of the $N = 2$ SCA in terms of the fields of the theory in which this algebra acts. We show that in the matter and ghost sectors of NSR string there is an $N = 2$ SCA.

For an arbitrary field V, with charge q under the current J, we can isolate the charged part

$$V = \hat{V}e^{i\frac{3q}{c}\phi}, \tag{28}$$

where $\hat{V}$ is neutral under $J(z)$.

$U(1)$ realization of Spectral flow

This procedure for $G^{\pm}(z)$ gives

$$G^{\pm} = \hat{G}^{\pm} e^{\pm \frac{3}{c}\phi}. \tag{29}$$

For every field V we can construct a field twisted by η

$$V_{\eta} = V e^{\eta\phi} = \hat{V} e^{(\frac{3q}{c}+\eta)\phi}. \tag{30}$$

One can show that charge of the twisted field is

$$q' = q + \frac{c}{3}\eta. \tag{31}$$

If the original field has an OPE with the $G^{\pm}$ in integer powers z^n, then the field V_{η} has OPE with $G^{\pm}$ in powers $z^{n\pm\eta}$. The additional power arises from the OPE of $\exp(\eta\phi)$ with $\exp(\pm\frac{3}{c}\phi)$.

$$\Delta' = \Delta + \frac{c}{6}(\frac{3q}{c} + \eta)^2 - \frac{3q^2}{2c} = \Delta + \eta q + \frac{1}{6}\eta^2 c. \tag{32}$$

Multiplication on the vertex $\exp \eta\phi$ realizes the action of Specral flow on the fields . We denote it as the U_{η}.

Chiral Fields in N=2 SCFT

For any NS-field Φ of the dimension Δ and $U(1)$-charge q we have

$$\left|G^{\mp}_{-1/2}|\Phi\rangle\right|^2 + \left|G^{\pm}_{1/2}|\Phi\rangle\right|^2 = \langle\Phi|\{G^{\pm}_{1/2}, G^{\mp}_{-1/2}\}|\Phi\rangle = (2\Delta \pm q)\langle\Phi|\Phi\rangle \geq 0. \tag{33}$$

It follows in a unitary theory in the NS-sector there is an inequality

$$2\Delta \geq |q|. \tag{34}$$

The fields with $2\Delta = q$ are called chiral fields. Such field Φ satisfies

$$G^{-}_{1/2}\Phi = G^{+}_{-1/2}\Phi = 0. \tag{35}$$

Using this and the relations of SCA , the restriction $2\Delta \geq |q|$, we can show that

$$\begin{aligned} &L_n\Phi = J_n\Phi = 0, \qquad n > 0, \\ &G^{+}_{r}\Phi = 0, r \geq -\frac{1}{2}, \quad G^{-}_{r}\Phi = 0, r > 0. \end{aligned} \tag{36}$$

Ramond vacua in N=2 SCFT

Since the $N=2$ superconformal algebra has the $N=1$ subalgebra, there is a restriction on the dimension $\Delta \geq \frac{c}{24}$ for the Ramond fields. Moreover, the Ramond field Φ with the dimension $\Delta = \frac{c}{24}$ satisfies

$$G_0 \Phi = 0. \tag{37}$$

Because of the restriction $\Delta \geq \frac{c}{24}$, Φ is annihilated by all the positive modes of the currents $G^{\pm}(z)$, $T(z)$, $J(z)$. Using the commutation relations of the $N=2$ algebra, one can show that if the field is annihilated by $G_0 = (G_0^+ + G_0^-)/\sqrt{2}$, then it is annihilated by $G_0^{\pm}$ separately. All this reads

$$\begin{aligned} L_n \Phi = J_n \Phi = 0, , \quad & n > 0, \\ G_n^{\pm} \Phi = 0, \quad & n \geq 0. \end{aligned} \tag{38}$$

Spectral flow of N=2 SCA and Space-time SUSY

- The physical states of the NS-sector are the space-time bosons.
- The states of the R-sector are fermions.
- The spectral flow with $\eta = \pm 1/2$ translates NS- into R- sector and backwards.
- It suggests that the corresponding vertex operator $\exp(\eta\phi)$ is the supercharge or at least its component.
- In what follows, we will show that it is true.

$N = 2$ in matter sector of NSR string.

Choose another basis in $N = 2$

$$\psi_k^{\pm} = \frac{1}{\sqrt{2}}(\psi^{2k} \pm i\psi^{2k+1}), \quad X_k^{\pm} = \frac{1}{\sqrt{2}}(X^{2k} \pm iX^{2k+1}). \tag{39}$$

Currents of $N = 2$ SCA

$$G_+^m = \sum_k i\sqrt{2}\psi_k^+ \partial X_k^-, \quad G_-^m = \sum_k i\sqrt{2}\psi_k^- \partial X_k^+, \quad J^m = \sum_k \psi_k^+ \psi_k^-,$$

Note, that the choice of the $N = 2$ superconformal algebra is not unique . One can take another $U(1)$-current

$$J^m = \psi_\mu \Lambda_{\mu\nu} \psi_\nu, \tag{40}$$

where $\Lambda_{\mu\nu}$ is a nondegenerate antisymmetric matrix with eigenvalues ± 1.
By a Lorentz transformation one can bring the $U(1)$-current to a such form .

Bosonization of Matter sector

Ten ψ_μ can be realized in terms of 5 bosons H_k with OPEs

$$H_a(z)H_b(0) \sim -\delta_{ab} \ln z, \tag{41}$$

via

$$\psi_k^{\pm} = e^{\pm i H_k} \tag{42}$$

In terms of bosons $U(1)$-charge takes form

$$J^m = \partial H^m, \tag{43}$$

where

$$H^m = \sum_k iH_k. \tag{44}$$

$N = 2$ in ghost sector of NSR string

There is $N = 2$ SCA in ghost system. The currents are

$$G_+^{gh} = \sqrt{2}\partial\beta c + \frac{3}{\sqrt{2}}\beta\partial c,$$
$$G_-^{gh} = -2\sqrt{2}b\gamma, \quad J^{gh} = -2cb + 3\beta\gamma. \tag{45}$$

Ghost system can be bosonized by three bosons ϕ, χ, σ

$$c = e^{\sigma}, \quad b = e^{-\sigma}.$$
$$\beta = e^{-\phi+\chi}\partial\chi, \quad \gamma = e^{\phi-\chi} \tag{46}$$

$N = 2$ in ghost sector of NSR string

In term of bosons

$$J^{gh} = \partial H^{gh}, \tag{47}$$

where

$$H^{gh} = 3\phi - 2\sigma. \tag{48}$$

The ghost energy-momentum tensor reads

$$T_{gh} = T_\phi + T_\chi + T_\sigma, \tag{49}$$

where

$$\begin{aligned} T_\phi &= -\frac{1}{2}\partial\phi\partial\phi - \partial^2\phi, \\ T_\chi &= \frac{1}{2}\partial\chi\partial\chi + \frac{1}{2}\partial^2\chi, \\ T_\sigma &= \frac{1}{2}\partial\sigma\partial\sigma + \frac{3}{2}\partial^2\sigma. \end{aligned} \tag{50}$$

$N = 2$ in ghost sector of NSR string

$\beta - \gamma$ system has vacua $V_q(z)$ parameterized by number q called the number of the picture , and the space of states generated by $\psi_\mu, \partial X_\mu, \beta, \gamma, b, c$ out of the vacuum V_q, is called the picture. The vacua V_q are determined by the conditions

$$\begin{aligned} \beta(z)V_q(0) &\sim O(z^q), \\ \gamma(z)V_q(0) &\sim O(z^{-q}). \end{aligned} \tag{51}$$

It follows that the translation of β, γ around the origin produces a phase $e^{2\pi iq}$. Therefore, q must be an integer in the NS-sector and half-integer in the R-sector.
The physical states in the different pictures are isomorphic to each other.
The isomorphism is given by the action of the so-called picture changing operator. It is convenient to choose the canonical pictures $q = -1/2$ in the R-sector and $q = -1$ in the NS-sector.

$N = 2$ in ghost sector of NSR string

The vacua V_q in terms of ϕ have the form

$$V_q = e^{q\phi}. \tag{52}$$

One can show that contraction of $\exp q\phi$ with β, γ has the proper powers of z .
The dimension of the vacuum, $\Delta(\exp(q\phi)) = -(q^2 + 2q)/2$.
Using the formulas

$$\begin{aligned} \beta\gamma &= \partial\phi, \\ bc &= -\partial\sigma, \end{aligned} \tag{53}$$

one can show that the $U(1)$-current is

$$J^{gh} = \partial H^{gh}, \tag{54}$$

where

$$H^{gh} = 3\phi - 2\sigma. \tag{55}$$

Vertex operators are massless bosons and fermions

The general form of a vertex in the NS-sector is

$$P(\psi_\mu, \partial X_\mu, \beta, \gamma, b, c) V_q e^{ik_\mu X^\mu}, \tag{56}$$

where P is a polynomial of its arguments, and k_μ is a momentum.
In the NS-sector q must be an integer.
The general form of a vertex in the R-sector is

$$P^\alpha(\psi_\mu, \partial X_\mu, \beta, \gamma, b, c) S_\alpha V_q e^{ik_\mu X^\mu}, \tag{57}$$

where P^α is a polynomial of its arguments and it transforms as a 32-component spinor. In the R-sector q is half-integer.
Two important examples of vertex operators are massless bosons and fermions in the pictures -1 and $-1/2$

$$\begin{aligned} V_{NS} &= \xi_\mu \psi^\mu V_{-1} e^{ik_\mu X^\mu} \\ V_R &= u^\alpha S_\alpha V_{-1/2} e^{ik_\mu X^\mu} \end{aligned} \tag{58}$$

where u_α, ξ_μ are polarizations.

Total $N = 2$ superconformal symmetry in NSR string

The total currents of $N = 2$ SCA in NSR string

$$\begin{aligned} T^{tot} &= T^m + T^{gh}, \\ G^{tot}_{\pm} &= G^m_{\pm} + G^{gh}_{\pm}, \\ J^{tot} &= J^m + J^{gh}. \end{aligned} \tag{59}$$

After bosonization the $U(1)$-current reads

$$J^{tot} = \partial H^{tot} = \partial H^m + \partial H^{gh} = \sum_k i\partial H_k + 3\partial\phi - 2\partial\sigma. \tag{60}$$

The spectral flow is a natural candidate for the SUSY generator

$$Q(z) = U_{-1/2} = \exp\left(-\frac{1}{2}H^{tot}\right). \tag{61}$$

Using the expression of H^{tot} in terms of the bosons, one can rewrite it in terms S_α is the spin field with all the spins down.

$$Q(z) = \exp\left(-\frac{1}{2}\sum iH_k - \frac{3}{2}\phi + \sigma\right) = cS_\alpha e^{-\frac{3}{2}\phi}, \tag{62}$$

GSO-projection

A general vertex operator of a physical state is

$$P(\partial X_\mu, \partial H_k, \partial\sigma, \partial\phi, \partial\chi) \exp\left[l\phi + r\chi + m\sigma + \sum_k is_k H_k + ik_\mu X^\mu \right], \tag{63}$$

SUSY generator

$$Q_\alpha = \int dz\, S_\alpha e^{-3/2\phi}. \tag{64}$$

Their mutual phase

$$2\pi i(3l/2 - \frac{1}{2}\sum s_k). \tag{65}$$

must be integer.
It is equivalent to the GSO-projection of this space.
So GSO-projection is necessary for the very possibility of determining the Space-Time supersymmetry action on the physical states.

Spectral flow of the massles states

The spectral flow $U_{\pm 1/2}$ transforms the Ramond vacua and Chiral fields into each other. Indeed, the spectral flow acts as

$$\begin{aligned} U_{1/2}\left|\begin{matrix}\Delta=\frac{q}{2}\\ Q=q\end{matrix}\right\rangle_{NS} &= \left|\begin{matrix}\Delta=\frac{c}{24}\\ Q=q-\frac{c}{6}\end{matrix}\right\rangle_{R}, \\ U_{-1/2}\left|\begin{matrix}\Delta=\frac{q}{2}\\ Q=-q\end{matrix}\right\rangle_{NS} &= \left|\begin{matrix}\Delta=\frac{c}{24}\\ Q=-q+\frac{c}{6}\end{matrix}\right\rangle_{R}. \end{aligned} \tag{66}$$

The vertex operators of the massless bosons are Chiral fields. And the vertex operators of the massless fermions are the Ramond vacua.

Massless boson vertex operator has form

$$V_{-1} = c\xi_\mu \psi^\mu e^{ikX} e^{-\phi}, \tag{67}$$

One can check that matter part ψ^μ as well as ghost part $ce^{-\phi}$ are Chiral fields.
Massless fermion vertex has form

$$V_{-1/2} = u^\alpha S_\alpha e^{ikX} e^{-\phi/2}. \tag{68}$$

One can see that matter part S_α as well as ghost part $ce^{-\phi/2}$ are Ramond vacua.
Therefore Spectral flow of $N = 2$ algebra interchanges massless fermion and boson states .
This means that these states form a supermultiplet !