Critical Behavior of a General $O(n)$ symmetric Model of two n Vector Fields in $D = 4 - 2\epsilon$

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Table of contents

Introduction I

- The renormalization group approach provides a natural framework for the understanding of critical properties of phase transitions. A very large variety of critical phenomena can be described by so called ϕ^4 models, $\phi = (\phi_1, \ldots, \phi_n)$. There are several ϕ^4 models:
  - The common $O(n)$ symmetric one field model (τ is a temperature-like parameter and $g > 0$):

$$S_{O(n)}(\phi) = \frac{1}{2}\left[(\nabla\phi)^2 + \tau\phi^2\right] + \frac{1}{4!}g\left(\phi^2\right)^2$$

  - Extended $O(n) + O(m)$ symmetric model

$$S_{O(n)+O(m)}(\phi_1, \phi_2) = \frac{1}{2}\left[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + \tau_1\phi_1^2 + \tau_2\phi_2^2\right] + \frac{1}{4!}\left[g_1(\phi_1^2)^2 + g_2(\phi_2^2)^2 + g_3(\phi_1^2)(\phi_2^2)\right]$$

Introduction I

- In the $O(n) + O(m)$ model six different fixpoints (FP) were found. Three of them are always unstable and the stability of three others depends on n and m (M.Fisher et al).
- The $O(n) + O(m)$ model has been used to describe multicritical phenomena. (The critical behavior of uniaxial antiferromagnets in a magnetic field parallel to the field direction, the $SO(5)$ theory of high T_c superconductors).
- Also interesting phenomena of inverse symmetry breaking, symmetry non restoration and reentrant phase transitions were reported (Weinberg, Ramos, Pinto).

Introduction II

- Recently frustrated spin systems with non collinear or canted spin ordering have been the object of intensive research (Kawamura, Pelissetto and Vicary et al., Holovatch) (Examples: Helical magnets and layered triangular Heisenberg antiferromagnets). Both fields have n components and the model possesses the $O(n) \times O(2)$ symmetry.

$$\begin{aligned} S_{O(n)\times O(2)}(\phi_1, \phi_2) =& \frac{1}{2}\left[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + \tau\left(\phi_1^2 + \phi_2^2\right)\right] \\ &+ \frac{1}{4!}u\left(\phi_1^2 + \phi_2^2\right)^2 + \frac{1}{4!}v\left[(\phi_1\phi_2)^2 - (\phi_1^2)(\phi_2^2)\right] \end{aligned}$$

- Here the scalar product $\phi_1\phi_2$ is present.
- In the $4 - 2\epsilon$ expansion, the number of FPs and their stability depend on n.

Introduction III

- We have studied the critical behavior of the O(n)-symmetric model with two n-vector fields within the RG field-theoretical approach in $4-2\epsilon$ expansion.

$$
\begin{aligned}
S_{O(n)}(\phi_1,\phi_2) = {} & \frac{1}{2}\left[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + \tau_1\phi_1^2 + \tau_2\phi_2^2 + 2\tau_3\phi_1\phi_2\right] \\
& + \frac{1}{8}\Big[g_{11}(\phi_1^2)^2 + g_{22}(\phi_2^2)^2 + 2g_{12}(\phi_1^2)(\phi_2^2) \\
& \qquad + 2g_{33}(\phi_1\phi_2)^2 + 2\sqrt{2}g_{13}(\phi_1)^2(\phi_1\phi_2) + 2\sqrt{2}g_{23}(\phi_2)^2(\phi_1\phi_2)\Big]
\end{aligned}
$$

- The model becomes O(n)+O(n) symmetric when $\tau_3 = g_{33} = g_{13} = g_{23} = 0$.
- Setting $\tau_1 = \tau_2,\ \tau_3 = g_{13} = g_{23} = 0,\ g_{11} = g_{22},\ g_{12} = g_{11} - g_{33}$ leads to the O(n)$\times$O(2) model of frustrated spins.

$4 - 2\epsilon$ Expansion I

- It is useful to rewrite the interaction part of $S_{O(n)}$ as

$$S_{\text{int}}(\phi_1, \phi_2, g) \quad = \tfrac{1}{8}\sum_{k,l=1}^{3} \mathcal{I}_k g_{kl} \mathcal{I}_l = \tfrac{1}{8}\mathcal{I} g \mathcal{I},$$

where

$$\begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \end{pmatrix} = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \\ \sqrt{2}\phi_1\phi_2 \end{pmatrix}.$$

$4-2\epsilon$ Expansion II

- The expression for the critical exponents can be taken from (Brézin, le Guillou, and Zinn-Justin in Domb-Green Vol.6).
- The six β functions $\beta_{ij} \equiv \mu\partial_\mu g_{ij}$, where μ is an auxiliar parameter with the critical dimension 1, can be written in 1-loop order

$$\beta_{ij} = -2\epsilon g_{ij} + \frac{1}{2}(n+8)g_{ik}g_{kl} + \frac{1}{2}C_{ij,kl,mn}g_{kl}g_{mn}$$

with

i,j	$C_{ij,kl,mn}g_{kl}g_{mn}$
$1,1$	$-8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2$
$1,2$	$-6g_{11}g_{12} - 6g_{12}g_{22} - 4g_{13}g_{23} + g_{11}g_{33} + 4g_{12}^2 + 2g_{13}^2 + g_{22}g_{33} + 2g_{23}^2 + g_{33}^2$
$1,3$	$-6g_{12}g_{23} - 3g_{13}g_{33} + 6g_{12}g_{13} + 3g_{23}g_{33}$
$2,2$	$-8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2$
$2,3$	$-6g_{12}g_{13} - 3g_{23}g_{33} + 6g_{12}g_{23} + 3g_{13}g_{33}$
$3,3$	$-2g_{13}^2 - 2g_{23}^2 - 6g_{33}^2 + 2g_{11}g_{33} + 8g_{12}g_{33} + 4g_{13}g_{23} + 2g_{22}g_{33}$

We have rescaled the couplings by a factor $8\pi^2$ as usual.

- The FPs g^* are the solutions of $\beta_{ij}(g^*) = 0$. $S_{O(n)}$ is symmetric under the simultaneous interchange of g_{11} with g_{22} and g_{13} with g_{23}. The simultaneous change of signs of g_{13} and g_{23} leaves the solution invariant.

$4 - 2\epsilon$ Expansion III

- The stability-matrix can be easily obtained:

$$\omega_{ij,kl} = \partial \beta_{ij}(g) / \partial g_{kl}|_{g=g^*}$$

Its eigenvalues are the critical exponents ω.

- The critical exponents η are obtained from the eigenvalues γ^*_Φ of the symmetric 2×2 matrix γ_Φ at $g = g^*$,

$$\begin{aligned}
\{\gamma_\Phi\}_{11} &= \frac{1}{16}\left(2(n+2)g_{11}^2 + (n+2)g_{23}^2 + (n+1)g_{33}^2 + 2ng_{12}^2 + 4g_{12}g_{33} + 3(n+2)g_{13}^2\right), \\
\{\gamma_\Phi\}_{21} &= \frac{\sqrt{2}(n+2)}{16}\left((g_{11} + g_{12} + g_{33})g_{13} + (g_{22} + g_{12} + g_{33})g_{23}\right), \\
\{\gamma_\Phi\}_{22} &= \frac{1}{16}\left((n+2)g_{13}^2 + 2(n+2)g_{22}^2 + 2ng_{12}^2 + 3(n+2)g_{23}^2 + (n+1)g_{33}^2 + 4g_{12}g_{33}\right),
\end{aligned}$$

calculated at the specific FP, with respect to $\eta_i = 2\gamma^*_\Phi$.

- The critical indices $1/\nu = 2 + \gamma^*_\tau$ are obtained from the eigenvalues γ^*_τ of (in one-loop order):

$$\gamma_\tau = -\frac{1}{2}\begin{pmatrix} (n+2)g_{11} & ng_{12} + g_{33} & (n+2)g_{13} \\ ng_{12} + g_{33} & (n+2)g_{22} & (n+2)g_{23} \\ (n+2)g_{13} & (n+2)g_{23} & 2g_{12} + (n+1)g_{33} \end{pmatrix}_{g=g^*}.$$

$4 - 2\epsilon$ Expansion IV

- The three crossover exponents are eigenvalues of the 3×3 matrix (in one-loop order)

$$\gamma_{\mathrm{cr,s}} = -\frac{1}{2} \left(\begin{array}{ccc} 2g_{11} & g_{33} & 2g_{13} \\ g_{33} & 2g_{22} & 2g_{23} \\ 2g_{13} & 2g_{23} & 2g_{12} + g_{33} \end{array} \right)_{g=g^*} .$$

- The fourth crossover exponent is (in one-loop order)

$$\gamma_{\mathrm{cr,a}} = -g_{12} + \frac{1}{2} g_{33}.$$

table of contents

Field rotations I

- The direct calculation of FPs from the β functions leads to more than 50 FPs and several lines of FPs!
- Some FPs are equivalent due to the internal rotation of the fields:

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

- Performing the rotation yields

$$\begin{pmatrix} \mathcal{I}_1' \\ \mathcal{I}_2' \\ \mathcal{I}_3' \end{pmatrix} = M \begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \end{pmatrix}, \quad M = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos(2\varphi) & \frac{1}{2} - \frac{1}{2}\cos(2\varphi) & \sqrt{\frac{1}{2}}\sin(2\varphi) \\ \frac{1}{2} - \frac{1}{2}\cos(2\varphi) & \frac{1}{2} + \frac{1}{2}\cos(2\varphi) & -\sqrt{\frac{1}{2}}\sin(2\varphi) \\ -\sqrt{\frac{1}{2}}\sin(2\varphi) & \sqrt{\frac{1}{2}}\sin(2\varphi) & \cos(2\varphi) \end{pmatrix}$$

- The matrix M is orthogonal and the interaction transforms according to

$$S_{\mathrm{int}}(\phi_1', \phi_2', g') = \frac{1}{8} \mathcal{I}'^T g' \mathcal{I}', \quad g' = M g M^T.$$

- Obviously both sets of couplings describe the same critical behavior.

Field rotations II

- One finds that the following is invariant under the rotations

$$a_1 = g_{11} + g_{22} + 2g_{12}, \quad a_2 = g_{11} + g_{22} + g_{33} \tag{1}$$

- whereas

$$\begin{aligned} a_{31} &= g_{11} - g_{22}, & a_{32} &= \sqrt{2}(g_{13} + g_{23}), \\ a_{41} &= -g_{11} + 2g_{12} - g_{22} + 2g_{33}, & a_{42} &= -\sqrt{8}(g_{13} - g_{23}) \end{aligned}$$

transform according to

$$\begin{pmatrix} a'_{31} \\ a'_{32} \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & \sin(2\varphi) \\ -\sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} a_{31} \\ a_{32} \end{pmatrix}$$

and

$$\begin{pmatrix} a'_{41} \\ a'_{42} \end{pmatrix} = \begin{pmatrix} \cos(4\varphi) & \sin(4\varphi) \\ -\sin(4\varphi) & \cos(4\varphi) \end{pmatrix} \begin{pmatrix} a_{41} \\ a_{42} \end{pmatrix}.$$

- For the interactions invariant under $O(n) \times O(2)$ the amplitudes a_{31}, a_{32}, a_{41}, a_{42} have to vanish. Otherwise we may choose φ.
- We will choose it so that $a_{42} = 0$, i.e $g_{23} = g_{13}$.
- From the FPs with the condition $g_{23} = g_{13}$, all other FPs can be obtained by means of the orthogonal transformations leaving the expressions (1) invariant.

table of contents

The classification of the fixed points in the large n limit

- In the large n limit we may neglect the last term in the β functions:

$$\beta_{ij} = -2\epsilon g_{ij} + \frac{1}{2}(n+8)g_{ik}g_{kl} + \frac{1}{2}C_{ij,kl,mn}g_{kl}g_{mn}$$

expressing g in terms of the matrix p,

$$g = 4\epsilon p/(n+8).$$

- At criticality ($\beta_{ij} \equiv 0$) and in the limit $n \to \infty$ the matrix p becomes idempotent: $p = p^2$.
- The only eigenvalues of idempotent matrices are 0 and 1. Thus depending on the number k of eigenvalues 1 there are four types of symmetric (3×3) idempotent matrices $p^{(k)}$

$$p_{ij}^{(0)} = 0, \quad p_{ij}^{(1)} = z_i z_j, \quad p_{ij}^{(2)} = \delta_{ij} - z_i z_j, \quad p_{ij}^{(3)} = \delta_{ij}; \quad i,j = 1,2,3,$$

with the restriction

$$z_1^2 + z_2^2 + z_3^2 = 1.$$

- Further conditions on z for the classes $p^{(1,2)}$ can be obtained by considering the first two orders in $1/(n+8)$ to g^*.

The classification of the FPs in the large n, class $p^{(0)}$

- This class consists of the trivial FP

$$g^* = 4\epsilon p^{(0)}/(n+8) = 0$$

  only.
- The stability-matrix is diagonal:

$$\omega_{ij} = -(2\epsilon)\delta_{ij}$$

  All its eigenvalues are negative and the FP is unstable.
- This FP is exact and remains invariant under the orthogonal transformations.

The classification of the FPs in the large n, class $p^{(1)}$

- The following ansatz (h is symmetric) is put into the β-functions

$$g^*_{ij} = \frac{4\epsilon}{(n+8)} z_i z_j + \frac{4\epsilon}{(n+8)^2} h_{ij} + O\left(\frac{1}{(n+8)^3}\right) \tag{2}$$

.

- We then obtain the following conditions on z:

$$(1 - z_{12}^2)(4 - z_{12}^2) z_{12}(z_1 - z_2) = 0 \quad (1 - z_{12}^2)(4 - z_{12}^2) z_{12} z_3^2 = 0, \tag{3}$$

where $z_{12} := z_1 + z_2$. Thus solutions are given by

$$z_{12} = 0, \pm 1, \pm 2, \pm\sqrt{2},$$

- the first solutions follow immediately from the eqs. (3), whereas the last pair follows from $z_1 - z_2 = 0$, $z_3 = 0$ and $z_1^2 + z_2^2 + z_3^2 = 1$ and describes an O(n)×O(2)-invariant interaction. Due to (2) a change of the sign of the zs does not alter the FP. Thus z_{12} and $-z_{12}$ yield the same class of FPs.

The classification of the FPs in the large n, class $p^{(1)}$

- The solutions $z_{12} = 0, \pm 1, \pm 2, \pm\sqrt{2}$ divide the class $p^{(1)}$ into subclasses. Each subclass has its own characteristic critical exponents
- While $z_{12} = z_1 + z_2$ stays constant, $z_1 - z_2$ and z_3 vary under rotation according to
$$(z_1 - z_2)^2 + 2z_3^2 = 2 - z_{12}^2.$$
- For $z_{12} \neq \pm\sqrt{2}$ one obtains a whole continuum of solutions, i.e. lines of FPs!
- Each subclass has its own characteristic critical exponents. In the limit of large n they are

$$\begin{aligned}
\omega = & \quad \{(2\epsilon), 0\ (2\times), -(2\epsilon)\ (3\times)\}\,, \quad \gamma_\tau^* = \{-(2\epsilon), 0\ (2\times)\}\,, \\
\gamma_{\mathrm{cr}}^* = & \quad \left\{\frac{(2\epsilon)}{n}(-1 \pm z_{12}\sqrt{2 - z_{12}^2}), \frac{(2\epsilon)}{n}(1 - z_{12}^2)\ (2\times)\right\}, \\
\gamma_\Phi^* = & \quad \left\{\frac{(2\epsilon)^2}{8n}(1 \pm z_{12}\sqrt{2 - z_{12}^2})\right\}.
\end{aligned}$$

The classification of the FPs in the large n, class $p^{(2)}$

- The following ansatz (h is symmetric) is put into the β-functions

$$g^*_{ij} = \frac{4\epsilon}{n+8}(\delta_{ij} - z_i z_j) + \frac{4\epsilon}{(n+8)^2} h_{ij} + O\left(\frac{1}{(n+8)^3}\right), \tag{4}$$

.

- We then obtain the following conditions on z:

$$(z_{12}^2 + 1) z_{12}^2 (z_1 - z_2) = 0, \quad (z_{12}^2 + 1) z_{12} z_3^2 = 0. \tag{5}$$

where $z_{12} := z_1 + z_2$. Thus solutions are given by

$$z_{12} = 0, \pm \mathrm{i}, \pm\sqrt{2},$$

where the first two solutions are immediately obvious from eqs. (5) and the last one follows from $z_1 = z_2$, $z_3 = 0$, and $z_1^2 + z_2^2 + z_3^2 = 1$. This last solution represents an O(n)$\times$ O(2)-invariant model.

The classification of the FPs in the large n, class $p^{(2)}$

- The solutions $z_{12} = 0, \pm\mathrm{i}, \pm\sqrt{2}$ divide the class $p^{(2)}$ into subclasses. Each subclass has its own characteristic critical exponents
- While $z_{12} = z_1 + z_2$ stays constant, $z_1 - z_2$ and z_3 vary under rotation according to
$$(z_1 - z_2)^2 + 2z_3^2 = 2 - z_{12}^2.$$
- For $z_{12} \neq \pm\sqrt{2}$ one obtains a whole continuum of solutions, i.e. lines of FPs!
- Each subclass has its own characteristic critical exponents. In the limit of large n they are

$$\begin{aligned}
\omega &= \{(2\epsilon)\ (3\times), 0\ (2\times), -(2\epsilon)\}\,, \quad \gamma_\tau^* = \{-(2\epsilon)\ (2\times), 0\}\,,\\
\gamma_{\mathrm{cr}}^* &= \left\{\frac{(2\epsilon)}{n}(-2 + z_{12}^2), \frac{(2\epsilon)}{n}(-1 \pm \sqrt{1 + 2z_{12}^2 - z_{12}^4}), \frac{(2\epsilon)}{n} z_{12}^2\right\},\\
\gamma_\Phi^* &= \left\{\frac{(2\epsilon)^2}{8n}(2 \pm z_{12}\sqrt{2 - z_{12}^2}),\right\}.
\end{aligned}$$

The classification of the FPs in the large n, class $p^{(3)}$

- In the large n limit one obtains

$$g^* = 4\epsilon p^{(3)}/(n+8) = 4\epsilon\delta_{ij}/(n+8),$$

which yields the exponents in leading order

$$\omega = \{(2\epsilon)\ (6\times)\}, \quad \gamma_\tau^* = \{-(2\epsilon)\ (3\times)\},$$

$$\gamma_{\rm cr}^* = \left\{\frac{-3(2\epsilon)}{n}, \frac{-(2\epsilon)}{n}\ (2\times), \frac{(2\epsilon)}{n}\right\}, \quad \gamma_\Phi^* = \left\{\frac{3(2\epsilon)^2}{8n}\ (2\times)\right\}.$$

table of contents

Solutions for finite n, Fixed Points and Critical Exponents

- The 'gauge' condition $g_{13} = g_{23}$ yields (simple) Representative Solutions (RS), all other solutions can be obtained by means of rotations.
- For each class $p^{(k)}$ there is one solution invariant under $\mathrm{O}(n)\times\mathrm{O}(2)$
- Solutions not invariant under $\mathrm{O}(n)\times\mathrm{O}(2)$ have one exponent $\omega = 0$ since the field rotations create lines of fixed points.
- All solutions with the exception of the trivial FP have one exponent $\omega = 2\epsilon$ independent of n in one-loop order, since $\beta_{ij} = -2\epsilon g_{ij} +$ term bilinear in the gs

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 0.1** This is the trivial (interaction free) fixed point. All anomalous exponents γ^* vanish

$$\gamma^*_\Phi = \{0\ (2\times)\}\,,\quad \gamma^*_\tau = \{0\ (3\times)\}\,,\quad \gamma^*_{\mathrm{cr}} = \{0\ (4\times)\}\,,\quad \omega = \{-(2\epsilon)\ (6\times)\}$$

- **RS 1.1** $g_{11} = \frac{4\epsilon}{n+8}$, other $g_{ij} = 0$. $z_{12} = \pm 1$. *RS* 1.1 represents the unstable n-Heisenberg-Gaussian FP of the $O(n) + O(n)$ model. The critical exponents are given by

$$\gamma^*_\tau = \left\{-\frac{(n+2)(2\epsilon)}{n+8}, 0\ (2\times)\right\},\quad \gamma^*_{\mathrm{cr}} = \left\{-\frac{2(2\epsilon)}{n+8}, 0\ (3\times)\right\},$$

$$\gamma^*_\Phi = \left\{\frac{(n+2)(2\epsilon)^2}{4(n+8)^2}, 0\right\},\quad \omega = \left\{(2\epsilon), -(2\epsilon)\ (2\times), -\frac{(n+6)(2\epsilon)}{n+8}, -\frac{6(2\epsilon)}{n+8}, 0\right\}.$$

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 1.2** $g_{11} = g_{22} = \frac{2n}{n^2+8}\epsilon$, $g_{12} = \frac{8-2n}{n^2+8}\epsilon$, other $g_{ij} = 0$. $z_{12} = 0$. *RS* 1.2 represents the biconical FP of the $O(n) + O(n)$ model (stable for $n = 3$ in the $O(n) + O(n)$ model). The critical exponents are

$$\gamma^*_\Phi = \left\{ \frac{n(n^2-3n+8)(2\epsilon)^2}{8(n^2+8)^2}(2\times) \right\}, \quad \gamma^*_\tau = \left\{ -\frac{3n(2\epsilon)}{n^2+8}, \frac{(1-n)n(2\epsilon)}{n^2+8}, \frac{(n-4)(2\epsilon)}{n^2+8} \right\},$$

$$\gamma^*_{\mathrm{cr}} = \left\{ -\frac{n(2\epsilon)}{n^2+8}\,(2\times), \frac{(n-4)(2\epsilon)}{n^2+8}(2\times) \right\},$$

$$\omega = \left\{ 0, (2\epsilon), \frac{8(n-1)(2\epsilon)}{n^2+8}, \frac{(4-n)(2+n)(2\epsilon)}{n^2+8}, \frac{(4-n)(n-2)(2\epsilon)}{n^2+8}, \frac{(2-n)(4+n)(2\epsilon)}{n^2+8} \right\}.$$

- **RS 1.3** Not only invariant under $O(n)\times O(2)$, but even under $O(2n)$. $g_{11} = g_{22} = \frac{2}{n+4}\epsilon$, $g_{12} = \frac{2}{n+4}\epsilon$, other $g_{ij} = 0$. $z_{12} = \pm\sqrt{2}$. *RS* 1.3 represents the for $n < 2$ stable (in all models) isotropic $2n-$Heisenberg FP. The critical exponents are

$$\gamma^*_\Phi = \left\{ \frac{(2n+2)(2\epsilon)^2}{4(2n+8)^2}\,(2\times) \right\}, \quad \gamma^*_\tau = \left\{ -2\frac{(2n+2)(2\epsilon)}{2n+8}, -\frac{2(2\epsilon)}{2n+8}\,(2\times) \right\},$$

$$\gamma^*_{\mathrm{cr}} = \left\{ -\frac{2(2\epsilon)}{2n+8}\,(4\times) \right\}, \quad \omega = \left\{ (2\epsilon), \frac{8(2\epsilon)}{2n+8}\,(2\times), \frac{(4-2n)(2\epsilon)}{2n+8}\,(3\times) \right\}.$$

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 1.4** $g_{11,22} = \frac{2}{n+8}\epsilon \pm \sqrt{\frac{32(1-n)}{(n+8)^3}}\epsilon$, $g_{12} = \frac{6}{n+8}\epsilon$, other $g_{ij} = 0$. $z_{12} = \pm 2$. $RS1.4$ also belongs to the $O(n) + O(n)$ model. This FP coincides with the biconical FP for $n = 1$. In one loop order one obtains the exponents

$$\gamma_\Phi^* = \left\{ \frac{(n^2+37n+16)(2\epsilon)^2}{8(n+8)^3} \pm (n+2)\frac{\sqrt{2(1-n)}(2\epsilon)^2}{2(n+8)^{5/2}} \right\},$$

$$\gamma_\tau^* = \left\{ -\frac{(2+n)(2\epsilon)}{2(n+8)} \pm \frac{\sqrt{n^3+48n^2+32}(2\epsilon)}{2(n+8)^{3/2}}, -\frac{3(2\epsilon)}{n+8} \right\},$$

$$\gamma_{\mathrm{cr}}^* = \left\{ -\frac{(2\epsilon)}{n+8} \pm \frac{2\sqrt{2(1-n)}(2\epsilon)}{(n+8)^{3/2}}, -\frac{3(2\epsilon)}{n+8}(2\times) \right\},$$

$$\omega = \left\{ 0, (2\epsilon), \frac{(6-n)(2\epsilon)}{n+8}, \frac{(10-n)(2\epsilon)}{n+8}, -\frac{(n+2)(2\epsilon)}{2(n+8)} \pm \frac{\sqrt{n^2-188n+196}(2\epsilon)}{2(n+8)} \right\}.$$

We considered the coupling in two loop order, since it yields in order ϵ the region in which the couplings are real. We obtained $n_c = 1 - (2\epsilon)/48 + O(2\epsilon)^2$.

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.1** ($z_{12} = 0$) Two of the exponents ω equal 0 for any n in one-loop order.
- One is due to the invariance under rotations between the fields ϕ. The other one indicates that there may branch off a second line of FPs.
- One finds besides the FP of two decoupled systems (*RS 2.1a*)

$$g_{11}^* = g_{22}^*, \quad \text{other } g_{ij} = 0$$

another solution (*RS 2.1b*) with

$$g_{11}^* = g_{22}^*, \qquad g_{12}^*, g_{33}^* = O(\epsilon^2), \qquad g_{13}^* = g_{23}^* = O(\epsilon^{3/2})$$

- Both types of FPs agree in one-loop order, but differ in the next order.
- In the following we give the FPs and critical exponents in two-loop order (for γ_Φ^* in three-loop order).

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.1a**

$$g_{11}^* = g_{22}^* = \frac{4}{n+8}\epsilon - \frac{4(n^2-2n-20)}{(n+8)^3}\epsilon^2, \quad \text{other } g_{ij} = 0.$$

This solution describes two independent O(n) models and is the decoupled $n-$Heisenberg$-n-$Heisenberg FP of the $O(n)+O(n)$ model.

$$\begin{aligned}
\gamma_\Phi^* &= \left\{ \frac{(n+2)}{4(n+8)^2}(2\epsilon)^2 - \frac{(n+2)(n^2-56n-272)}{16(n+8)^4}(2\epsilon)^3\ (2\times) \right\}, \\
\gamma_\tau^* &= \left\{ -\frac{n+2}{2(n+8)^2}(2\epsilon)^2,\ -\frac{n+2}{n+8}(2\epsilon) - \frac{(n+2)(13n+44)}{2(n+8)^3}(2\epsilon)^2\ (2\times) \right\}, \\
\gamma_{\mathrm{cr}}^* &= \left\{ -\frac{2}{n+8}(2\epsilon) + \frac{(n+4)(n-22)}{2(n+8)^3}(2\epsilon)^2\ (2\times), -\frac{n+2}{2(n+8)^2}(2\epsilon)^2\ (2\times) \right\}, \\
\omega &= \left\{ (2\epsilon) - \frac{3(3n+14)}{(n+8)^2}(2\epsilon)^2\ (2\times), \frac{n-4}{n+8}(2\epsilon) + \frac{(n+2)(13n+44)}{(n+8)^3}(2\epsilon)^2, \right. \\
&\quad \left. -\frac{n+4}{n+8}(2\epsilon) - \frac{(n+4)(n-22)}{(n+8)^3}(2\epsilon)^2, \frac{n+2}{2(n+8)^2}(2\epsilon)^2, 0 \right\}.
\end{aligned}$$

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.1b** is new to our best knowledge. It agrees with *RS 2.1a*, which describes two uncoupled systems, in one-loop order:

$$
\begin{aligned}
g_{11}^* = g_{22}^* = &\quad \frac{4}{n+8}\epsilon - \frac{9n^3+98n^2-400n-2272}{2(n+8)^3(n+14)}\epsilon^2,\\
g_{13}^* = g_{23}^* = &\quad \pm\frac{\sqrt{2(n+4)(n+2)(n-4)}}{(n+8)^2\sqrt{n+14}}\epsilon^{3/2},\\
g_{12}^* = &\quad -\frac{n+2}{2(n+8)(n+14)}\epsilon^2,\\
g_{33}^* = &\quad \frac{(n+2)(n-4)}{(n+8)^2(n+14)}\epsilon^2.
\end{aligned}
$$

In the limit $D = 4$ it is real for $n \geq 4$.

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.1b** Its critical exponents are

$$\gamma_\Phi^* = \left\{ \frac{(n+2)}{4(n+8)^2}(2\epsilon)^2 \pm \frac{(n+2)\sqrt{2(n-4)(n+2)(n+4)}}{16(n+8)^3\sqrt{n+14}}(2\epsilon)^{5/2} - \frac{(n+2)(n^2-56n-272)}{16(n+8)^4}(2\epsilon)^3 \right\},$$

$$\gamma_\tau^* = \left\{ -\frac{n+2}{(n+8)}(2\epsilon) - \frac{(n+2)(29n^2+470n+1256)}{4(n+14)(n+8)^3}(2\epsilon)^2, -\frac{n+2}{n+8}(2\epsilon) - \frac{(n+2)(23n^2+434n+1208)}{4(n+8)^3(n+14)}(2\epsilon)^2, -\frac{3(n+2)(n^2+10n+64)}{4(n+8)^3(n+14)}(2\epsilon)^2 \right\},$$

$$\gamma_{\mathrm{cr}}^* = \left\{ -\frac{2}{n+8}(2\epsilon) + \frac{n^3-12n^2-660n-2416}{4(n+8)^3(n+14)}(2\epsilon)^2, -\frac{2}{n+8}(2\epsilon) + \frac{3n^3-4n^2-700n-2512}{4(n+8)^3(n+14)}(2\epsilon)^2, -\frac{(n+2)(n+6)(n+32)}{4(n+8)^3(n+14)}(2\epsilon)^2, -\frac{(n+2)(n+26)}{4(n+8)^2(n+14)}(2\epsilon)^2 \right\},$$

$$\omega = \left\{ (2\epsilon) - \frac{3(3n+14)}{(n+8)^2}(2\epsilon)^2 \ (2\times), -\frac{n+2}{(n+8)^2}(2\epsilon)^2, 0, \frac{n-4}{n+8}(2\epsilon) + \frac{(n+2)(15n^3+242n^2+656n+32)}{n(n+8)^3(n+14)}(2\epsilon)^2, -\frac{n+4}{n+8}(2\epsilon) - \frac{3n^4+12n^3-332n^2-1252n+64}{n(n+8)^3(n+14)}(2\epsilon)^2 \right\}.$$

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.3** ($z_{12} = \pm \mathrm{i}$) also is new to our knowledge.

$$g_{11,22} = \frac{2}{n+8}\epsilon \pm \sqrt{\frac{-4(3n+22)(n-2)(n+2)(n+4)(n+14)}{(n+8)^3(n^2+4n+20)^2}}\epsilon,$$

$$g_{12}^* = \frac{4(n+6)(n+4)}{(n+8)(n^2+4n+20)}\epsilon, \quad g_{33}^* = \frac{4(n^2-36)}{(n+8)(n^2+4n+20)}\epsilon, \quad g_{13,23}^* = 0.$$

- We consider the coupling in two loop order to obtain in order ϵ the region in which the couplings are real:

$$n_{\mathrm{c}} = 2 - (2\epsilon)/140 + O(2\epsilon)^2$$

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.3** ($z_{12} = \pm\mathrm{i}$) The critical exponents are

$$\gamma_\Phi^* = \left\{ \frac{(2n^6+37n^5+348n^4+2360n^3+9376n^2+13904n-9152)(2\epsilon)^2}{8(n+8)^3(n^2+4n+20)^2} \pm \frac{(n+2)\sqrt{-(3n+22)(n-2)(n+2)(n+4)(n+14)(2\epsilon)^2}}{8(n+8)^{5/2}(n^2+4n+20)} \right\},$$

$$\gamma_\tau^* = \left\{ -\frac{(2\epsilon)(n-1)(n-2)(n+6)}{(n+8)(n^2+4n+20)}, -\frac{(n+2)(2\epsilon)}{2(n+8)} \pm \frac{(2\epsilon)\sqrt{n^7+32n^6+512n^5+3792n^4+10064n^3-3548n^2-21376n+61184}}{2(n+8)^{3/2}(n^2+4n+20)} \right\},$$

$$\gamma_{\mathrm{cr}}^* = \left\{ -\frac{(2\epsilon)}{n+8} \pm \frac{\sqrt{-2(n^5+34n^4+312n^3+752n^2-1776n-7648)(2\epsilon)}}{(n+8)^{3/2}(n^2+4n+20)}, -\frac{(n+6)(3n+2)(2\epsilon)}{(n+8)(n^2+4n+20)}, -\frac{(n+6)(n+14)(2\epsilon)}{(n+8)(n^2+4n+20)} \right\},$$

$$\omega = \left\{ 0, (2\epsilon), \frac{(2\epsilon)(n^3+10n^2-4n-232)}{(n+8)(n^2+4n+20)}, \frac{(2\epsilon)\lambda'}{2(n+8)(n^2+4n+20)} \right\}.$$

where λ' is solution of the equation

$$\lambda'^3 + 16(n^2+4n+20)\lambda'^2 - 4(n+4)(n^5-18n^4-392n^3-1648n^2-496n+8928)\lambda' - 16(3n+22)(n-2)(n+6)(n-6)(n+4)(n+2)(n+14)^2 = 0.$$

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.2** ($z_{12} = \pm\sqrt{2}$) and **RS 3.1** are solutions of one and the same quadratic equation and correspond to the antichiral and chiral FP of the $O(2) \times O(n)$ model, respectively:

$$g_{11,22} = \frac{3n^2 - 2n + 24 + s(n-6)\sqrt{n^2 - 24n + 48}}{n^3 + 4n^2 - 24n + 144}\epsilon,$$

$$g_{12} = \frac{-n^2 - 6n + 72 + s(n+6)\sqrt{n^2 - 24n + 48}}{n^3 + 4n^2 - 24n + 144}\epsilon,$$

$$g_{33} = \frac{4(n^2 + n - 12 - s3\sqrt{n^2 - 24n + 48})}{n^3 + 4n^2 - 24n + 144}\epsilon, \quad g_{13,23} = 0,$$

  where $s = +1$ corresponds to *RS 3.1* and $s = -1$ to *RS 2.2*

- Both fixed points are O(n)×O(2) invariant.

Solutions for finite n, Fixed Points and Critical Exponents

- **RS 2.2** and **RS 3.1** The critical exponents are

$$\gamma_\Phi^* = \left\{ \frac{(5n^5 - 3n^4 - 16n^3 - 656n^2 + 3072n - 1152 + s(n-3)(n+4)w^{3/2})(2\epsilon)^2}{16N^2} \;(2\times) \right\},$$

$$\gamma_\tau^* = \left\{ -\frac{(n(48+n+n^2) - s(n-3)(4+n)\sqrt{w})(2\epsilon)}{2N}, \frac{(-2n^3 - 3n^2 + 28n - 48 + 5sn\sqrt{w})(2\epsilon)}{2N} \;(2\times) \right\}.$$

$$\gamma_{\rm cr}^* = \left\{ \frac{(-5n^2 - s(n-12)\sqrt{w})(2\epsilon)}{2N}, \frac{(-n^2 + 4n - 48 - sn\sqrt{w})(2\epsilon)}{2N} \;(2\times), \frac{(3n^2 + 8n - 96 - s(n+12)\sqrt{w})(2\epsilon)}{2N} \right\}.$$

$$\omega = \left\{ \frac{(n+4)\big((n+4)(n-3) - 3s\sqrt{w}\big)(2\epsilon)}{N} \;(2\times), \frac{(n^3 + 14n^2 + 56n - 96 + s(n+8)(n-6)\sqrt{w})(2\epsilon)}{2N} \;(2\times), \frac{(-3(n^2 - 24n + 48) + s(n+4)(n-3)\sqrt{w})(2\epsilon)}{N}, \;(2\epsilon) \right\}.$$

where $N = n^3 + 4n^2 - 24n + 144$, and $w = n^2 - 24n + 48$,

Solutions for finite n, Fixed Points and Critical Exponents

- The FP 3.1 is stable for large n.
- Two loop calculation gives the range where the FPs are real:

$$\begin{aligned} n &> 21.8 - 23.4(2\epsilon) + O(2\epsilon)^2 \\ n &< 2.20 - 0.57(2\epsilon) + O(2\epsilon)^2 \end{aligned}$$

- The question of the range of stability in $D = 3$ is under debate
- The $1/n$ expansion of the general $O(n)$ symmetric two field model gives in first order

$$\eta = \frac{6\Gamma(D-2)\sin(\frac{D\pi}{2})}{\pi\Gamma(D/2-2)\Gamma(1+D/2)n}$$

$$\gamma_\tau^* - (D-4) = \left\{ \frac{2(2-D)(1-D)\eta}{4-D}, \frac{2(2-D)(3-2D)\eta}{3(4-D)} \right\}.$$

- Both expansions agree with each other!

Summary and Conclusions

- The general $O(n)$ symmetric Hamiltonian has three different mass terms. It gives rise to a variety of critical and multicritical behaviors generalizing the $O(n) + O(n)$ and $O(2) \times O(n)$ models.
- We gave the expressions for the β functions and the matrices γ_Φ, γ_τ,$\gamma_{\rm cr,s}$ and ω, and $\gamma_{\rm cr,a}$ for the general $O(n)$ model from which the critical exponents are obtained in one-loop order (for η in two-loop order).
- A classification of the FPs in the large n limit was given. Two types of FPs emerge: Four of them are invariant under O(n)$\times$O(2). The other six FPs are not invariant under O(2) and yield lines of FPs.
- Under the numerous FPs the corresponding FPs of the well-known models were found.
- To our best knowledge the FPs *RS 2.1b* and *2.3* are new. *RS 2.1b* agrees with *RS 2.1a*, which describes two uncoupled systems, in one-loop order.