Critical Behavior of a General $O(n)$ symmetric Model of two $n$ Vector Fields in $D = 4 - 2\epsilon$

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The renormalization group approach provides a natural framework for the understanding of critical properties of phase transitions. A very large variety of critical phenomena can be described by so called $\phi^4$ models, $\phi = (\phi_1, \ldots, \phi_n)$. There are several $\phi^4$ models:

- The common $O(n)$ symmetric one field model ($\tau$ is a temperature-like parameter and $g > 0$):

  $$S_{O(n)}(\phi) = \frac{1}{2} \left[ (\nabla \phi)^2 + \tau \phi^2 \right] + \frac{1}{4!} g (\phi^2)^2$$

- Extended $O(n) + O(m)$ symmetric model

  $$S_{O(n)+O(m)}(\phi_1, \phi_2) = \frac{1}{2} \left[ (\nabla \phi_1)^2 + (\nabla \phi_2)^2 + \tau_1 \phi_1^2 + \tau_2 \phi_2^2 \right]$$

  $$+ \frac{1}{4!} \left[ g_1(\phi_1^2)^2 + g_2(\phi_2^2)^2 + g_3(\phi_1^2)(\phi_2^2) \right]$$
In the $O(n) + O(m)$ model six different fixpoints (FP) were found. Three of them are always unstable and the stability of three others depends on $n$ and $m$ (M. Fisher et al).

The $O(n) + O(m)$ model has been used to describe multicritical phenomena. (The critical behavior of uniaxial antiferromagnets in a magnetic field parallel to the field direction, the $SO(5)$ theory of high $T_c$ superconductors).

Also interesting phenomena of inverse symmetry breaking, symmetry non restoration and reentrant phase transitions were reported (Weinberg, Ramos, Pinto).
Recently frustrated spin systems with non collinear or canted spin ordering have been the object of intensive research (Kawamura, Pelissetto and Vicary et al., Holovatch) (Examples: Helical magnets and layered triangular Heisenberg antiferromagnets). Both fields have $n$ components and the model possesses the $O(n) \times O(2)$ symmetry.

\[
S_{O(n) \times O(2)}(\phi_1, \phi_2) = \frac{1}{2} \left[ (\nabla \phi_1)^2 + (\nabla \phi_2)^2 + \tau (\phi_1^2 + \phi_2^2) \right] \\
+ \frac{1}{4!} u (\phi_1^2 + \phi_2^2)^2 + \frac{1}{4!} v \left[ (\phi_1 \phi_2)^2 - (\phi_1^2)(\phi_2^2) \right]
\]

Here the scalar product $\phi_1 \phi_2$ is present.

In the $4 - 2\epsilon$ expansion, the number of FPs and their stability depend on $n$. 

We have studied the critical behavior of the O($n$)-symmetric model with two $n$-vector fields within the RG field-theoretical approach in $4 - 2\epsilon$ expansion.

\[
S_{O(n)}(\phi_1, \phi_2) = \frac{1}{2} \left[ (\nabla \phi_1)^2 + (\nabla \phi_2)^2 + \tau_1 \phi_1^2 + \tau_2 \phi_2^2 + 2\tau_3 \phi_1 \phi_2 \right] \\
+ \frac{1}{8} \left[ g_{11}(\phi_1^2)^2 + g_{22}(\phi_2^2)^2 + 2g_{12}(\phi_1^2)(\phi_2^2) \\
+ 2g_{33}(\phi_1 \phi_2)^2 + 2\sqrt{2}g_{13}(\phi_1)^2(\phi_1 \phi_2) + 2\sqrt{2}g_{23}(\phi_2)^2(\phi_1 \phi_2) \right]
\]

The model becomes O($n$)+O($n$) symmetric when $\tau_3 = g_{33} = g_{13} = g_{23} = 0$.

Setting $\tau_1 = \tau_2$, $\tau_3 = g_{13} = g_{23} = 0$, $g_{11} = g_{22}$, $g_{12} = g_{11} - g_{33}$ leads to the O($n$)×O(2) model of frustrated spins.
Introduction and Descriptive Overview

4 – 2\(\epsilon\) Expansion

Field rotations

The classification of the fixed points in the large \(n\) limit

Solutions for finite \(n\), Fixed Points and Critical Exponents

Summary and Conclusions
It is useful to rewrite the interaction part of $S_{O(n)}$ as

$$S_{\text{int}}(\phi_1, \phi_2, g) = \frac{1}{8} \sum_{k,l=1}^{3} I_k g_{kl} I_l = \frac{1}{8} I g I,$$

where

$$\begin{pmatrix}
I_1 \\
I_2 \\
I_3
\end{pmatrix}
= \begin{pmatrix}
\phi_1^2 \\
\phi_2^2 \\
\sqrt{2} \phi_1 \phi_2
\end{pmatrix}.$$
The expression for the critical exponents can be taken from (Brézin, le Guillou, and Zinn-Justin in Domb-Green Vol.6).

The six $\beta$ functions $\beta_{ij} \equiv \mu \partial_{\mu} g_{ij}$, where $\mu$ is an auxiliary parameter with the critical dimension 1, can be written in 1-loop order

$$\beta_{ij} = -2\epsilon g_{ij} + \frac{1}{2} (n + 8) g_{ik} g_{kl} + \frac{1}{2} C_{ij,kl,mng_{kl}g_{mn}}$$

with

- $i, j \quad C_{ij,kl,mng_{kl}g_{mn}}$
- $1, 1 \quad -8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2$
- $1, 2 \quad -6g_{11}g_{12} - 6g_{12}g_{22} - 4g_{13}g_{23} + g_{11}g_{33} + 4g_{12}^2 + 2g_{13}^2 + g_{22}g_{33} + 2g_{23}^2 + g_{33}^2$
- $1, 3 \quad -6g_{12}g_{23} - 3g_{13}g_{33} + 6g_{12}g_{13} + 3g_{23}g_{33}$
- $2, 2 \quad -8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2$
- $2, 3 \quad -6g_{12}g_{13} - 3g_{23}g_{33} + 6g_{12}g_{23} + 3g_{13}g_{33}$
- $3, 3 \quad -2g_{13}^2 - 2g_{23}^2 - 6g_{33}^2 + 2g_{11}g_{33} + 8g_{12}g_{33} + 4g_{13}g_{23} + 2g_{22}g_{33}$

We have rescaled the couplings by a factor $8\pi^2$ as usual.

The FPs $g^*$ are the solutions of $\beta_{ij}(g^*) = 0$. $S_{O(n)}$ is symmetric under the simultaneous interchange of $g_{11}$ with $g_{22}$ and $g_{13}$ with $g_{23}$. The simultaneous change of signs of $g_{13}$ and $g_{23}$ leaves the solution invariant.
The stability-matrix can be easily obtained:

$$\omega_{ij,kl} = \partial \beta_{ij}(g)/\partial g_{kl}|_{g=g^*}$$

Its eigenvalues are the critical exponents $\omega$.

The critical exponents $\eta$ are obtained from the eigenvalues $\gamma^*_\Phi$ of the symmetric $2 \times 2$ matrix $\gamma_\Phi$ at $g = g^*$,

$$\{\gamma^*_\Phi\}_{11} = \frac{1}{16} \left( 2(n+2)g_{11}^2 + (n+2)g_{23}^2 + (n+1)g_{33}^2 + 2ng_{12}^2 + 4g_{12}g_{33} + 3(n+2)g_{13}^2 \right),$$

$$\{\gamma^*_\Phi\}_{21} = \frac{\sqrt{2}(n+2)}{16} \left( (g_{11} + g_{12} + g_{33})g_{13} + (g_{22} + g_{12} + g_{33})g_{23} \right),$$

$$\{\gamma^*_\Phi\}_{22} = \frac{1}{16} \left( (n+2)g_{13}^2 + 2(n+2)g_{22}^2 + 2ng_{12}^2 + 3(n+2)g_{23}^2 + (n+1)g_{33}^2 + 4g_{12}g_{33} \right),$$

calculated at the specific FP, with respect to $\eta_i = 2\gamma^*_\Phi$.

The critical indices $1/\nu = 2 + \gamma^*_\tau$ are obtained from the eigenvalues $\gamma^*_\tau$ of (in one-loop order):

$$\gamma_\tau = -\frac{1}{2} \begin{pmatrix}
(n+2)g_{11} & ng_{12} + g_{33} & (n+2)g_{13} \\
ng_{12} + g_{33} & (n+2)g_{22} & (n+2)g_{23} \\
(n+2)g_{13} & (n+2)g_{23} & 2g_{12} + (n+1)g_{33}
\end{pmatrix}_{g=g^*}.$$
The three crossover exponents are eigenvalues of the $3 \times 3$ matrix (in one-loop order)

$$\gamma_{cr,s} = -\frac{1}{2} \begin{pmatrix} 2g_{11} & g_{33} & 2g_{13} \\ g_{33} & 2g_{22} & 2g_{23} \\ 2g_{13} & 2g_{23} & 2g_{12} + g_{33} \end{pmatrix} g = g^*.$$ 

The fourth crossover exponent is (in one-loop order)

$$\gamma_{cr,a} = -g_{12} + \frac{1}{2}g_{33}.$$
Field rotations I

- The direct calculation of FPs from the $\beta$ functions leads to more than 50 FPs and several lines of FPs!
- Some FPs are equivalent due to the internal rotation of the fields:

$$
\begin{pmatrix}
\phi'_1 \\
\phi'_2
\end{pmatrix} =
\begin{pmatrix}
\cos(\phi) & \sin(\phi) \\
-\sin(\phi) & \cos(\phi)
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}.
$$

- Performing the rotation yields

$$
\begin{pmatrix}
\mathcal{I}'_1 \\
\mathcal{I}'_2 \\
\mathcal{I}'_3
\end{pmatrix} = M
\begin{pmatrix}
\mathcal{I}_1 \\
\mathcal{I}_2 \\
\mathcal{I}_3
\end{pmatrix}, \\
M =
\begin{pmatrix}
\frac{1}{2} + \frac{1}{2} \cos(2\phi) & \frac{1}{2} - \frac{1}{2} \cos(2\phi) & \sqrt{\frac{1}{2}} \sin(2\phi) \\
\frac{1}{2} - \frac{1}{2} \cos(2\phi) & \frac{1}{2} + \frac{1}{2} \cos(2\phi) & -\sqrt{\frac{1}{2}} \sin(2\phi) \\
-\sqrt{\frac{1}{2}} \sin(2\phi) & \sqrt{\frac{1}{2}} \sin(2\phi) & \cos(2\phi)
\end{pmatrix}.
$$

- The matrix $M$ is orthogonal and the interaction transforms according to

$$
S_{\text{int}}(\phi'_1, \phi'_2, g') = \frac{1}{8} \mathcal{I}'^T g' \mathcal{I}', \\
g' = MgM^T.
$$

- Obviously both sets of couplings describe the same critical behavior.
Field rotations II

- One finds that the following is invariant under the rotations

\[ a_1 = g_{11} + g_{22} + 2g_{12}, \quad a_2 = g_{11} + g_{22} + g_{33} \]  \hspace{1cm} (1)

- whereas

\[ a_{31} = g_{11} - g_{22}, \quad a_{32} = \sqrt{2}(g_{13} + g_{23}), \]
\[ a_{41} = -g_{11} + 2g_{12} - g_{22} + 2g_{33}, \quad a_{42} = -\sqrt{8}(g_{13} - g_{23}) \]

transform according to

\[
\begin{pmatrix}
  a'_{31} \\
  a'_{32}
\end{pmatrix} =
\begin{pmatrix}
  \cos(2\varphi) & \sin(2\varphi) \\
  -\sin(2\varphi) & \cos(2\varphi)
\end{pmatrix}
\begin{pmatrix}
  a_{31} \\
  a_{32}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  a'_{41} \\
  a'_{42}
\end{pmatrix} =
\begin{pmatrix}
  \cos(4\varphi) & \sin(4\varphi) \\
  -\sin(4\varphi) & \cos(4\varphi)
\end{pmatrix}
\begin{pmatrix}
  a_{41} \\
  a_{42}
\end{pmatrix}.
\]

- For the interactions invariant under O(n)×O(2) the amplitudes \(a_{31}, a_{32}, a_{41}, a_{42}\) have to vanish. Otherwise we may choose \(\varphi\).

- We will choose it so that \(a_{42} = 0\), i.e \(g_{23} = g_{13}\).

- From the FPs with the condition \(g_{23} = g_{13}\), all other FPs can be obtained by means of the orthogonal transformations leaving the expressions (1) invariant.
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The classification of the fixed points in the large $n$ limit

- In the large $n$ limit we may neglect the last term in the $\beta$ functions:

$$\beta_{ij} = -2\epsilon g_{ij} + \frac{1}{2} (n+8) g_{ik} g_{kl} + \frac{1}{2} C_{ij,kl,mn} g_{kl} g_{mn}$$

expressing $g$ in terms of the matrix $p$,

$$g = \frac{4\epsilon p}{n+8}.$$ 

- At criticality ($\beta_{ij} \equiv 0$) and in the limit $n \to \infty$ the matrix $p$ becomes idempotent: $p = p^2$.

- The only eigenvalues of idempotent matrices are 0 and 1. Thus depending on the number $k$ of eigenvalues 1 there are four types of symmetric $(3 \times 3)$ idempotent matrices $p^{(k)}$:

$$p^{(0)}_{ij} = 0, \quad p^{(1)}_{ij} = z_i z_j, \quad p^{(2)}_{ij} = \delta_{ij} - z_i z_j, \quad p^{(3)}_{ij} = \delta_{ij}; \quad i, j = 1, 2, 3,$$

with the restriction

$$z_1^2 + z_2^2 + z_3^2 = 1.$$ 

- Further conditions on $z$ for the classes $p^{(1,2)}$ can be obtained by considering the first two orders in $1/(n+8)$ to $g^*$. 

The classification of the FPs in the large $n$, class $p^{(0)}$

- This class consists of the trivial FP
  \[ g^* = 4\epsilon p^{(0)}/(n+8) = 0 \]
  only.
- The stability-matrix is diagonal:
  \[ \omega_{ij} = -(2\epsilon)\delta_{ij} \]
  All its eigenvalues are negative and the FP is unstable.
- This FP is exact and remains invariant under the orthogonal transformations.
The classification of the FPs in the large $n$, class $p^{(1)}$

- The following ansatz ($h$ is symmetric) is put into the $\beta$-functions

$$g_{ij}^* = \frac{4\epsilon}{(n+8)} z_i z_j + \frac{4\epsilon}{(n+8)^2} h_{ij} + O\left(\frac{1}{(n+8)^3}\right)$$

(2)

- We then obtain the following conditions on $z$:

$$(1 - z_{12}^2)(4 - z_{12}^2)z_{12}(z_1 - z_2) = 0 \quad (1 - z_{12}^2)(4 - z_{12}^2)z_{12}z_3^2 = 0,$$

(3)

where $z_{12} := z_1 + z_2$. Thus solutions are given by

$$z_{12} = 0, \pm 1, \pm 2, \pm \sqrt{2},$$

- the first solutions follow immediately from the eqs. (3), whereas the last pair follows from $z_1 - z_2 = 0$, $z_3 = 0$ and $z_1^2 + z_2^2 + z_3^2 = 1$ and describes an $O(n) \times O(2)$-invariant interaction. Due to (2) a change of the sign of the $z$s does not alter the FP. Thus $z_{12}$ and $-z_{12}$ yield the same class of FPs.
The classification of the FPs in the large \( n \), class \( p^{(1)} \)

- The solutions \( z_{12} = 0, \pm 1, \pm 2, \pm \sqrt{2} \) divide the class \( p^{(1)} \) into subclasses. Each subclass has its own characteristic critical exponents.
- While \( z_{12} = z_1 + z_2 \) stays constant, \( z_1 - z_2 \) and \( z_3 \) vary under rotation according to
  \[
  (z_1 - z_2)^2 + 2z_3^2 = 2 - z_{12}^2. 
  \]
- For \( z_{12} \neq \pm \sqrt{2} \) one obtains a whole continuum of solutions, i.e. lines of FPs!
- Each subclass has its own characteristic critical exponents. In the limit of large \( n \) they are
  \[
  \omega = \{(2\varepsilon), 0 \,(2\times), -(2\varepsilon) \,(3\times)\}, \quad \gamma_{\tau}^* = \{-\,(2\varepsilon), 0 \,(2\times)\}, \\
  \gamma_{cr}^* = \left\{ \frac{(2\varepsilon)}{n}(-1 \pm z_{12}\sqrt{2 - z_{12}^2}), \frac{(2\varepsilon)}{n}(1 - z_{12}^2) \,(2\times) \right\}, \\
  \gamma_{\Phi}^* = \left\{ \frac{(2\varepsilon)^2}{8n}(1 \pm z_{12}\sqrt{2 - z_{12}^2}) \right\}. 
  \]
The classification of the FPs in the large $n$, class $p^{(2)}$

- The following ansatz ($h$ is symmetric) is put into the $\beta$-functions

$$
g^{*}_{ij} = \frac{4\epsilon}{n+8}(\delta_{ij} - z_{i}z_{j}) + \frac{4\epsilon}{(n+8)^2}h_{ij} + O\left(\frac{1}{(n+8)^3}\right), \tag{4}
$$

- We then obtain the following conditions on $z$:

$$(z_{12}^2 + 1)z_{12}^2(z_{1} - z_{2}) = 0, \quad (z_{12}^2 + 1)z_{12}z_{3}^2 = 0. \tag{5}$$

where $z_{12} := z_{1} + z_{2}$. Thus solutions are given by

$$z_{12} = 0, \pm i, \pm \sqrt{2},$$

where the first two solutions are immediately obvious from eqs. (5) and the last one follows from $z_{1} = z_{2}$, $z_{3} = 0$, and $z_{1}^2 + z_{2}^2 + z_{3}^2 = 1$. This last solution represents an $O(n) \times O(2)$-invariant model.
The classification of the FPs in the large $n$, class $p^{(2)}$

- The solutions $z_{12} = 0, \pm i, \pm \sqrt{2}$ divide the class $p^{(2)}$ into subclasses. Each subclass has its own characteristic critical exponents.
- While $z_{12} = z_1 + z_2$ stays constant, $z_1 - z_2$ and $z_3$ vary under rotation according to
  \[ (z_1 - z_2)^2 + 2z_3^2 = 2 - z_{12}^2. \]
- For $z_{12} \neq \pm \sqrt{2}$ one obtains a whole continuum of solutions, i.e. lines of FPs!
- Each subclass has its own characteristic critical exponents. In the limit of large $n$ they are
  \[
  \omega = \{ (2\epsilon) (3\times), 0 (2\times), -(2\epsilon) \}, \quad \gamma^* = \{ -(2\epsilon) (2\times), 0 \}, \\
  \gamma_{cr}^* = \left\{ \frac{(2\epsilon)}{n} (-2 + z_{12}^2), \frac{(2\epsilon)}{n} (-1 \pm \sqrt{1 + 2z_{12}^2 - z_{12}^4}), \frac{(2\epsilon)}{n} z_{12}^2 \right\}, \\
  \gamma_{\Phi}^* = \left\{ \frac{(2\epsilon)^2}{8n} (2 \pm z_{12} \sqrt{2 - z_{12}^2}) \right\}.
  \]
In the large $n$ limit one obtains

$$g^* = 4\epsilon p^{(3)}/(n + 8) = 4\epsilon\delta_{ij}/(n + 8),$$

which yields the exponents in leading order

$$\omega = \{(2\epsilon) (6\times)\}, \quad \gamma^*_\tau = \{- (2\epsilon) (3\times)\},$$

$$\gamma^*_\text{cr} = \left\{ \frac{-3(2\epsilon)}{n}, \frac{-2\epsilon}{n} (2\times), \frac{2\epsilon}{n} \right\}, \quad \gamma^*_\Phi = \left\{ \frac{3(2\epsilon)^2}{8n} (2\times) \right\}.$$
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The 'gauge' condition $g_{13} = g_{23}$ yields (simple) Representative Solutions (RS), all other solutions can be obtained by means of rotations.

For each class $p^{(k)}$ there is one solution invariant under $O(n) \times O(2)$.

Solutions not invariant under $O(n) \times O(2)$ have one exponent $\omega = 0$ since the field rotations create lines of fixed points.

All solutions with the exception of the trivial FP have one exponent $\omega = 2\epsilon$ independent of $n$ in one-loop order, since $\beta_{ij} = -2\epsilon g_{ij} + \text{term bilinear in the } g$s.
Solutions for finite $n$, Fixed Points and Critical Exponents

- **RS 0.1** This is the trivial (interaction free) fixed point. All anomalous exponents $\gamma^*$ vanish

$$
\gamma^*_\Phi = \{0 \ (2 \times)\}, \quad \gamma^*_\tau = \{0 \ (3 \times)\}, \quad \gamma^*_\text{cr} = \{0 \ (4 \times)\}, \quad \omega = \{-(2\epsilon) \ (6 \times)\}
$$

- **RS 1.1** $g_{11} = \frac{4\epsilon}{n+8}$, other $g_{ij} = 0$. $z_{12} = \pm 1$. RS 1.1 represents the unstable $n$-Heisenberg-Gaussian FP of the $O(n) + O(n)$ model. The critical exponents are given by

$$
\gamma^*_\tau = \begin{cases} 
-\frac{(n+2)(2\epsilon)}{n+8}, 0 \ (2 \times) \end{cases}, \quad \gamma^*_\text{cr} = \begin{cases} 
-\frac{2(2\epsilon)}{n+8}, 0 \ (3 \times) \end{cases},
$$

$$
\gamma^*_\Phi = \begin{cases} 
\frac{(n+2)(2\epsilon)^2}{4(n+8)^2}, 0 \end{cases}, \quad \omega = \begin{cases} 
(2\epsilon), -(2\epsilon) \ (2 \times), -\frac{(n+6)(2\epsilon)}{n+8}, -\frac{6(2\epsilon)}{n+8}, 0 \end{cases}.
$$
Solutions for finite $n$, Fixed Points and Critical Exponents

**RS 1.2** $g_{11} = g_{22} = \frac{2n}{n^2+8} \epsilon$, $g_{12} = \frac{8-2n}{n^2+8} \epsilon$, other $g_{ij} = 0$. $z_{12} = 0$. RS 1.2 represents the biconical FP of the $O(n)+O(n)$ model (stable for $n = 3$ in the $O(n)+O(n)$ model). The critical exponents are

$$\gamma^*_\Phi = \left\{ \frac{n(n^2 - 3n + 8)(2\epsilon)^2}{8(n^2 + 8)^2} (2\times) \right\}, \quad \gamma^*_\tau = \left\{ -\frac{3n(2\epsilon)}{n^2 + 8}, \frac{(1-n)n(2\epsilon)}{n^2 + 8}, \frac{(n-4)(2\epsilon)}{n^2 + 8} \right\},$$

$$\gamma^*_\text{cr} = \left\{ -\frac{n(2\epsilon)}{n^2 + 8} (2\times), \frac{(n-4)(2\epsilon)}{n^2 + 8} (2\times) \right\},$$

$$\omega = \left\{ 0, (2\epsilon), \frac{8(n-1)(2\epsilon)}{n^2 + 8}, \frac{(4-n)(2+n)(2\epsilon)}{n^2 + 8}, \frac{(4-n)(n-2)(2\epsilon)}{n^2 + 8}, \frac{(2-n)(4+n)(2\epsilon)}{n^2 + 8} \right\}.$$

**RS 1.3** Not only invariant under $O(n)\times O(2)$, but even under $O(2n)$. $g_{11} = g_{22} = \frac{2}{n+4} \epsilon$, $g_{12} = \frac{2}{n+4} \epsilon$, other $g_{ij} = 0$. $z_{12} = \pm \sqrt{2}$. RS 1.3 represents the for $n < 2$ stable (in all models) isotropic $2n$–Heisenberg FP. The critical exponents are

$$\gamma^*_\Phi = \left\{ \frac{(2n+2)(2\epsilon)^2}{4(2n+8)^2} (2\times) \right\}, \quad \gamma^*_\tau = \left\{ -\frac{2(2n+2)(2\epsilon)}{2n+8}, -\frac{2(2\epsilon)}{2n+8} (2\times) \right\},$$

$$\gamma^*_\text{cr} = \left\{ -\frac{2(2\epsilon)}{2n+8} (4\times), \omega = \left\{ (2\epsilon), \frac{8(2\epsilon)}{2n+8} (2\times), \frac{(4-2n)(2\epsilon)}{2n+8} (3\times) \right\}. $$
 Solutions for finite $n$, Fixed Points and Critical Exponents

- **RS 1.4** $g_{11,22} = \frac{2}{n+8} \epsilon \pm \sqrt{\frac{32(1-n)}{(n+8)^3}} \epsilon$, $g_{12} = \frac{6}{n+8} \epsilon$, other $g_{ij} = 0$.
  
  $z_{12} = \pm 2$. RS1.4 also belongs to the $O(n) + O(n)$ model. This FP coincides with the biconical FP for $n = 1$. In one loop order one obtains the exponents

  \[
  \gamma^* = \begin{cases}
  \left\{ \frac{(n^2 + 37n + 16)(2\epsilon)^2}{8(n + 8)^3} \pm (n + 2) \frac{\sqrt{2(1-n)(2\epsilon)^2}}{2(n + 8)^{5/2}} \right\}, \\
  \left\{ -\frac{(2 + n)(2\epsilon)}{2(n + 8)} \pm \frac{\sqrt{n^3 + 48n^2 + 32(2\epsilon)}}{2(n + 8)^{3/2}}, -\frac{3(2\epsilon)}{n + 8} \right\}, \\
  \left\{ -\frac{(2\epsilon)}{n + 8} \pm \frac{2\sqrt{2(1-n)(2\epsilon)}}{(n + 8)^{3/2}}, -\frac{3(2\epsilon)}{n + 8} (2\epsilon) \right\}, \\
  0, \frac{(6 - n)(2\epsilon)}{n + 8}, \frac{(10 - n)(2\epsilon)}{n + 8}, -\frac{(n + 2)(2\epsilon)}{2(n + 8)} \pm \frac{\sqrt{n^2 - 188n + 196(2\epsilon)}}{2(n + 8)} \}.
  \]

  We considered the coupling in two loop order, since it yields in order $\epsilon$ the region in which the couplings are real. We obtained $n_c = 1 - (2\epsilon)/48 + O(2\epsilon)^2$. 

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Solutions for finite $n$, Fixed Points and Critical Exponents

- **RS 2.1 ($z_{12} = 0$)** Two of the exponents $\omega$ equal 0 for any $n$ in one-loop order.

- One is due to the invariance under rotations between the fields $\phi$. The other one indicates that there may branch off a second line of FPs.

- One finds besides the FP of two decoupled systems (**RS 2.1a**) $g_{11}^* = g_{22}^*$, other $g_{ij} = 0$

another solution (**RS 2.1b**) with

$$
g_{11}^* = g_{22}^*, \quad g_{12}^*, g_{33}^* = O(\epsilon^2), \quad g_{13}^* = g_{23}^* = O(\epsilon^{3/2})
$$

- Both types of FPs agree in one-loop order, but differ in the next order.

- In the following we give the FPs and critical exponents in two-loop order (for $\gamma^*_\Phi$ in three-loop order).
Solutions for finite $n$, Fixed Points and Critical Exponents

- **RS 2.1a**

$$g_{11}^* = g_{22}^* = \frac{4}{n+8}\epsilon - \frac{4(n^2 - 2n - 20)}{(n+8)^3}\epsilon^2,$$ other $g_{ij} = 0$.

This solution describes two independent $O(n)$ models and is the decoupled $n$–Heisenberg–$n$–Heisenberg FP of the $O(n) + O(n)$ model.

$$\gamma^*_\Phi = \begin{cases} 
\frac{(n+2)}{4(n+8)^2}(2\epsilon)^2 - \frac{(n+2)(n^2 - 56n - 272)}{16(n+8)^4}(2\epsilon)^3 \quad {2\times}, \\
\end{cases}$$

$$\gamma^*_\tau = \begin{cases} 
- \frac{n+2}{2(n+8)^2}(2\epsilon)^2, - \frac{n+2}{n+8}(2\epsilon) - \frac{(n+2)(13n + 44)}{2(n+8)^3}(2\epsilon)^2 \quad {2\times}, \\
\end{cases}$$

$$\gamma^*_c = \begin{cases} 
- \frac{2}{n+8}(2\epsilon) + \frac{(n+4)(n-22)}{2(n+8)^2}(2\epsilon)^2 \quad {2\times}, - \frac{n+2}{2(n+8)^2}(2\epsilon)^2 \quad {2\times}, \\
\end{cases}$$

$$\omega = \begin{cases} 
(2\epsilon) - \frac{3(3n + 14)}{(n+8)^2}(2\epsilon)^2 \quad {2\times}, \frac{n - 4}{n+8}(2\epsilon) + \frac{(n+2)(13n + 44)}{(n+8)^3}(2\epsilon)^2, \\
- \frac{n+4}{n+8}(2\epsilon) - \frac{(n+4)(n-22)}{(n+8)^3}(2\epsilon)^2, \frac{n+2}{2(n+8)^2}(2\epsilon)^2, 0 \end{cases}.$$
RS 2.1b is new to our best knowledge. It agrees with RS 2.1a, which describes two uncoupled systems, in one-loop order:

\[
\begin{align*}
g_{11}^* &= g_{22}^* = \\ &= \frac{4}{n + 8} \epsilon - \frac{9n^3 + 98n^2 - 400n - 2272}{2(n + 8)^3(n + 14)} \epsilon^2, \\
g_{13}^* &= g_{23}^* = \\ &= \pm \sqrt{2(n + 4)(n + 2)(n - 4)} \frac{1}{(n + 8)^2 \sqrt{n + 14}} \epsilon^{3/2}, \\
g_{12}^* = \\ &= -\frac{n + 2}{2(n + 8)(n + 14)} \epsilon^2, \\
g_{33}^* = \\ &= \frac{(n + 2)(n - 4)}{(n + 8)^2(n + 14)} \epsilon^2.
\end{align*}
\]

In the limit \(D = 4\) it is real for \(n \geq 4\).
Solutions for finite $n$, Fixed Points and Critical Exponents

- **RS 2.1b** Its critical exponents are

\[
\gamma^*_\Phi = \left\{ \frac{(n+2)}{4(n+8)^2} (2\epsilon)^2 \pm \frac{(n+2)\sqrt{2(n-4)(n+2)(n+4)}}{16(n+8)^3 \sqrt{n+14}} (2\epsilon)^{5/2} - \frac{(n+2)(n^2 - 56n - 272)}{16(n+8)^4} (2\epsilon)^3 \right\},
\]

\[
\gamma^*_\tau = \left\{ -\frac{n+2}{n+8} (2\epsilon) - \frac{(n+2)(29n^2 + 470n + 1256)}{4(n+14)(n+8)^3} (2\epsilon)^2, -\frac{n+2}{n+8} (2\epsilon) - \frac{(n+2)(23n^2 + 434n + 1208)}{4(n+8)^3(n+14)} (2\epsilon)^2, \right. \\
\left. -\frac{3(n+2)(n^2 + 10n + 64)}{4(n+8)^3(n+14)} (2\epsilon)^2 \right\},
\]

\[
\gamma^*_{cr} = \left\{ -\frac{2}{n+8} (2\epsilon) + \frac{n^3 - 12n^2 - 660n - 2416}{4(n+8)^3(n+14)} (2\epsilon)^2, -\frac{2}{n+8} (2\epsilon) + \frac{3n^3 - 4n^2 - 700n - 2512}{4(n+8)^3(n+14)} (2\epsilon)^2, \right. \\
\left. -\frac{(n+2)(n+6)(n+32)}{4(n+8)^3(n+14)} (2\epsilon)^2, -\frac{(n+2)(n+26)}{4(n+8)^2(n+14)} (2\epsilon)^2 \right\},
\]

\[
\omega = \left\{ (2\epsilon) - \frac{3(3n+14)}{(n+8)^2} (2\epsilon)^2 (2\times), -\frac{n+2}{n+8} (2\epsilon)^2, 0, -\frac{n+2}{n+8} (2\epsilon) + \frac{(n+2)(15n^3 + 242n^2 + 656n + 32)}{n(n+8)^3(n+14)} (2\epsilon)^2, \right. \\
\left. -\frac{n+4}{n+8} (2\epsilon) - \frac{3n^4 + 12n^3 - 332n^2 - 1252n + 64}{n(n+8)^3(n+14)} (2\epsilon)^2 \right\}.
\]
RS 2.3 \((z_{12} = \pm i)\) also is new to our knowledge.

\[
g_{11,22} = \frac{2}{n+8} \epsilon \pm \sqrt{-\frac{4(3n+22)(n-2)(n+2)(n+4)(n+14)}{(n+8)^3(n^2+4n+20)^2}} \epsilon,
\]

\[
g^*_{12} = \frac{4(n+6)(n+4)}{(n+8)(n^2+4n+20)} \epsilon, \quad g^*_{33} = \frac{4(n^2-36)}{(n+8)(n^2+4n+20)} \epsilon, \quad g^*_{13,23} = 0.
\]

We consider the coupling in two loop order to obtain in order \(\epsilon\) the region in which the couplings are real:

\[
n_c = 2 - (2\epsilon)/140 + O(2\epsilon)^2
\]
Solutions for finite $n$, Fixed Points and Critical Exponents

\textbf{RS 2.3} ($z_{12} = \pm i$) The critical exponents are

$$\gamma^*_\Phi = \left\{ \frac{(2n^6 + 37n^5 + 348n^4 + 2360n^3 + 9376n^2 + 13904n - 9152)(2\epsilon)^2}{8(n + 8)^3(n^2 + 4n + 20)^2} \right.$$

$$\pm \frac{(n + 2)\sqrt{-(3n + 22)(n - 2)(n + 2)(n + 4)(n + 14)(2\epsilon)^2}}{8(n + 8)^{5/2}(n^2 + 4n + 20)} \left\}, \right.$$

$$\gamma^*_\tau = \left\{ \frac{-(2\epsilon)(n - 1)(n - 2)(n + 6)}{(n + 8)(n^2 + 4n + 20)}, \frac{-(n + 2)(2\epsilon)}{2(n + 8)} \right\},$$

$$\gamma^*_\text{cr} = \left\{ \frac{-(2\epsilon)(n + 6)}{(n + 8)(n^2 + 4n + 20)}, \frac{-(n + 6)(n + 14)(2\epsilon)}{(n + 8)(n^2 + 4n + 20)} \right\},$$

$$\omega = \left\{ 0, (2\epsilon), \frac{(2\epsilon)(n^3 + 10n^2 - 4n - 232)}{(n + 8)(n^2 + 4n + 20)}, \frac{(2\epsilon)\lambda'}{2(n + 8)(n^2 + 4n + 20)} \right\}.$$ 

where $\lambda'$ is solution of the equation

$$\lambda'^3 + 16(n^2 + 4n + 20)\lambda'^2 - 4(n + 4)(n^5 - 18n^4 - 392n^3 - 1648n^2 - 496n + 8928)\lambda'$$

$$- 16(3n + 22)(n - 2)(n + 6)(n - 6)(n + 4)(n + 2)(n + 14)^2 = 0.$$
RS 2.2 \((z_{12} = \pm \sqrt{2})\) and RS 3.1 are solutions of one and the same quadratic equation and correspond to the antichiral and chiral FP of the \(O(2) \times O(n)\) model, respectively:

\[
g_{11,22} = \frac{3n^2 - 2n + 24 + s(n - 6)\sqrt{n^2 - 24n + 48}}{n^3 + 4n^2 - 24n + 144} \epsilon, \\
g_{12} = \frac{-n^2 - 6n + 72 + s(n + 6)\sqrt{n^2 - 24n + 48}}{n^3 + 4n^2 - 24n + 144} \epsilon, \\
g_{33} = \frac{4(n^2 + n - 12 - s3\sqrt{n^2 - 24n + 48})}{n^3 + 4n^2 - 24n + 144} \epsilon, \quad g_{13,23} = 0,
\]

where \(s = +1\) corresponds to RS 3.1 and \(s = -1\) to RS 2.2.

Both fixed points are \(O(n) \times O(2)\) invariant.
RS 2.2 and RS 3.1 The critical exponents are

\[ \gamma^*_\Phi = \left\{ \frac{(5n^5 - 3n^4 - 16n^3 - 656n^2 + 3072n - 1152 + s(n - 3)(n + 4)w^{3/2})(2\epsilon)^2}{16N^2} \right\}, \]

\[ \gamma^*_\tau = \left\{ -\frac{(n(48 + n + n^2) - s(n - 3)(4 + n)\sqrt{w})(2\epsilon)}{2N}, -\frac{(-2n^3 - 3n^2 + 28n - 48 + 5sn\sqrt{w})(2\epsilon)}{2N} \right\} \cdot \]

\[ \gamma^*_\text{cr} = \left\{ -\frac{(-5n^2 - s(n - 12)\sqrt{w})(2\epsilon)}{2N}, \frac{(-n^2 + 4n - 48 - sn\sqrt{w})(2\epsilon)}{2N} \right\} \cdot \]

\[ \omega = \left\{ \frac{(n + 4)((n + 4)(n - 3) - 3s\sqrt{w})(2\epsilon)}{N}, \frac{(n^3 + 14n^2 + 56n - 96 + s(n + 8)(n - 6)\sqrt{w})(2\epsilon)}{2N} \right\} \cdot \]

\[ \frac{(-3(n^2 - 24n + 48) + s(n + 4)(n - 3)\sqrt{w})(2\epsilon)}{N} \cdot \]
The FP 3.1 is stable for large \( n \).

Two loop calculation gives the range where the FPs are real:

\[
\begin{align*}
  n &> 21.8 - 23.4(2\epsilon) + O(2\epsilon)^2 \\
  n &< 2.20 - 0.57(2\epsilon) + O(2\epsilon)^2
\end{align*}
\]

The question of the range of stability in \( D = 3 \) is under debate.

The \( 1/n \) expansion of the general \( O(n) \) symmetric two field model gives in first order

\[
\eta = \frac{6\Gamma(D - 2) \sin\left(\frac{D\pi}{2}\right)}{\pi\Gamma(D/2 - 2)\Gamma(1 + D/2)n}
\]

\[
\gamma^*_\tau - (D - 4) = \left\{ \frac{2(2 - D)(1 - D)\eta}{4 - D}, \frac{2(2 - D)(3 - 2D)\eta}{3(4 - D)} \right\}.
\]

Both expansions agree with each other!
1. Introduction and Descriptive Overview
2. $4 - 2\epsilon$ Expansion
3. Field rotations
4. The classification of the fixed points in the large $n$ limit
5. Solutions for finite $n$, Fixed Points and Critical Exponents
6. Summary and Conclusions
Summary and Conclusions

- The general $O(n)$ symmetric Hamiltonian has three different mass terms. It gives rise to a variety of critical and multicritical behaviors generalizing the $O(n) + O(n)$ and $O(2) \times O(n)$ models.

- We gave the expressions for the $\beta$ functions and the matrices $\gamma_\Phi$, $\gamma_\tau$, $\gamma_{cr,s}$ and $\omega$, and $\gamma_{cr,a}$ for the general $O(n)$ model from which the critical exponents are obtained in one-loop order (for $\eta$ in two-loop order).

- A classification of the FPs in the large $n$ limit was given. Two types of FPs emerge: Four of them are invariant under $O(n) \times O(2)$. The other six FPs are not invariant under $O(2)$ and yield lines of FPs.

- Under the numerous FPs the corresponding FPs of the well-known models were found.

- To our best knowledge the FPs $RS\ 2.1b$ and 2.3 are new. $RS\ 2.1b$ agrees with $RS\ 2.1a$, which describes two uncoupled systems, in one-loop order.