An "analytic" walk in QCD perturbation theory

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Collaborators and Publications

♣ "Fractional Analytic Perturbation Theory in Minkowski space and application to Higgs boson decay into a $b\bar{b}$ pair"
A. P. Bakulev, S. V. Mikhailov, N. G. Stefanis,
Phys. Rev. D **75**, 056005 (2007) [arXiv:hep-ph/0607040]

♣ "QCD analytic perturbation theory: From integer powers to any power of the running coupling"
A. P. Bakulev, S. V. Mikhailov, N. G. Stefanis,
Phys. Rev. D **72**, 074014 (2005) [arXiv:hep-ph/0506311]

♣ "Analyticity properties of three-point functions in QCD beyond leading order"
A. P. Bakulev, A. I. Karanikas, N. G. Stefanis,
Phys. Rev. D **72**, 074015 (2005) [arXiv:hep-ph/0504275]

♣ "Pion form factor analysis using NLO analytic perturbation theory"
N. G. Stefanis,
Nucl. Phys. Proc. Suppl. **152**, 245 (2006) [arXiv:hep-ph/0410245]

♣ "Pion form factor in QCD: From nonlocal condensates to NLO analytic perturbation theory"
A. P. Bakulev, K. Passek-Kumerički, W. Schroers, N. G. Stefanis,
Phys. Rev. D **70**, 033014 (2004) [arXiv:hep-ph/0405062]

♣ "Perturbative logarithms and power corrections in QCD hadronic functions: A unifying approach"
N. G. Stefanis,
Lect. Notes Phys. **616**, 153 (2003) [arXiv:hep-ph/0203103]

♣ "Analyticity and power corrections in hard-scattering hadronic functions"
A. I. Karanikas, N. G. Stefanis,
Phys. Lett. B **504**, 225 (2001) [arXiv:hep-ph/0101031]

♣ "Analytic coupling and Sudakov effects in exclusive processes: Pion and $\gamma^*\gamma \to \pi^0$ form factors"
N. G. Stefanis, W. Schroers, H. C. Kim,
Eur. Phys. J. C **18**, 137 (2000) [arXiv:hep-ph/0005218]

♣ "Pion form factors with improved infrared factorization"
N. G. Stefanis, W. Schroers, H. C. Kim,
Phys. Lett. B **449**, 299 (1999) [arXiv:hep-ph/9807298]

Benchmarks of presentation

- UV freedom and Landau singularity
- First remedies in the IR: Color saturation, effective gluon mass
- Shirkov-Solovtsov analytic coupling — Euclidean and Minkowski space
- From a recipe to a paradigm: APT
- More scales, more riddles: Logs of factorization (evolution) scale
- Generalization of analyticity concept: From the coupling to the whole amplitude, Naive and Maximal analytization
- Creation of FAPT in spacelike and timelike regions
- To Do List: Series resummation, Sudakov resummation (exponentiation vs. analytization), power corrections

Ultraviolet freedom and Landau singularity

QCD has provided successful microscopic theory of strong interactions from a few GeV to highest measured energies.

♣ At short distance it is asymptotically free, i.e., $\alpha_s(Q^2) = g_s^2(Q^2)/4\pi$ becomes small: $\alpha_s(Q^2) \sim 1/\ln Q^2 \to 0$ as $Q^2/\Lambda_{\rm QCD}^2 \to \infty$

♣ $\Lambda_{\rm QCD}$ is intrinsic (scheme-dependent) QCD scale extracted from experimental data

♣ **BUT** at $Q^2 = \Lambda_{\rm QCD}^2$, running strong coupling has Landau singularity that spoils analyticity. To restore analyticity and ensure causality in whole Q^2 plane, this ghost singularity has to be averted (removed or regularized).

First remedies in the infrared

- Cutoff regularization of running coupling at some value for which perturbation theory works, e.g., $\alpha_s^{\text{cutoff}} = 0.5$.
- Assuming that below some momentum scale there is color saturation with spontaneous chiral symmetry breaking and quarks and gluons being confined within color-singlet states. This entails minimum momentum scale m_g that can be conceived of as an effective gluon mass. Then, at one loop, $\alpha_s^{\text{sat}} = \frac{4\pi}{\beta_0 \ln[(Q^2+\lambda)/\Lambda^2_{\text{QCD}}]}$ with $\lambda = 4m_g^2$
- To be consistent with asymptotic freedom (and the RG), m_g should vanish asymptotically, **Cornwall, PRD26 (1982) 1453**: $m_{\text{g}}^2(Q^2) = m_{\text{g}}^2 \left[\ln\left(\frac{Q^2+4m_{\text{g}}^2}{\Lambda^2_{\text{QCD}}}\right) \Big/ \ln\left(\frac{4m_{\text{g}}^2}{\Lambda^2_{\text{QCD}}}\right) \right]^{-12/11}$

- Screening of α_s singularities by Sudakov factor $\mathrm{e}^{-S} \to 0$. Because $\mathrm{e}^{-S} \to 0$ drops to 0 faster than any power of $\ln\left(\mu^2/\Lambda^2_{\mathrm{QCD}}\right)$, this provides *in situ* IR protection against Landau singularities (**Botts+Sterman**).
- In axial gauge, all Sudakov contributions due to unintegrated transverse momenta of gluon propagators exponentiate into suppressing Sudakov factors (**Li+Sterman**).
- Resummation of IR-renormalon asymptotic series which are defined as an integral of the running coupling over the IR region (**Krasnikov+Pivovarov**).
- Using Λ-parametrization for $\alpha_s(Q^2)$ in spacelike region, construct for $R(q^2)$ in the timelike region expansion in which all $(\pi^2/L^2)^N$-terms ($L \equiv \ln s/\Lambda^2_{\mathrm{QCD}}$) are summed explicitly (**Radyushkin**).

- Shirkov+Solovtsov (1996) invented analytic coupling based solely on RG invariance and causality (spectrality).
- No extraneous IR regulators necessary and no ad hoc cutoff procedures involved except $\Lambda_{\rm QCD}$.
- Analyticity is ensured in complex Q^2 plane by means of the Källen-Lehmann representation:

$$\left[f(Q^2)\right]_{\rm an} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

- Same spectral density $\rho_f(\sigma) = \mathbf{Im}[\mathbf{f}(-\sigma)]/\pi$ defines running coupling in timelike region, by taking recourse to dispersion relation for the Adler function (**Milton–Solovtsov–Solovtsova** — **see Solovtsova's talk**).

At one-loop level, one obtains in

Euclidean region

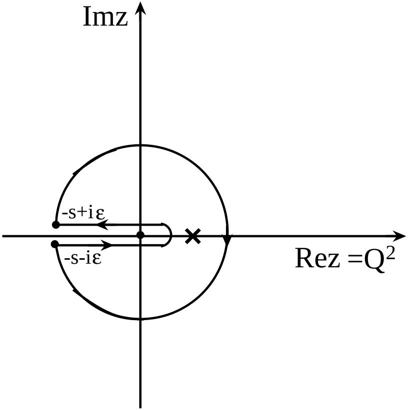
$$\mathcal{A}_1(Q^2) \;=\; \int_0^\infty \frac{\rho(\sigma)}{\sigma+Q^2}\,d\sigma \;=\; \frac{1}{L} - \frac{1}{e^L-1}$$

Minkowski region

$$\mathfrak{A}_1(s) \;=\; \int_s^\infty \frac{\rho(\sigma)}{\sigma}\,d\sigma \;=\; \frac{1}{\pi}\arccos\frac{L_s}{\sqrt{\pi^2+L_s^2}}$$

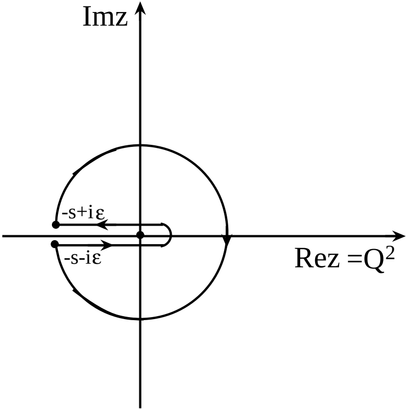
with $L=\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)$ and $L_s=\ln\left(s/\Lambda^2_{\rm QCD}\right)$

Graphical representation of analytization: PT



Problem: Amending Landau singularity in fixed-order PT.

Graphical representation of analytization: APT



No problem: Landau singularity absent by construction in APT.

Analytic **P**erturbation **T**heory

In **APT** hadronic quantities (in Minkowski region)

$$\oint f(z)R(z)dz \ ,$$

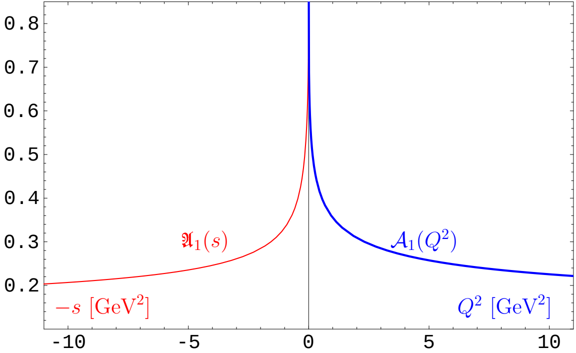
with

$$R^{\text{PT}}(z) = \sum_n d_m \alpha_s^m(z) \xrightarrow{\textbf{APT}} \mathcal{R}^{\text{APT}}(z) = \sum_n d_m \mathfrak{A}_m(z)$$

have no Landau singularity in Euclidean region (denoted by cross in Figure above), because new spacelike couplings $\mathcal{A}_n(z)$ are analytic functions

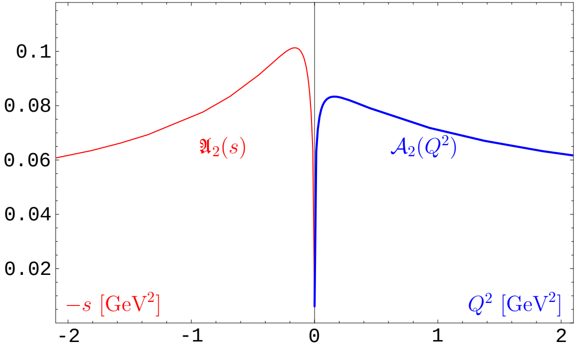
Distorted mirror symmetry (**Shirkov–Solovtsov**)

First analytic couplings
$\mathcal{A}_1(Q^2)$ (Euclidean) and $\mathfrak{A}_1(s)$ (Minkowski) space



Distorted mirror symmetry (**Shirkov–Solovtsov**)

Second analytic couplings
$\mathcal{A}_2(Q^2)$ (Euclidean) and $\mathfrak{A}_2(s)$ (Minkowski) space



APT formalism at 1-loop

[Radyushkin (1982), Shirkov (1999)]

$$\begin{pmatrix} a^n(k) \\ \mathcal{A}_n(k) \\ \mathfrak{A}_n(k) \end{pmatrix} = \frac{1}{(n-1)!}\left(-\frac{d}{dk}\right)^{n-1} \begin{pmatrix} a^1(k) \\ \mathcal{A}_1(k) \\ \mathfrak{A}_1(k) \end{pmatrix} \qquad (1)$$

- a^n (standard, $n \in \mathbb{R}$: power)
- $\mathcal{A}_n$ (analytic in spacelike region, $n \in \mathbb{R}$: index); $k = L \equiv \ln(Q^2/\Lambda^2)$
- $\mathfrak{A}_n$ (analytic in timelike region, $n \in \mathbb{R}$: index); $k = L_s \equiv \ln(s/\Lambda^2)$

APT formalism at 1-loop from FAPT for $\nu = n \in \mathbb{N}$

- Euclidean space:

$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2} = \frac{1}{L^n} - \frac{F(e^{-L}, 1-n)}{\Gamma(n)}$$

- Minkowski space:

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma} = \frac{\sin\left[(n-1)\arccos\left(L_s/\sqrt{L_s^2+\pi^2}\right)\right]}{(n-1)\,\pi\left[\sqrt{L_s^2+\pi^2}\right]^{n-1}}$$

- Spectral density:

$$\rho_n(\sigma) = \frac{1}{\pi}\mathbf{Im}[a^n(-\sigma)] = \frac{\sin\left[n\arccos\left(L_s/\sqrt{L_s^2+\pi^2}\right)\right]}{n\,\pi\left[\sqrt{L_s^2+\pi^2}\right]^{n}}$$

$(a = 1/L;\ a_s = 1/L_s)$

Some Remarks

- Analytization $\mathbf{A_E}$ in Euclidean space: subtraction of Landau pole (at one loop)
- Analytization $\mathbf{A_M}$ in Minkowski space: summation of π^2 terms
- Two-loop expressions for analytic couplings possible via **Lambert function** (**Magradze (2000)**)
- Higher orders can be obtained with approximate spectral density and numerical integration (**Shirkov (1999)**)
- **Elimination of ghost singularities in analytic approach appears as a result of causality (spectrality) and RG invariance, i.e., pole remover not introduced by hand**
- Spectral density modified by (nonperturbative) power corrections (**Alekseev (2006); Nesterenko+Papavassiliou (2005); Cvetič+Valenzuela (2005)**)

- Analytization of multi-scale hadronic amplitudes beyond LO of pQCD involves additional logarithms depending on scale that serves as factorization or renormalization scale [**Karanikas+Stefanis, PLB504(2001)225**]
- Evolution induces non-integer, i.e., fractional, powers of coupling constant
- Resummation of gluon radiative corrections, gives rise to Sudakov factors that have to be included into analytization procedure [**Stefanis+Schroers+Kim, PLB449(1999)299; EPJC18(2000)137**]
- Naive analytization; maximal analytization vs. exact analytization [**Bakulev+Passek+Schroers+Stefanis, 2004**]

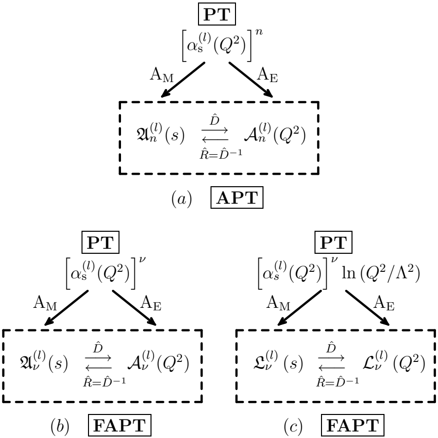
To accommodate analytization of terms like

- $Z[L] = \mathrm{e}^{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)} da} \to [a_s(L)]^{\gamma_0/2\beta_0}$ $\longleftrightarrow$ RG at one loop
- $[a_s(L)]^n \ln[a_s(L)]$ $\longleftrightarrow$ RG at two loops
- $[a_s(L)]^n L^m$ $\longleftrightarrow$ Factorization
- $\exp\left[-a_s(L)F(x)\right]$ $\longleftrightarrow$ Sudakov resummation

typically appearing in perturbative calculations beyond LO, analyticity requirement has to be applied to whole QCD amplitude.

♣ Though such terms do not modify ghost singularities, they do contribute to spectral density and are tantamount to *fractional (real) powers* of the strong coupling. To include them into the dispersion integral, we apply the **Karanikas-Stefanis [PLB 504 (2001) 225]** analyticity requirement.

Different analytization concepts



Euclidean-Minkowski connection

Linear operations $\mathbf{A}_{\rm E}$ and $\mathbf{A}_{\rm M}$ define, respectively, analytic running couplings in Euclidean (spacelike). Here index $\nu \in \mathbb{R}$

$$\mathbf{A}_{\rm E}\left[a^{\nu}_{(l)}\right] \quad = \quad \mathcal{A}^{(l)}_{\nu} \quad \text{with} \quad \mathcal{A}^{(l)}_{\nu}(Q^2) \equiv \int_0^\infty \frac{\rho^{(l)}_{\nu}(\sigma)}{\sigma + Q^2}\, d\sigma \qquad (2)$$

and Minkowski (timelike) region

$$\mathbf{A}_{\rm M}\left[a^{\nu}_{(l)}\right] = \mathfrak{A}^{(l)}_{\nu} \quad \text{with} \quad \mathfrak{A}^{(l)}_{\nu}(s) \equiv \int_s^\infty \frac{\rho^{(l)}_{\nu}(\sigma)}{\sigma}\, d\sigma\,. \qquad (3)$$

Integral transformations are iterrelated:

$$\hat{D}\hat{R} = \hat{R}\hat{D} = 1 \qquad (4)$$

In spacelike region, analytic images of the coupling can be expressed in terms of reduced transcendental **Lerch function**

♣ $F(z,\nu)$: $(L \equiv \ln(Q^2/\Lambda^2))$

$$\mathcal{A}_{\nu}(L) = \frac{1}{L^{\nu}} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}\,, \qquad (5)$$

♣ First term corresponds to pQCD; second one entailed by pole remover $(1/(\mathrm{e}^L - 1)$ at one loop). ♣ This function is entire function in index ν and has the properties: ♣ $\mathcal{A}_0(L) = 1$,
♣ $\mathcal{A}_{-m}(L) = L^m$ for $m \in \mathbb{N}$, and ♣ $\mathcal{A}_m(\pm\infty) = 0$ for $m \geq 2$, $m \in \mathbb{N}$, while for $|L| < 2\pi$,

♣ $\mathcal{A}_{\nu}(L) = -\,[1/\Gamma(\nu)]\sum_{r=0}^{\infty}\zeta(1-\nu-r)\,[(-L)^r/r!]\,.$

In timelike region, these images are completely determined by elementary functions [**Bakulev+Mikhailov+Stefanis, PRD 72 (2005) 074014**] ($L_s \equiv \ln(s/\Lambda^2)$):

$$\mathfrak{A}_\nu(L_s) = \frac{\sin\left[(\nu-1)\arccos\left(L_s/\sqrt{\pi^2+L_s^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L_s^2\right)^{(\nu-1)/2}} \qquad (6)$$

Main properties:

- $\mathfrak{A}_0(L) = 1$;
- $\mathfrak{A}_{-1}(L) = L$;
- $\mathfrak{A}_{-2}(L) = L^2 - \frac{\pi^2}{3}$;
- $\mathfrak{A}_m(L) = (-1)^m \mathfrak{A}_m(-L)$ for $m \geq 2\,,\ m \in \mathbb{R}$;
- $\mathcal{A}_m(\pm\infty) = 0$

Compilation of salient FAPT features

QCD Scheme	**PT**	**APT**	**FAPT**
Space	$\{a^\nu\}_{\nu\in\mathbb{R}}$	$\{\mathcal{A}_m\}_{m\in\mathbb{N}}$	$\{\mathcal{A}_\nu\}_{\nu\in\mathbb{R}}$
Series expansion	$\sum_m f_m\, a^m(L)$	$\sum_m f_m\, \mathcal{A}_m(L)$	$\sum_m f_m\, \mathcal{A}_m(L)$
Inverse powers	$[a(L)]^{-m}$	—	$\mathcal{A}_{-m}(L) = L^m$
Multiplication	$a^\mu a^\nu = a^{\mu+\nu}$	—	—
Index derivative	$a^\nu \ln^k a$	—	$\frac{d^k \mathcal{A}_\nu}{d\nu^k} = \left[a^\nu \ln^k(a)\right]_{\rm an}$

[**Bakulev+Mikhailov+Stefanis, PRD 72 (2005) 074014**]

♣ First process to be considered is factorizable part of pion's electromagnetic form factor at NLO accuracy in Euclidean space. At leading twist, one has the convolution $[A(z) \otimes zB(z) \equiv \int_0^1 dz A(z) B(z)]$

$$F_{\pi}^{\rm Fact}(Q^2) = \varphi_{\pi}(x, \mu_{\rm F}^2) \otimes T_{\rm H}^{\rm NLO}\left(x, y, Q^2; \mu_{\rm F}^2, \mu_{\rm R}^2\right) \otimes \varphi_{\pi}(y, \mu_{\rm F}^2)$$

♣ Twist-2 pion distribution amplitude (using $\bar{x} \equiv 1 - x$)

$$\varphi_{\pi}(x, \mu^2) = 6x\bar{x}\left[1 + a_2(\mu^2)\, C_2^{3/2}(2x-1) + a_4(\mu^2)\, C_4^{3/2}(2x-1) + \ldots\right]$$

contains all non-perturbative information on pion quark structure in terms of Gegenbauer coefficients a_n at scale $\mu^2 \approx 1$ GeV^2

Differences among the various analytization schemes

♣ Beyond LO pQCD $F_{\pi}^{\rm Fact}(Q^2)$ depends on factorization scale $\mu_{\rm F}$ and renormalization scale $\mu_{\rm R}$

Naive Analytization

$$\left[Q^2 T_{\rm H}\left(x, y, Q^2; \mu_{\rm F}^2, \lambda_{\rm R} Q^2\right)\right]_{\rm Naive-An} = \mathcal{A}_1^{(2)}(\lambda_{\rm R} Q^2)\, t_{\rm H}^{(0)}(x, y) + \frac{\left[\mathcal{A}_1^{(2)}(\lambda_{\rm R} Q^2)\right]^2}{4\pi}\, t_{\rm H}^{(1)}\left(x, y; \lambda_{\rm R}, \frac{\mu_{\rm F}^2}{Q^2}\right)$$

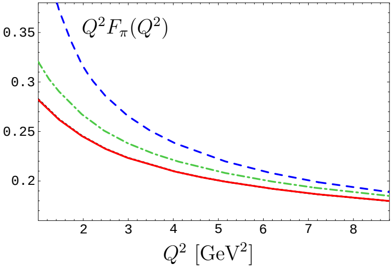
Maximal Analytization

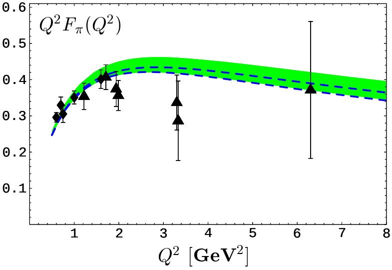
$$\left[Q^2 T_{\rm H}\left(x, y, Q^2; \mu_{\rm F}^2, \lambda_{\rm R} Q^2\right)\right]_{\rm Max-An} = \mathcal{A}_1^{(2)}(\lambda_R Q^2)\, t_{\rm H}^{(0)}(x, y) + \frac{\mathcal{A}_2^{(2)}(\lambda_{\rm R} Q^2)}{4\pi}\, t_{\rm H}^{(1)}\left(x, y; \lambda_{\rm R}, \frac{\mu_{\rm F}^2}{Q^2}\right)$$

Remarks

- Naive Analytization just replaces strong coupling and its powers by their corresponding analytic images Stefanis+Schroers+Kim. Incorrect because $[\mathcal{A}_1(L)]^n \neq [a_s^n(L)]_{\rm An}$, but phenomenologically rather good.
- Maximal Analytization associates to powers of running coupling their own dispersive images, i.e., $[a_s^n(L)]_{\rm Max-An} = \mathcal{A}_n(L)$.
- Crucial advantage of FAPT analysis is that dependence of prediction for $F_\pi^{\rm Fact}(Q^2)$ on perturbative scheme and scale setting is diminished already at NLO.

$Q^2 F_\pi^{\text{Fact}}(Q^2)$ vs. Q^2 with $\mu_{\text{R}}^2 = Q^2$, $\mu_{\text{F}}^2 = 5.76$ GeV^2





LEFT:

♣ **pQCD** (dashed line); ♣ Naive Analytization (dash-dotted line);
♣ Maximal Analytization (solid line)

RIGHT: ♣ $Q^2 F_\pi^{\text{Fact}}(Q^2)$ vs. exp. data.

Effect of amplitude analytization

KS analytization demands inclusion of the logarithmic term $\ln(Q^2/\mu_{\rm F}^2) = \ln(\lambda_{\rm R} Q^2/\Lambda^2) - \ln(\lambda_{\rm R}\mu_{\rm F}^2/\Lambda^2)$, so that we obtain

$$\begin{aligned}\left[Q^2 T_{\rm H}(x,y,Q^2;\mu_{\rm F}^2,\lambda_{\rm R}Q^2)\right]_{\rm KS}^{\rm An} &= \mathcal{A}_1^{(2)}(\lambda_{\rm R}Q^2)\, t_{\rm H}^{(0)}(x,y) \\ &+ \frac{\mathcal{A}_2^{(2)}(\lambda_{\rm R}Q^2)}{4\pi}\, t_{\rm H}^{(1)}\left(x,y;\lambda_{\rm R},\frac{\mu_{\rm F}^2}{Q^2}\right) \\ &+ \frac{\Delta_2^{(2)}\left(\lambda_{\rm R}Q^2\right)}{4\pi}\left[C_{\rm F}\, t_{\rm H}^{(0)}(x,y)\,\left(6+2\ln(\bar{x}\bar{y})\right)\right],\end{aligned}$$

with $\Delta_2^{(2)}\left(Q^2\right) \equiv \mathcal{L}_2^{(2)}\left(Q^2\right) - \mathcal{A}_2^{(2)}\left(Q^2\right)\,\ln\left[Q^2/\Lambda^2\right]$, where

$$\mathcal{L}_2^{(2)}\left(Q^2\right) \equiv \left[\left(\alpha_s^{(2)}\left(Q^2\right)\right)^2 \ln\left(\frac{Q^2}{\Lambda^2}\right)\right]_{\rm KS}^{\rm An} = \frac{4\pi}{b_0}\left[\frac{\left(\alpha_s^{(2)}(Q^2)\right)^2}{\alpha_s^{(1)}(Q^2)}\right]_{\rm KS}^{\rm An}$$

Performing the **KS analytization** [**Bakulev + Karanikas + Stefanis, PRD 72 (2005) 074015**], we find **Deviation from Max analytization**:

$$\mathcal{L}_2^{(2)}\left(Q^2\right) = \frac{4\pi}{b_0}\left[\mathcal{A}_1^{(2)}\left(Q^2\right) + c_1\,\frac{4\pi}{b_0}\,f_{\mathcal{L}}\left(Q^2\right)\right],$$

where [$\zeta(z)$ is the Riemann zeta-function]

$$f_{\mathcal{L}}\left(Q^2\right) = \sum_{n\geq 0}\left[\psi(2)\zeta(-n-1) - \frac{d\zeta(-n-1)}{dn}\right]\frac{\left[-\ln\left(Q^2/\Lambda^2\right)\right]^n}{\Gamma(n+1)}$$

♣ Expression for $F_\pi^{\rm Fact}(Q^2)$ found to be **extremely stable against variations of factorization scale**

♣ **Sensitivity to renormalization scale and scheme significantly reduced** relative to standard pQCD

Higgs-boson decay

♣ Consider decay of a scalar Higgs boson to a $b\bar{b}$ pair at the four-loop level of the quantity R_{S} from which one can obtain the width $\Gamma(\mathrm{H} \to \mathrm{b}\bar{\mathrm{b}})$.

♣ In that case, no ghost singularities present, but analytic continuation from spacelike to timelike region will entail so-called "kinematical" π^2 terms that may be comparable with expansion coefficients.

♣ Starting point is the correlator of two scalar currents $J_b^{\mathrm{S}} = \bar{\Psi}_b \Psi_b$ for bottom quarks with mass m_b, coupled to the scalar Higgs boson with mass M_{H} and where $Q^2 = -q^2$:

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iq\cdot x} \langle 0| \ T[\ J_b^{\mathrm{S}}(x) J_b^{\mathrm{S}}(0)\ ]\ |0\rangle \,.$$

Then, $R_{\rm S}(s) = \textbf{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)$ and one can express the width in terms of $R_{\rm S}$, i.e.,

$$\Gamma(\mathrm{H} \to b\bar{b}) = \frac{G_{\rm F}}{4\sqrt{2}\pi} M_{\rm H} m_b^2(M_{\rm H}) R_{\rm S}(s = M_{\rm H}^2)$$

$R_{\rm S}$ is obtained via analytic continuation of Adler function D into Euclidean space using $\mathbf{A}_{\rm M}$ (equivalently, integral transformation $\hat{R}$)

♣ One has to calculate [**Chetyrkin+Kniehl+Sirlin (1997)**]

$$\widetilde{R}_{\sf S}(s) \equiv \widetilde{R}_{\sf S}(Q^2 = s, \mu^2 = s) = 3m_b^2(s)\left[1 + \sum_{n\geq 1} r_n\ a_s^n(s)\right]$$

Expansion coefficients r_n contain π^2 terms originating from integral transformation $\hat{R}$ of the powers of the logarithms entering $\widetilde{D}_{\rm S}$.

♣ Running mass in l-loop approximation, $m_{(l)}$, can be cast in terms of RG invariant quantity $\hat{m}_{(l)}$ to read

$$m^2_{(l)}(Q^2) \;\; = \;\; \hat{m}^2_{(l)} \left[a_s(Q^2)\right]^{\nu_0} f_{(l)}(a_s(Q^2))\,,$$

where expansion of $f_{(l)}(x)$ at 3-loop order is given by

$$\begin{aligned} f_{(l)}(a_s) \;\; = \;\; & 1 + a_s \frac{b_1}{2b_0}\left(\frac{\gamma_1}{b_1} - \frac{\gamma_0}{b_0}\right) \\ & + a_s^2 \frac{b_1^2}{16\, b_0^2}\left[\frac{\gamma_0}{b_0} - \frac{\gamma_1}{b_1} + 2\left(\frac{\gamma_0}{b_0} - \frac{\gamma_1}{b_1}\right)^2 \right. \\ & \left. + \frac{b_0\, b_2}{b_1^2}\left(\frac{\gamma_2}{b_2} - \frac{\gamma_0}{b_0}\right)\right] + O\left(a_s^3\right) \end{aligned}$$

Analytization of Adler function

♣ Expanding the running mass in a power series, according to

$$m^2_{(l)}(Q^2) \;=\; \hat{m}^2_{(l)}\,\left(a_s(Q^2)\right)^{\nu_0}\left[1+\sum_{m\geq 1}^{\infty} e^{(l)}_m\,\left(a_s(Q^2)\right)^m\right],$$

and choosing $\mu^2 = Q^2$, we use

$$\Delta^{(l)}_m = e^{(l)}_m + \sum_{k\geq 1}^{\min[l,m-1]} d_k\, e^{(l)}_{m-k}$$

to find

$$\left[3\,\hat{m}_b^2\right]_{(l)}^{-1}\ \widetilde{D}_{\rm S}^{(l)}(Q^2) = \left(a_s^{(l)}(Q^2)\right)^{\nu_0} + \sum_{n\geq 1}^{l} d_n \left(a_s^{(l)}(Q^2)\right)^{n+\nu_0} + \sum_{m\geq 1}^{\infty} \Delta_m^{(l)}\ \left(a_s^{(l)}(Q^2)\right)^{m+\nu_0}$$

- ♣ Second term contains original series expansion of D (truncated at $n = l$)
- ♣ Third term collects mass-evolution effects

♣ Finally, we obtain $\widetilde{R}_{\mathrm{S}}^{\mathrm{MFAPT}}$ from $\widetilde{D}_{\mathrm{S}}^{(l)}(Q^2)$ by applying the analytization operation $\mathbf{A}_{\mathrm{M}}$:

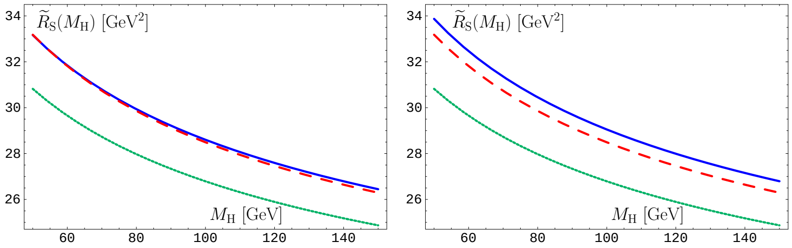
$$\widetilde{R}_{\mathrm{S}}^{(l)\mathrm{MFAPT}} = \mathbf{A}_{\mathrm{M}}[D_{\mathrm{S}}^{(l)}]$$
$$= 3\,\hat{m}_{(l)}^2 \left[\mathfrak{a}_{\nu_0}^{(l)} + \sum_{n\geq 1}^{l} d_n \mathfrak{a}_{n+\nu_0}^{(l)} + \sum_{m\geq 1} \Delta_m^{(l)} \mathfrak{a}_{m+\nu_0}^{(l)}\right],$$

where we have used the short-hand notation

$$[a_s(s)^{\nu}]_{\mathrm{an}} = \mathfrak{a}_{\nu}^{(l)}(s) \equiv \left(\frac{4}{b_0}\right)^{\nu} \mathfrak{A}_{\nu}^{(l)}(s)$$

and $b_0 = \frac{11}{3} C_{\mathrm{A}} - \frac{4}{3} T_{\mathrm{R}} N_f$ with $C_{\mathrm{A}} = N_c = 3,\ T_{\mathrm{R}} = \frac{1}{2}$.

Results for $\widetilde{R}_{\rm S}(M_{\rm H}^2)$, calculated within different approaches in the $\overline{\rm MS}$ scheme, versus the Higgs mass M_H:



♣ Long-dashed curve: results of **Baikov et al.** in pQCD at $l = 4$

♣ Solid curve: FAPT results with evolution effects up to $m = l + 4$ and $N_f = 5$ (second sum).

♣ Green curve: results of **Broadhurst et al.** ("naive non-Abelianization")

- KS analyticity requirement proven successful in describing hadronic observables at partonic level at NLO and beyond
- Including into dispersion relations contributions stemming from all terms that affect spectral density, makes it possible to treat processes containing two large momentum scales.
- By the same token, KS principle enables generalization of APT to any real power of strong coupling leading to FAPT in Euclidean and Minkowski space
- Resummation in FAPT including can be considered with improvement of convergence (cf. **Bakulev's talk**)
- Heavy-quark effects can be included in evolution (cf. **Bakulev's talk**)
- Low-energy behavior of (F)APT can be improved (cf. **Nesterenko's talk; Valenzuela's talk**)
- Future tasks: Sudakov gluon resummation and inclusion of power corrections in different QCD processes