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New type of self-organization on complex networks

PROBLEM

We consider a critical dynamics of a very specific system on complex networks. The system under consideration is **closed**. It has **two thresholds** for the main dynamical variable z and the **total value of z is conserved and equal to zero**.

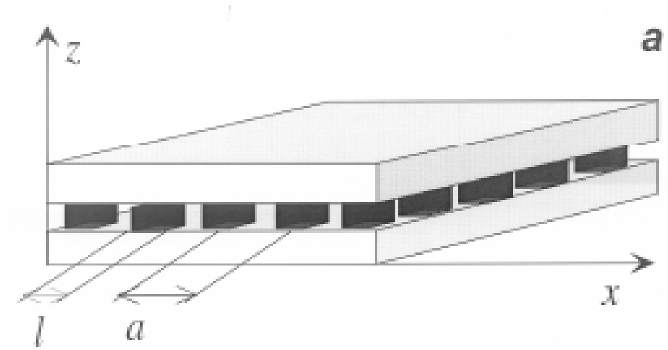
The question is:

Does such a system demonstrate the self-organized behavior or not?

PHYSICAL ANALOG

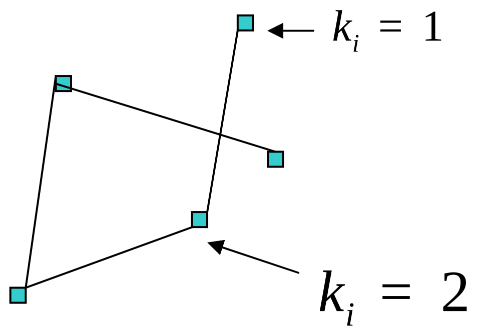
The physical prototype of our system is discrete superconductor (SQUID) placed in external magnetic field.

1. The junction current in superconductor has two threshold values: positive and negative ones.
2. It is clear that the total current induced by an external magnetic field is equal to zero. However the formation of sets of junctions with positive and negative currents is not forbidden.
3. Positive and negative currents can annihilate with each other.
4. For discrete superconductor the closed boundary conditions are natural.



Scale-free network

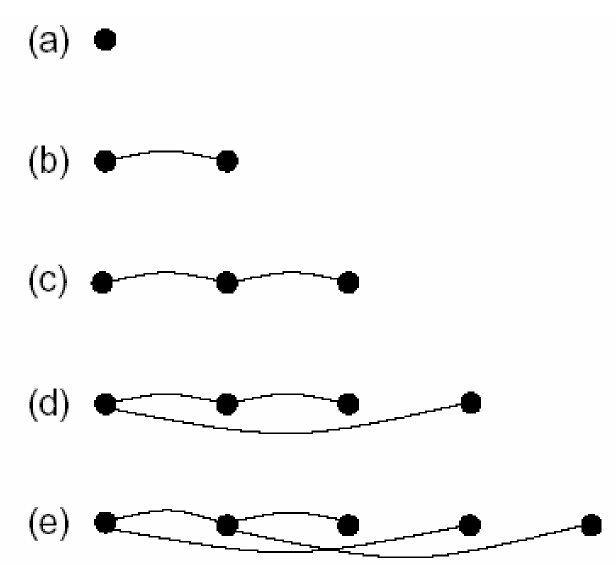
Network is a set of nodes connected by links.
k_i (degree) is a number of links for i-th node



CONSTRUCTION PROCESS

At each time step a new node attaches to one (or more) of existing nodes by m links.

ATTACHMENT RULES
What is a linear preferential attachment?



LINEAR PREFERENTIAL ATTACHMENT

Nodes for linking are chosen with probability proportional to a special function *f(k)*. Lets this function is linear, for example:

$$f(k,t) = k/(2t)$$

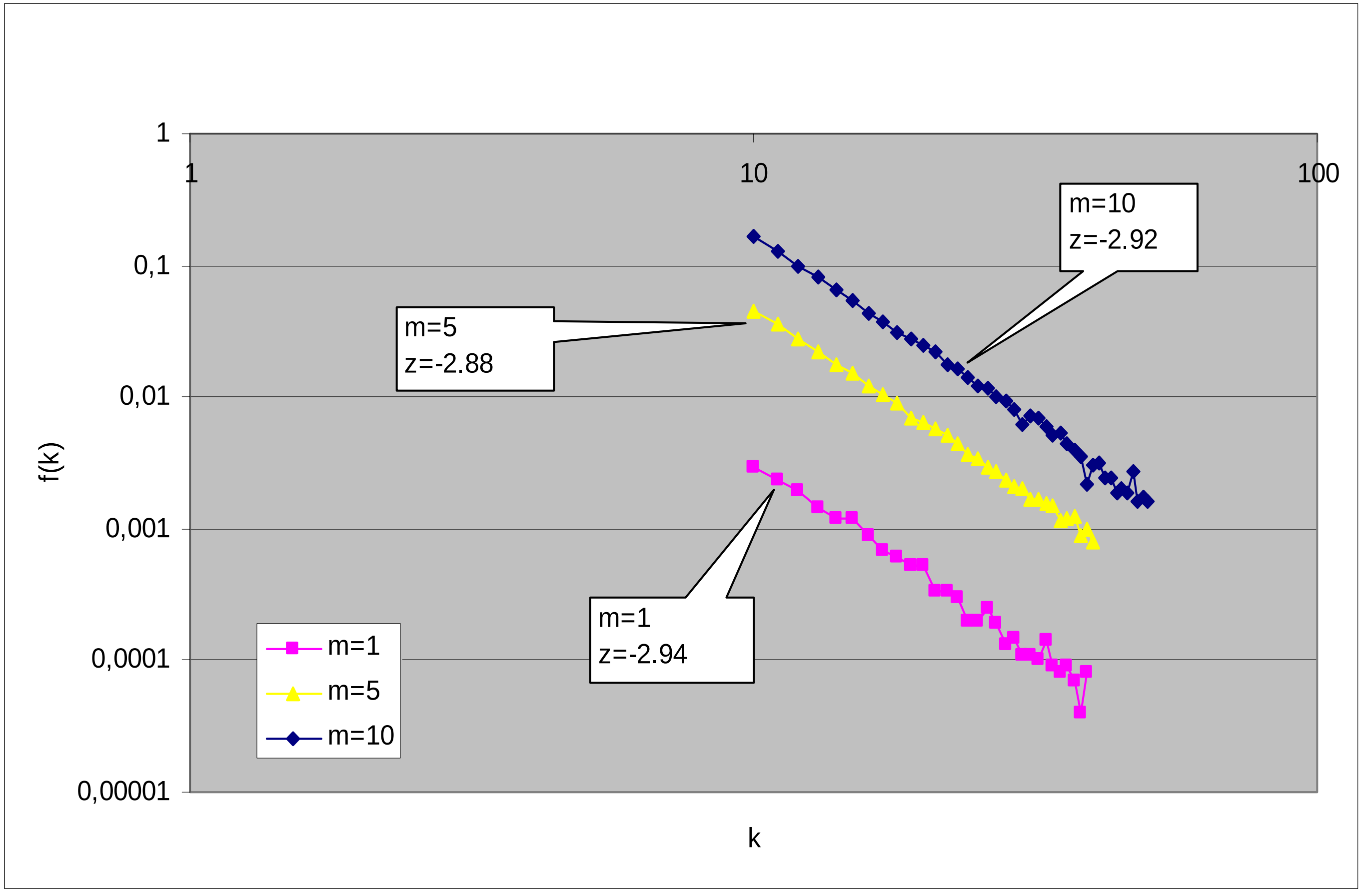
Such a type of attachment is linear preferential one (LPA).

If we use LPA that the probability for node to have k links is described by following formula:

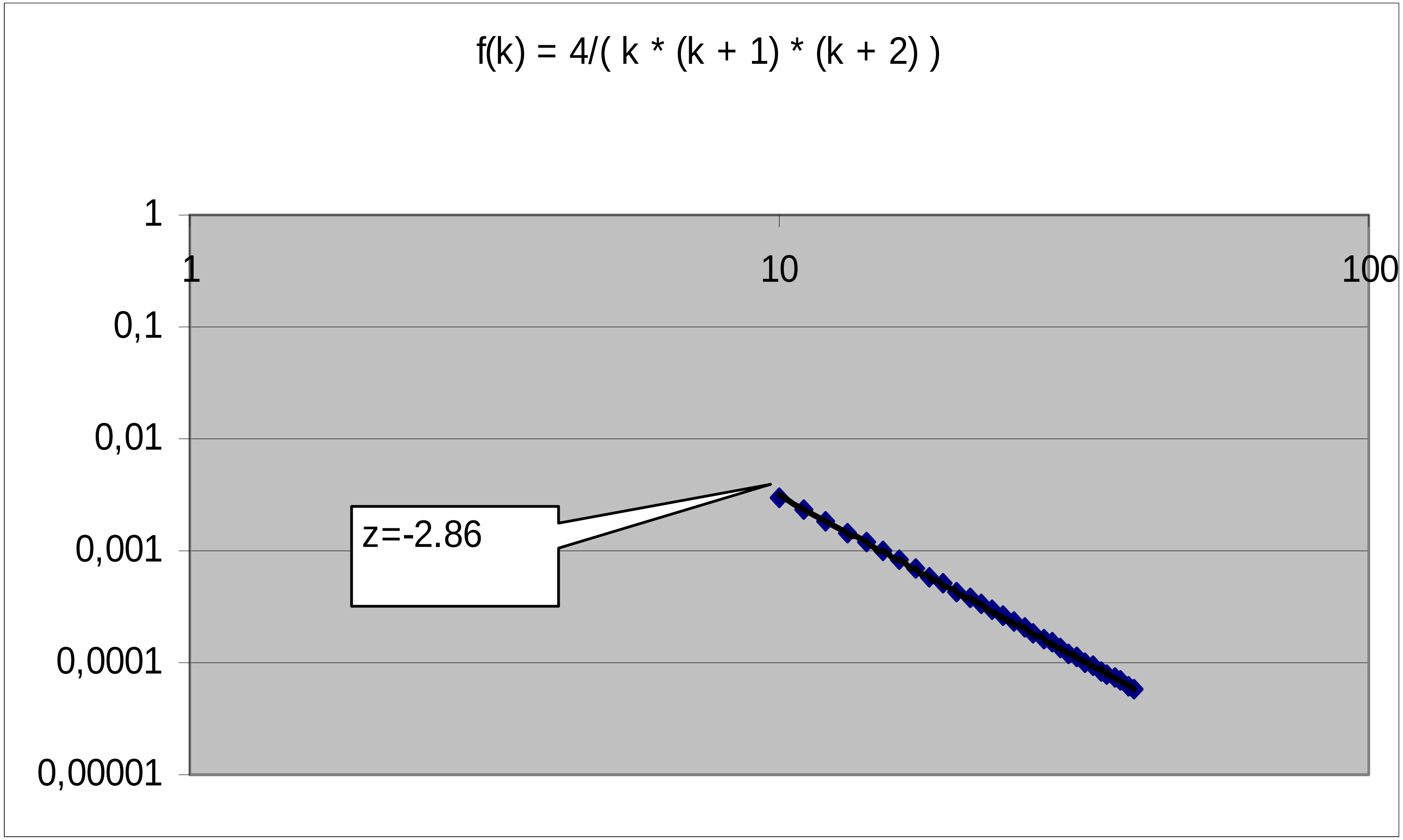
$$P(k) = \begin{cases} \dfrac{4}{k(k+1)(k+2)} & \text{-exact solution} \\[2ex] \dfrac{c}{k^3} & \text{-continuous approximation} \end{cases}$$

We see that scale-free networks are generated with linear preferential attachment.

PROBABILITY FUNCTION *P(k)*



PROBABILITY FUNCTION P(k)



COMPUTER SIMULATIONS. ALGORITHM

0. Using LPA we generate the network with size N=10 000 and with prefixed value of m (**m is number of links owned to node at the moment of its birth**). We numbered the nodes in order of theirs birth. Dynamics on the network is described by following rules:

$$z_n > z_c \Rightarrow z_n \to z_n - k_n$$

$$z_{\mathrm{i}} \to z_i + 1$$

$$z_n < -z_c \Rightarrow z_n \to z_n + k_n$$

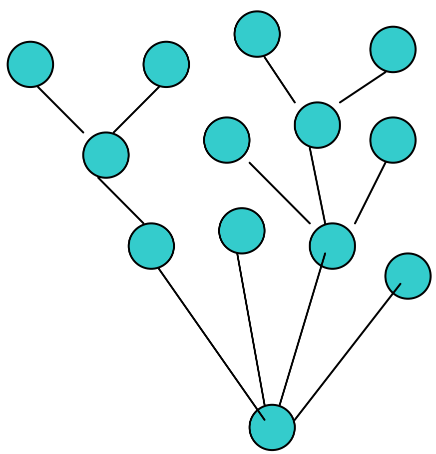
$$z_{\mathrm{i}} \to z_i - 1$$

Index *i* denotes nodes linked with n-th node . Kn is the number of nodes connected with n-th node.

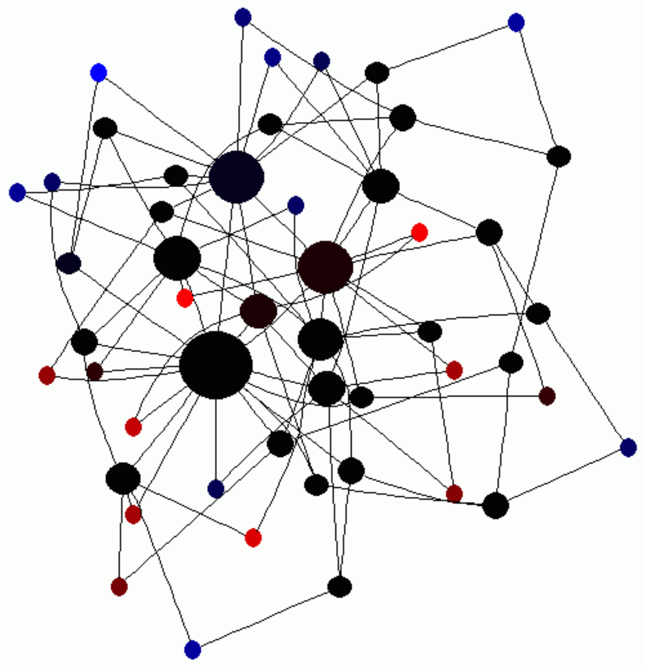
This algorithm is a result of reduction of equations describing a phases dynamics in discrete superconductors

RESULTS. NETWORK (ILLUSTRATION)

Fragments of networks with m=1 and m=2. In the case of m=1 we have “tree” without loops, and in the case of m>1 there are closed passes on the network.

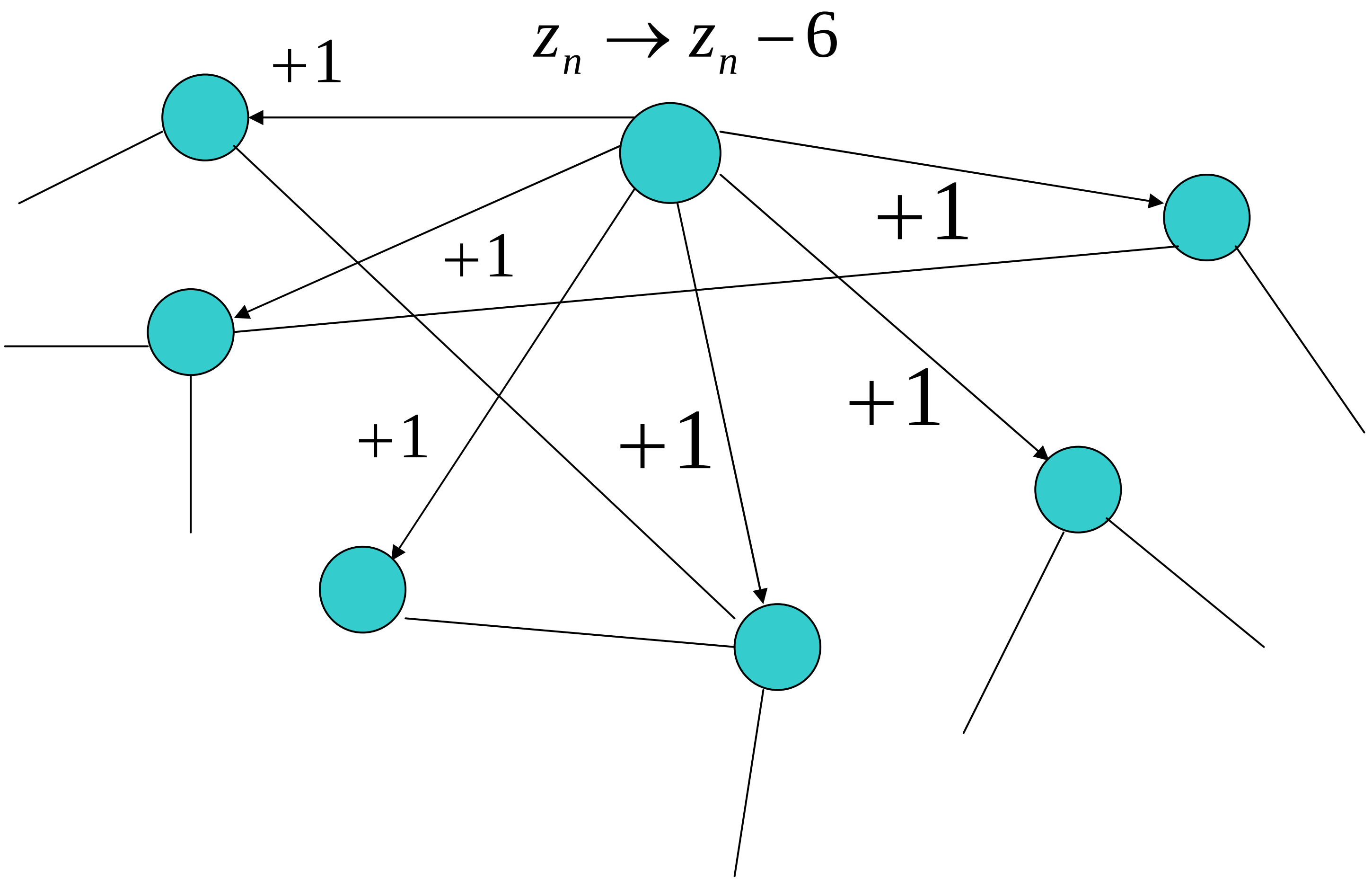


m=1



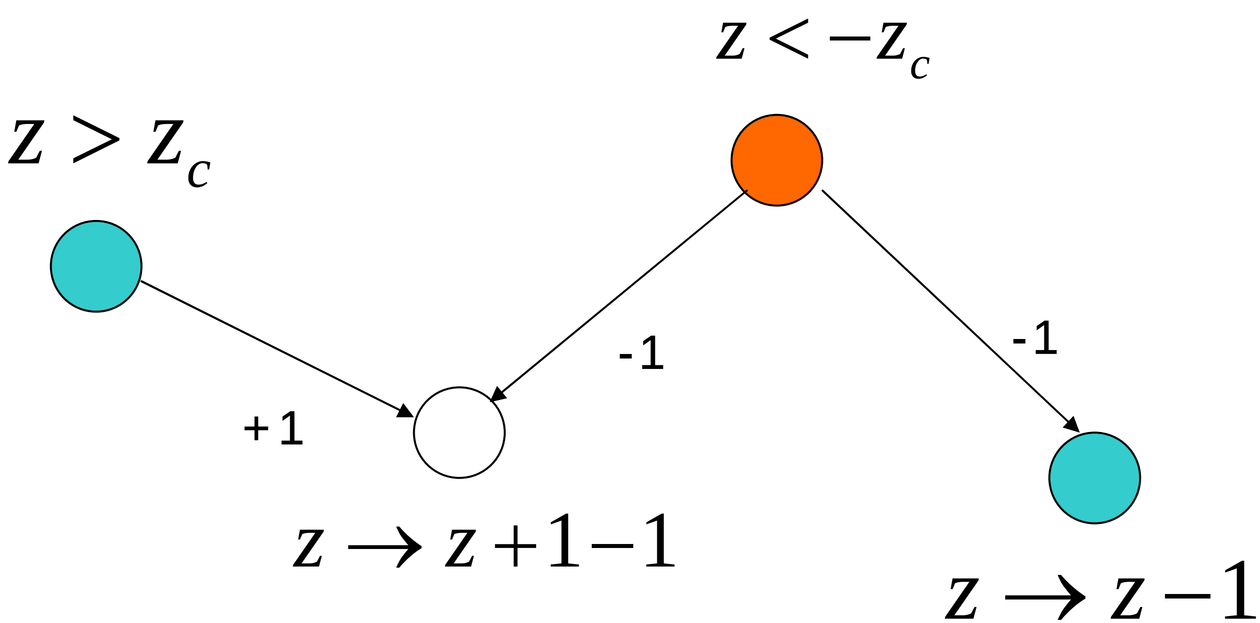
m=2

ALGORITHM. ILLUSTRATION



REDISTRIBUTION PROCESS

ALGORITHM. ILLUSTRATION



Two types of annihilation process:

1. Positive and negative units topple to the same node. The value of z on this node fluctuates around zero.
2. Positive or negative unit comes to node with large value of |z|. It leads to decreasing of |z|.

COMPUTER SIMULATION PROCESS

1. The system under consideration **is perturbed** at the boundary. We define **the boundary** of our network as a group of nodes where k=m (k is a node degree, m is a number of links owned to the node in the moment of its birth). We **divide** the boundary of network into **two subset**: "positive" and "negative". Before every perturbation we choose (randomly and independently) one node from each subset. We add $+\Delta h$ to the node from "positive" subset and $-\Delta h$ to the node from "negative" subset.

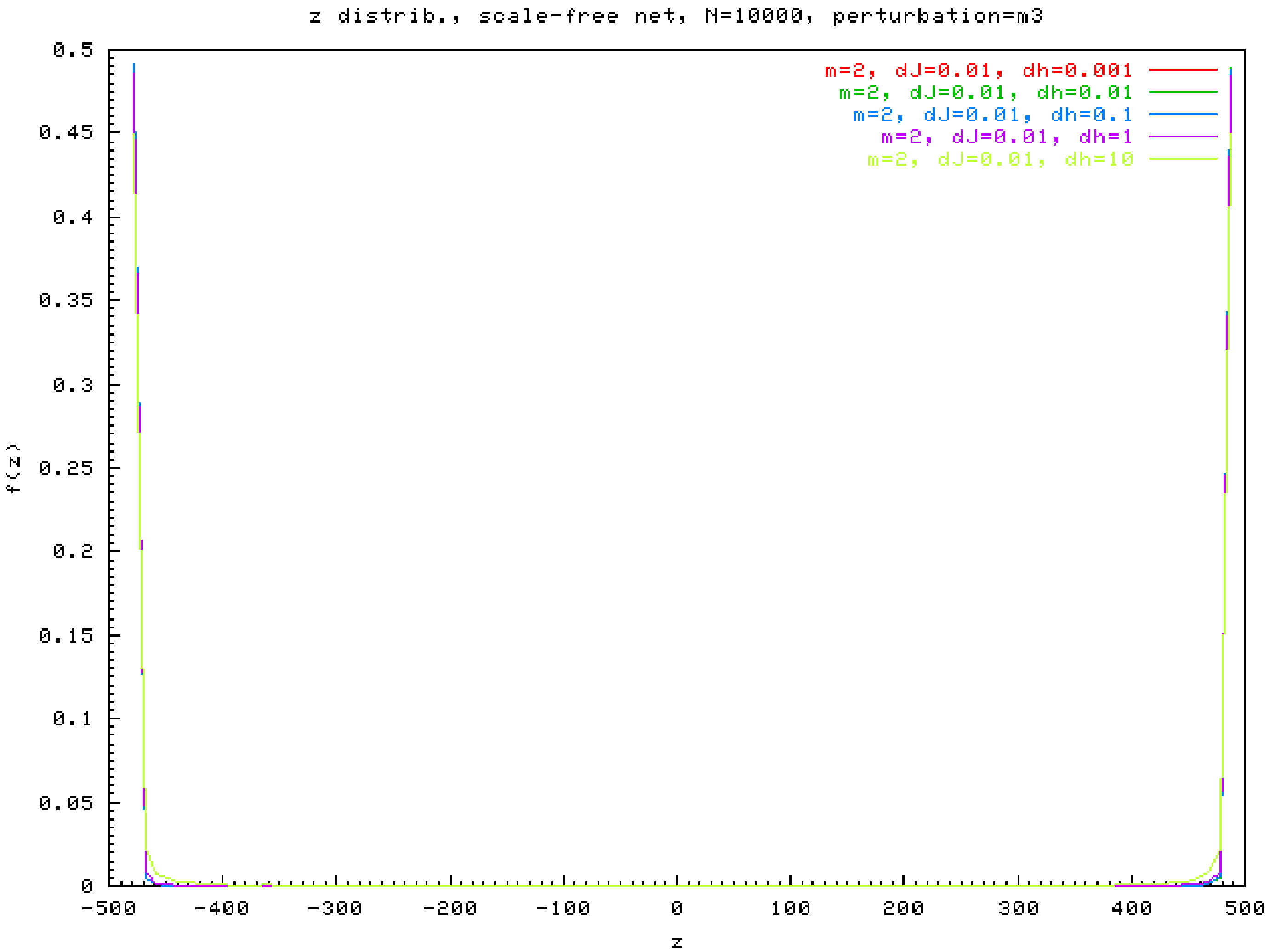
Note, that the fraction of boundary nodes is great enough. For example, if m=1 that the number of nodes m=k is 2/3 of total amount of network nodes.

2. After perturbation the system is allowed to relax to a stable state where all |z|<zc. When the dynamics stops we perturb the system again.

RESULTS. CRITICAL STATE

After transition period the system under consideration reaches a critical state. This state consists of a large number of metastable states that have the same structure. Each metastable state is **a collection of the steady sets of nodes where values of z are closed to positive or negative threshold.** During the system evolution the values of z fluctuate. The system comes from one metastable state to another by means of **avalanches** launched by perturbations. Positive and negative variables **can annihilate with each other and this process successfully substitutes the outlet process.**

CRITICAL STATE (illustration)



RESULTS. AVALANCHES IN THE SYSTEM

The system migrates from one metastable state to another by means of avalanches. During the avalanche the value of z on any node can change due to the toppling of this node or its neighbors. We calculate a size of an avalanche as a total number of toppling events during the relaxation process and normalize obtained quantity on size of the network (N=10000).

We study the distribution function of avalanche sizes for N=10000 and $\Delta h = 1$. Dependence of this function on the value of m is also investigated. We attempt to approximate obtained data by power-law function in order to understand if the self-organized criticality realizes in our system or not.

RESULTS. AVALANCHE STATISTICS

We consider the networks with m=1 and m=2. In these cases the distribution functions for avalanche sizes are studied. The data obtained can be approximated by function:

$$\rho(s) \sim s^{-\tau} \exp(-s / s_c)$$

Why we don’t observe a power-law distribution relevant to self-organized criticality? It can be explain by the fact that the annihilation process occurs on a large number of nodes. It was shown earlier that the amount of sites where the outlet process takes place affects on the form of distribution function for avalanche sizes. The increasing of this number leads to deviation of distribution function from power-law dependence. However we don’t consider this fact as a loss of self-organization. The systems remains critical and demonstrates an avalanche-like dynamics. We call such a state as a new type of self-organization.

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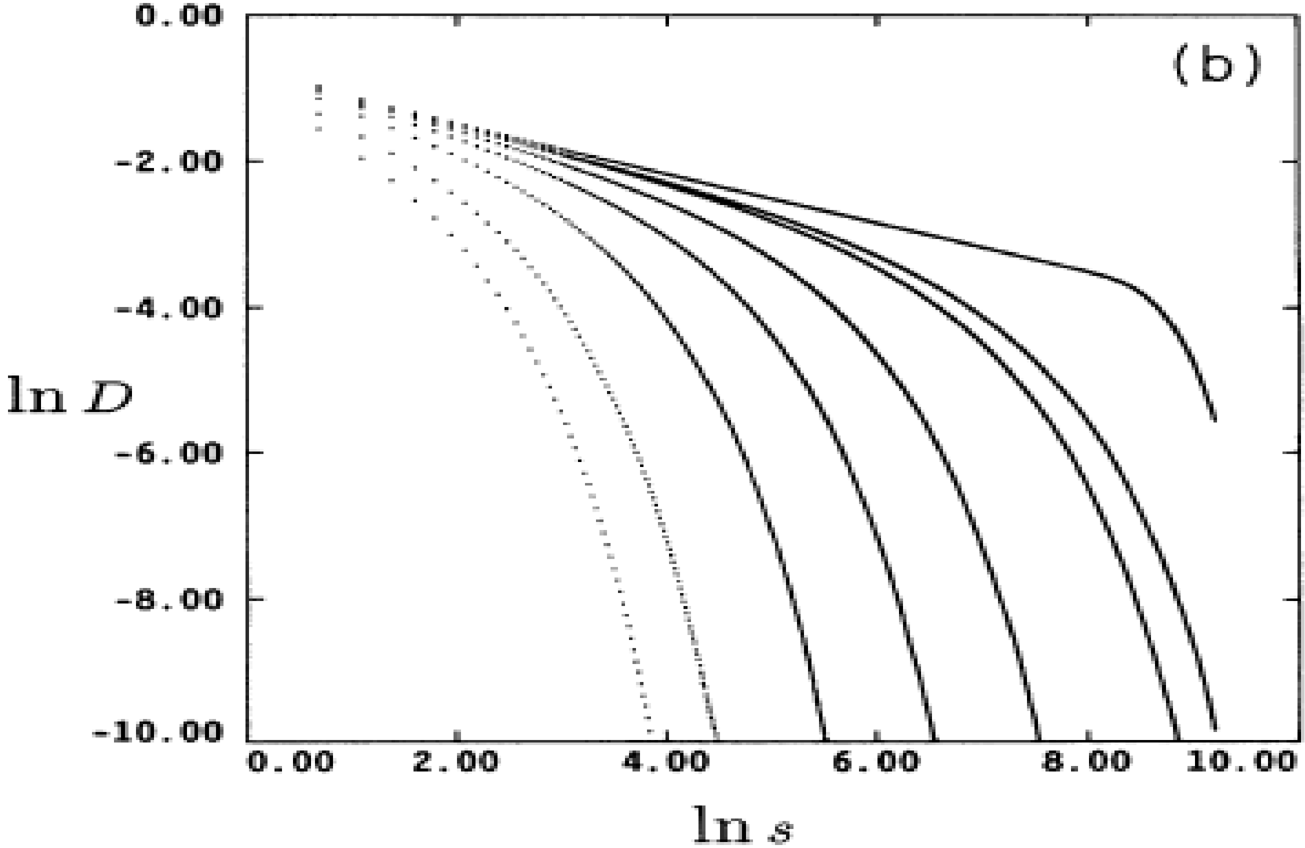
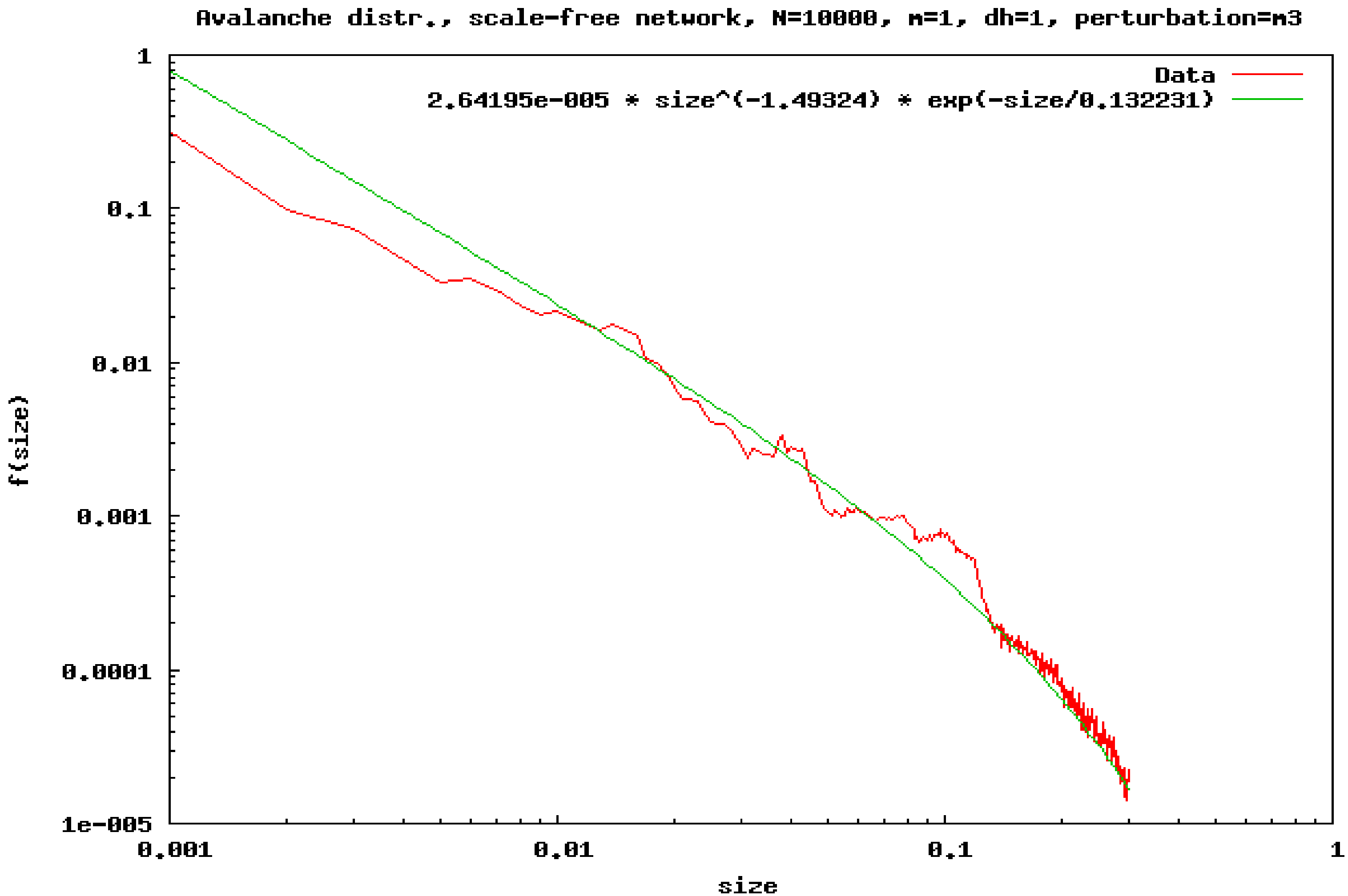
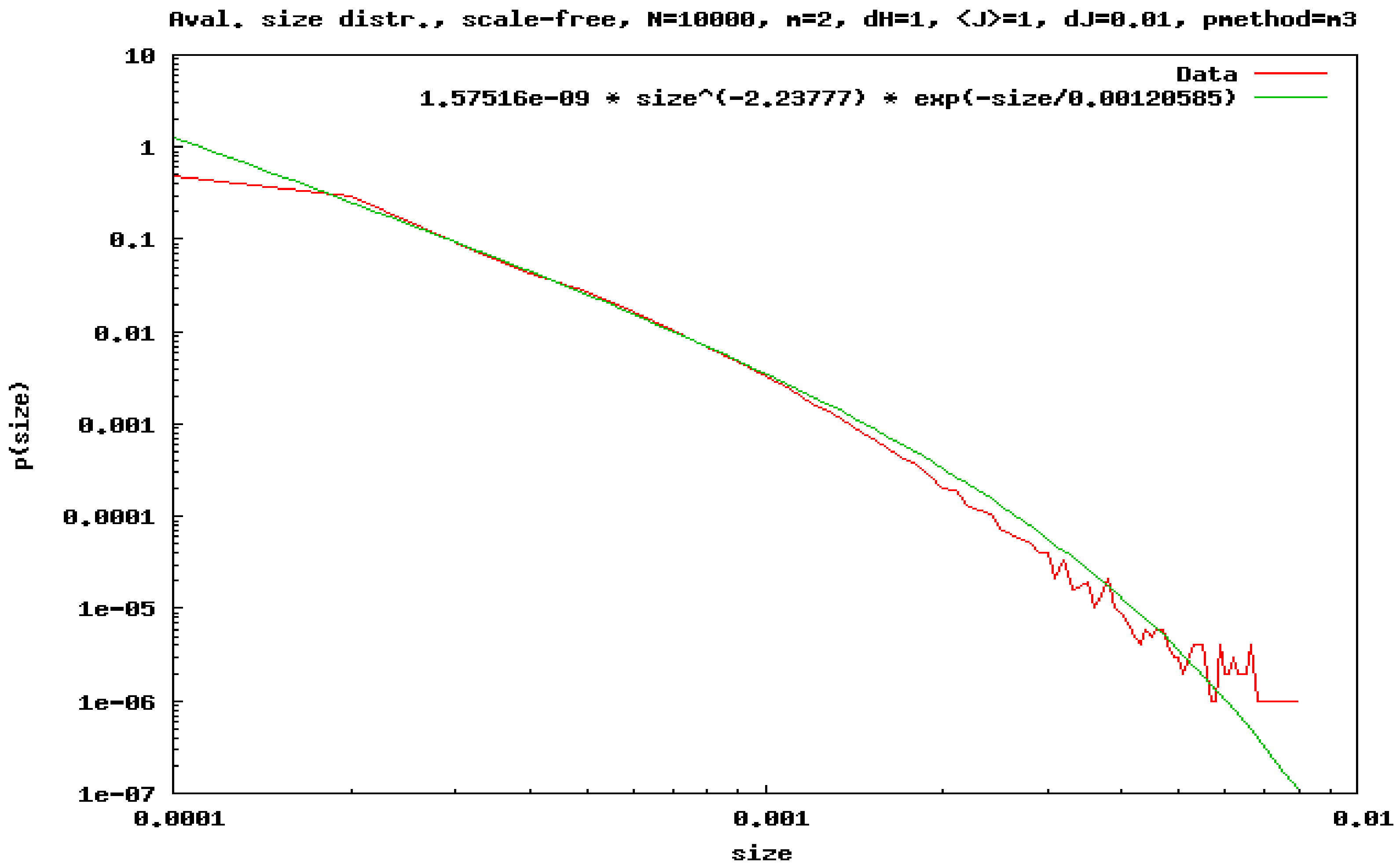


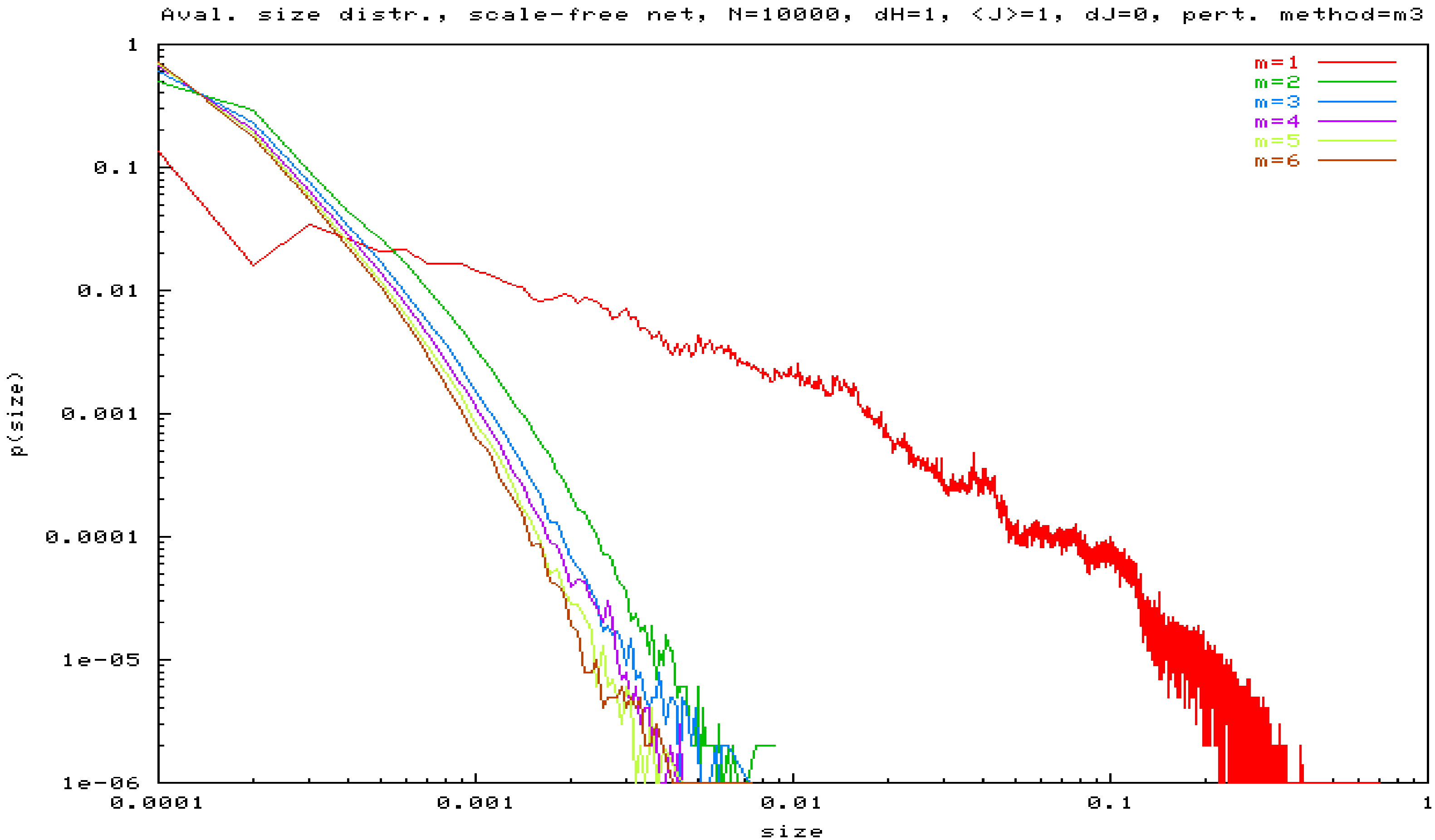
AVALANCHE DISTRIBUTION (illustration)



AVALANCHE DISTRIBUTION (illustration)



Distribution function depends on value of m



The shape of distribution function for m=1 differs from ones for other values of m. The reason is the absence of loops in the system for m=1.

RESUME:

1. Critical state in the system under consideration is the self-organized one in the following sense. It is a set of metastable states. The structures of all metastable states are the same. Each of them is a collection of sets of nodes where the values of node variables z are closed to positive or negative threshold. During its evolution the system migrates from one metastable state to another by means of avalanche process. During the avalanche the values of z fluctuate.
2. The distribution function for avalanche sizes can be approximated by following dependence:

$$\rho(s) \sim s^{-\tau} \exp(-s/s_c)$$

3.Avalanche distributions for networks with m=1 and m>1 are crucially different. It is a result of network structure.
In the case of m=1 we have a tree-like networks without loops, but for m>1 there are cycles on the network.