From logarithmic scale invariance to logarithmic Conformal Field Theory

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Renormalization Group & Related Topics

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Plan

1. Logarithmic scaling
   another (generalized) way to realize scale invariance
2. Logarithmic CFT
   field theoretic realization of this new form of scaling

Bottom line:

scaling becomes non-diagonalizable !

History

Logarithmic scaling is not really new: (about) first appearance in polymers and percolation (Saleur '92), but first systematic study of logCFT by Gurarie in '93.

Since then, they have played a prominent role in many topics:

- percolation
- polymers
- WZW models
- 2d turbulence
- disordered systems
- sandpile models
- spanning trees
- quantum Hall effect
- string theory
- dimer models
- logarithmic minimal models
- W-algebras ...

RG transformations

Depends on rescaling parameter b,

$$K' = \mathcal{R}_b(K) \qquad \text{with fixed point} \quad K^* = \mathcal{R}_b(K^*).$$

Linearization around K^* yields

$$K'_\alpha - K^*_\alpha = \sum_\beta \mathcal{L}_{\alpha\beta}\,(K_\beta - K^*_\beta).$$

If $\mathcal{L}$ is diagonalizable, we form scaling variables (eigenvectors)

$$u_i = \sum_\alpha c_i^\alpha\,(K_\alpha - K^*_\alpha) \quad \Longrightarrow \quad \boxed{u'_i = \lambda_i u_i} \quad \lambda_i = \lambda_i(b)$$

Semi-group property $\mathcal{L}(b)\mathcal{L}(b') = \mathcal{L}(bb')$ implies $\boxed{\lambda_i = b^{y_i}}$. The exponents are directly related to the critical exponents.

Scaling operators

Scaling operators couple to scaling variables

$$\mathcal{H} = \sum_i u_i \, \phi_i = \sum_i u_i \sum_{\vec{r}} \phi_i(\vec{r}) \sim \sum_i u_i \int \mathrm{d}\vec{r} \, \phi_i(\vec{r})$$

Invariance of $\mathcal{H}$ requires that under $r \to r' = r/b$, they transform as

$$\boxed{\phi_i(r) \xrightarrow{\text{RG}} \phi_i'(r') = b^{d-y_i} \, \phi_i(r/b)}$$

Set $x_i = d - y_i$ the scaling dimension of ϕ_i.

Scaling (and translation) invariance implies that correlators obey

$$\langle \phi_i(r_1) \, \phi_j(r_2) \rangle = b^{-x_i - x_j} \, \langle \phi_i(r_1/b) \, \phi_j(r_2/b) \rangle = \frac{a_{ij}}{|r_1 - r_2|^{x_i + x_j}}.$$

Power laws, (and more) well accounted for by ordinary CFTs.

Conformal symmetry

Note that previous correlator is also invariant under special conformal

$$r' = \frac{r + a\, r^2}{1 + 2a \cdot r + a^2\, r^2}\,,$$

provided

$$\phi_i(r) \longrightarrow \left|\frac{Dr'}{Dr}\right|^{x_i/d} \phi_i(r') = (1 + 2a \cdot r + a^2\, r^2)^{-x_i}\, \phi_i(r')\,.$$

With rotations, these transformations form the global conformal group $SO(d+1,1)$ (Euclidean).

In $d = 2$, this global invariance can be supplemented with local conformal covariance, leading to CFTs. Then ϕ above is **primary field.**

So far

Scale transformations $r \to r/b$ are 'diagonalized' :

- Scaling variables transform multiplicatively

$$u_i' = b^{y_i} \, u_i$$

- Conjugate operators transform homogeneously

$$\phi_i'(r') = b^{x_i} \, \phi_i(r/b)$$

- Assuming local conformal symmetry, higher correlators can be computed, and have algebraic singularities only.

Question is ...

What happens if linearized RG transformations

are no longer diagonalizable ??

Jordan ...

Natural to think of Jordan blocks for degenerate eigenvalues:

$$\text{canonical form is} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{for rank 2.}$$

Assume two scaling variables have same eigenvalue and transform non-diagonally (in the Jordan way).

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$$\begin{pmatrix} v' \\ u' \end{pmatrix} = \begin{pmatrix} b^y & * \\ 0 & b^y \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

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RG composition law requires $f(b) + f(b') = f(bb')$, namely

$$\boxed{f(b) = A \log b}$$

Logarithmic scaling

Non-diagonal scaling for two degenerate scaling variables takes the logarithmic form

$$\begin{pmatrix} v' \\ u' \end{pmatrix} = b^y \begin{pmatrix} 1 & A \log b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

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Hamiltonian containing operators coupling to u, v

$$\mathcal{H} = u \int \mathrm{d}r\, \phi(r) + v \int \mathrm{d}r\, \psi(r) + \ldots$$

is invariant provided

$$\begin{pmatrix} \psi' \\ \phi' \end{pmatrix} = b^x \begin{pmatrix} 1 & -A\log b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

Consequences on correlators

Scaling form for 2-pt correlators $\langle\phi\,\phi\rangle$, $\langle\phi\,\psi\rangle$, $\langle\psi\,\psi\rangle$?

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$$\langle\phi(r_1)\,\phi(r_2)\rangle = \frac{a}{|r_1 - r_2|^{2x}},$$

$$\begin{aligned}\langle\phi(r_1)\,\psi(r_2)\rangle &= b^{-2x}\Big\{\langle\phi(r_1/b)\,\psi(r_2/b)\rangle + \langle\phi(r_1/b)\,\phi(r_2/b)\rangle\,\log b\Big\}\\ &= b^{-2x}\,\langle\phi(r_1/b)\,\psi(r_2/b)\rangle + \frac{a}{|r_1 - r_2|^{2x}}\,\log b\end{aligned}$$

Consequences on correlators

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\begin{aligned}
\langle\phi(r_1)\,\phi(r_2)\rangle &= \frac{a}{|r_1 - r_2|^{2x}}, \\
\langle\phi(r_1)\,\psi(r_2)\rangle &= \frac{a' - a\log|r_1 - r_2|}{|r_1 - r_2|^{2x}}, \\
\langle\psi(r_1)\,\psi(r_2)\rangle &= \frac{a'' - 2a'\log|r_1 - r_2| + a\log^2|r_1 - r_2|}{|r_1 - r_2|^{2x}},
\end{aligned}
$$

Now contain logarithmic singularities !

Forms dictated by translation (L_{-1}) and scale invariance (L_0) only. Not conformally invariant yet ...

Under conformal transformations

If we assume invariance under special conformal transformations (L_1), it implies $a = 0$ in the previous formulas, which simplify to

$$\langle \phi(r_1)\,\phi(r_2)\rangle = 0 \qquad \longleftarrow \qquad \text{true for } n\text{-pt !!}$$

$$\langle \phi(r_1)\,\psi(r_2)\rangle = \frac{a'}{|r_1 - r_2|^{2x}}$$

$$\langle \psi(r_1)\,\psi(r_2)\rangle = \frac{a'' - 2a' \log|r_1 - r_2|}{|r_1 - r_2|^{2x}}$$

The $\log^2 r$ term disappears.

Generic 2-pt functions in LogCFT for pair of fields transforming in the Jordan way: **the fields (ϕ, ψ) make up a logarithmic pair**; ϕ is the primary field, ψ is the logarithmic partner of ϕ.

Higher rank

Easily generalized to higher rank cells. F.i. the rank 3 case

$$\begin{pmatrix} w' \\ v' \\ u' \end{pmatrix} = b^y \begin{pmatrix} 1 & A_1 \log b & \frac{A_1 A_2}{2} \log^2 b + A_3 \log b \\ 0 & 1 & A_2 \log b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ v \\ u \end{pmatrix}$$

involves $\log b$ and $\log^2 b$ terms.

In general, rank r Jordan cells lead to

- $\log b$ terms to maximal power $r-1$ in RG transformations,
- $\log |r_1 - r_2|$ terms to maximal power $r-1$ in 2-pt functions,
- n-pt correlator of primary partner $\langle \phi(r_1) \ldots \phi(r_n) \rangle = 0$.

Summary

Usually (no log)

Diagonal scaling $\Leftrightarrow$ RG transformations diagonalizable

Homogeneous transfos for scaling parameters u_i (and b^{y_i})

Tensorial transfos for scaling operators $\longrightarrow$ power laws

Realized by CFTs in the continuum (local scale inv/cov in $d = 2$)

Jordan cells

Logarithmic scaling $\Leftrightarrow$ RG no longer diagonalizable

Inhomogeneous transfos for scaling parameters u_i with $\log b$ factors

Inhomogeneous transfos for scaling operators $\rightarrow$ power laws $+$ logs

Principles of local scale inv $\longrightarrow$ LogCFTs (more complicated)

Example

Simplest and most studied LogCFT.

$$S = \frac{1}{\pi} \int \partial\theta \bar{\partial}\tilde{\theta} \qquad \text{(symplectic fermions)}$$

- θ and $\tilde{\theta}$ are scalar, anticomm. fields, with canonical dimension 0 $\longrightarrow$ four fields $\mathbb{I}$, θ, $\tilde{\theta}$, $\omega =: \tilde{\theta}\theta :$ of dimension 0, two are bosonic
- Wick contraction $\underbrace{\theta(z,\bar{z})\,\tilde{\theta}}(w,\bar{w}) = -\log|z-w|$
- stress-energy tensor $T(z) = -2 :\partial\theta\,\partial\tilde{\theta}:$
- Virasoro algebra has central charge $c = -2$
- may be thought of as minimal model $(p,p') = (1,2)$, $c = 1 - \frac{6(p-p')^2}{pp'}$

Jordan cell

The identity $\mathbb{I}$ and $\omega = :\theta\tilde{\theta}:$ form a logarithmic pair with $x = 0$.

From OPE $T(z)\omega(w)$, one finds, under infinitesimal dilation,

$$L_0\mathbb{I} = 0\,, \qquad L_0\omega = \mathbb{I}$$

Likewise, $\phi = \partial\bar{\partial}(\tilde{\theta}\theta)$ and $\psi = \tilde{\theta}\theta\,\partial\bar{\partial}(\tilde{\theta}\theta)$ form another logarithmic pair with $x = 2$

$$L_0\phi = \phi\,, \qquad L_0\psi = \psi + \phi$$

Each of these pairs ($+$ many more) generates an **indecomposable representation** of the Virasoro algebra (because of Jordan cell).

Correlators ?

Can we understand the structure of 2-pt functions ?

$$\langle \phi(r_1)\, \phi(r_2) \rangle = 0, \qquad \longleftarrow \qquad \text{true for } n\text{-pt !!}$$

$$\langle \phi(r_1)\, \psi(r_2) \rangle = \frac{a'}{|r_1 - r_2|^{2x}}, \qquad \langle \psi(r_1)\, \psi(r_2) \rangle = \frac{a'' - 2a' \log|r_1 - r_2|}{|r_1 - r_2|^{2x}}$$

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Because of zero modes of $\theta, \tilde{\theta}$ (remember $\int \mathrm{d}\theta_0 = 0$)

$$\langle \, \mathbb{I} \, \rangle = 0 \qquad [\text{and also } \langle \phi(1)\phi(2)\ldots \rangle = 0 \text{ for } \phi = \partial\bar{\partial}(\tilde{\theta}\theta)]$$

However since $\int \mathrm{d}\theta_0 \, \theta_0 = 1$, one has

$$\langle \omega(z) \rangle = \langle \tilde{\theta}\theta \rangle = 1, \qquad \langle \omega(z)\omega(w) \rangle = -2 \log |z - w|.$$

Exactly match above formulae for $x = 0$!

CFTs vs Log CFTs

Usual features of rational CFTs:

1. finite number of Virasoro representations
2. Vir representations are highest weight, completely reducible
3. Vir representations mainly identified by a conformal weight (L_0 diagonalizable)
4. conformal weights are bounded below
5. full, non-chiral theory basically reduces to chiral parts
6. correlation functions only have algebraic singularities
7. finite fusion (or quasi-rational)
8. chiral characters transform linearly under modular group of torus

CFTs vs Log CFTs

Typical features of Log CFTs:

1. finite number of Virasoro representations NO
2. Vir representations are highest weight, completely reducible NO
3. Vir representations mainly identified by a conformal weight NO (L_0 diagonalizable)
4. conformal weights are bounded below YES
5. full, non-chiral theory basically reduces to chiral parts NO
6. correlation functions only have algebraic singularities NO, Log^k
7. finite fusion (or quasi-rational) YES
8. chiral characters transform linearly under modular group NO

Recent developments

- Many highly non-trivial checks of Log CFT in sandpile model (Jeng, Grigorev, Mahieu, Moghimi-Araghi, Poghosyan, Priezzhev, Piroux, Rajabpour, Rouhani, PR, ... 2001-2008)
- Infinite series of lattice models: logarithmic extension of minimal models (p, p'); log Ising model, ... (Pearce, Rasmussen, Zuber 2006)
- percolation might involve rank 3 Jordan cells (Rasmussen & Pearce 2007); see Saleur & Read 2007, Mathieu & Ridout 2007 for alternatives.
- abstract Log CFTs: check Flohr, Feigin, Fuchs, Gaberdiel, Gainutdinov, Kausch, Runkel, Semikhatov, Tipunin, ... 2003-2008

and yet, many open questions ...