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ON THE INFRARED BEHAVIOR OF THE ADLER FUNCTION

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ANALYTIC APPROACH TO QCD

The relevant dispersion relation provides definite analytic properties in a kinematic variable of a quantity in hand.

BASIC IDEA

perturbation theory + RG method + analytic properties in Q^2

QED: ***Redmond, Uretsky (1958); Bogoliubov, Logunov, Shirkov (1959).***

QCD: ***D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett. 79, 1209 (1997).***

Advantages:

- **no unphysical singularities**
- **mild scheme dependence**
- **no free parameters**
- **higher loop stability**

ADLER FUNCTION

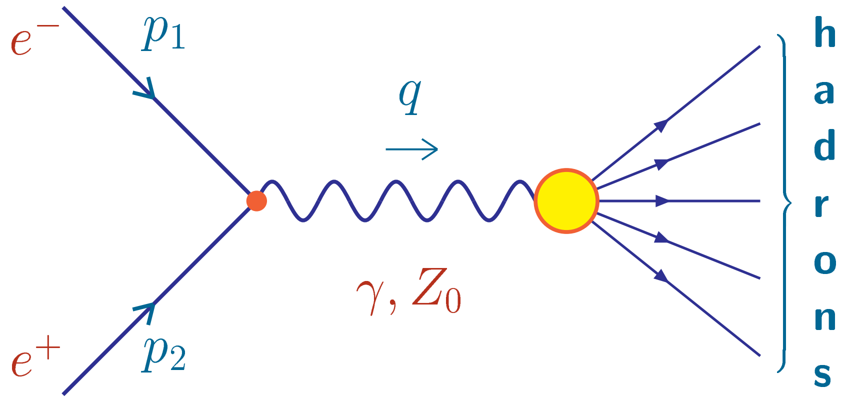
Hadronic vacuum polarization function $\Pi(q^2)$ plays a crucial role in various issues of elementary particle physics. Indeed, the theoretical description of some strong interaction processes and hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

- electron–positron annihilation into hadrons
- hadronic τ lepton decay
- muon anomalous magnetic moment
- running of the electromagnetic coupling

The cross–section of e^+e^- annihilation into hadrons reads

$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0$,



$$L_{\mu\nu} = \frac{1}{2}\Big[q_\mu q_\nu - g_{\mu\nu} q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\Big],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma) \langle 0|J_\mu(-q)|\Gamma\rangle \langle \Gamma|J_\nu(q)|0\rangle,$$

Γ **denotes a final hadron state, and** $J_\mu = \sum_f Q_f :\bar{q}\,\gamma_\mu\, q:$ **stands for the electromagnetic quark current.**

It is worth stressing that $\Delta_{\mu\nu}(q^2)$ **exists only for** $q^2 \geq 4m_\pi^2$**, since otherwise no hadron state** Γ **could be excited:**

- ***R.P. Feynman (1972); S.L. Adler, PRD10 (1974).***

The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

$$\Pi_{\mu\nu}(q^2) = i\int e^{iqx}\langle 0\big|\, T\big\{J_\mu(x)\, J_\nu(0)\big\}\big|0\rangle\, d^4x = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2).$$

The hadronic vacuum polarization function $\Pi(q^2)$ **satisfies the once–subtracted dispersion relation (cut for** $q^2 \geq 4m_\pi^2$**)**

$$\Pi(q^2) = \Pi(q_0^2) - \left(q^2 - q_0^2\right)\int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s-q^2)(s-q_0^2)}\, ds,$$

where $m_\pi = 135\,\mathrm{MeV}$ **is the mass of the** π **meson and** $R(s)$ **denotes the measurable ratio of two cross–sections:**

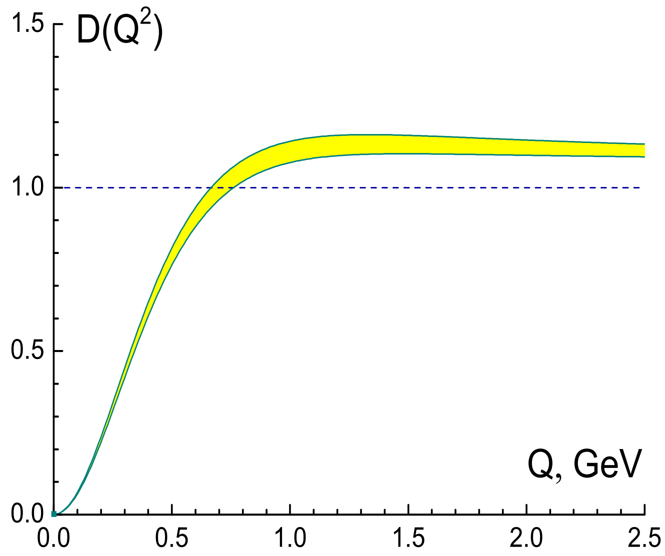
$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s-i\varepsilon) - \Pi(s+i\varepsilon)\Big] = \frac{\sigma\left(e^+e^- \to \mathrm{hadrons}; s\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-; s\right)}.$$

It is worth noting here that $R(s) \equiv 0$ **for** $s < 4m_\pi^2$ **because of the kinematic restrictions mentioned above:**

- ***R.P. Feynman (1972).***

For practical purposes it is convenient to deal with the Adler function

$$D(Q^2) = \frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad Q^2 = -q^2 \geq 0,$$



which satisfies the dispersion relation

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

and plays an indispensable role for the congruous processing of the timelike and spacelike experimental data:

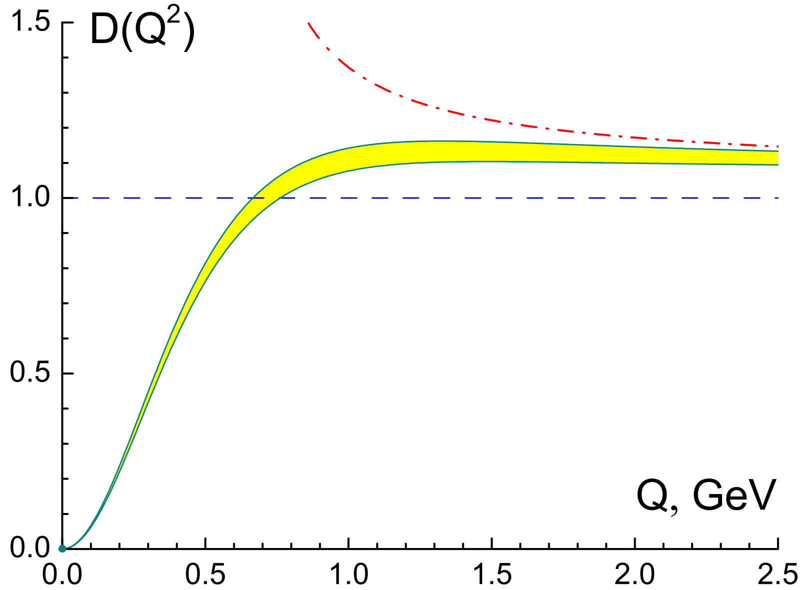
- ***S.L. Adler (1974); A. De Rujula, H. Georgi, PRD13 (1976); J.D. Bjorken (1989).***

The inverse relation between $D(Q^2)$ and $R(s)$ reads

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}$$

- ***A.V. Radyushkin (1982); N.V. Krasnikov, A.A. Pivovarov, PLB116 (1982).***

On the one hand, perturbation theory provides an explicit expression for the Adler function valid at high energies (an overall factor $N_c \sum_f Q_f^2$ is omitted throughout):



$$D^{(\ell)}_{\text{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[\alpha_s^{(\ell)}(Q^2)\right]^j, \qquad Q^2 \to \infty.$$

On the other hand, this perturbative approximation is inconsistent with the dispersion relation for $D(Q^2)$ due to unphysical singularities of the strong running coupling $\alpha_s(Q^2)$:

$$D^{(1)}_{\text{pert}}(Q^2) = 1 + d_1\, \alpha_s^{(1)}(Q^2), \qquad \alpha_s^{(1)}(Q^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln(Q^2/\Lambda^2)},$$

where $d_1 = 1/\pi$ and $\beta_0 = 11 - 2n_f/3$.

Dispersion relation imposes stringent constraints on $D(Q^2)$:

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

- **Since $R(s)$ assumes finite values and $R(s) \to \text{const}$ when $s \to \infty$, then $D(Q^2) = 0$ at $Q^2 = 0$ (holds for $m_\pi \neq 0$ only)**

- **Adler function possesses the only cut $Q^2 \leq -4m_\pi^2$ along the negative semiaxis of real Q^2**

PRIMARY OBJECTIVE: **to merge these nonperturbative constraints with perturbative result for the Adler function.**

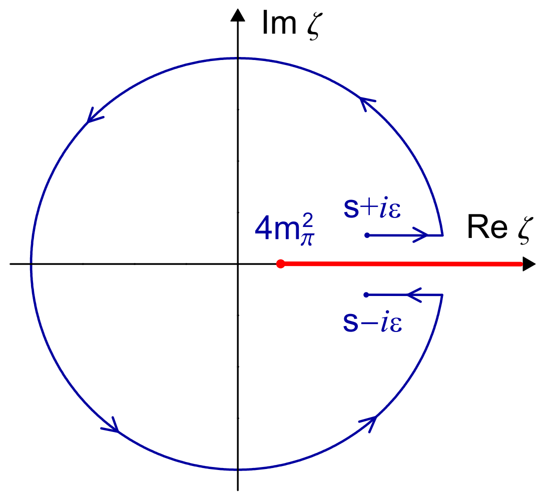
NEW INTEGRAL REPRESENTATION FOR $D(Q^2)$

This objective can be achieved by deriving the integral representations for the Adler function and $R(s)$–ratio, which involve the common spectral function.

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

$$\updownarrow$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}$$



Parton model prediction + kinematic restriction on $R(s)$:

$$R_0(s) = \theta(s - 4m_\pi^2) \quad \longleftrightarrow \quad D_0(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2}$$

■ ***R.P. Feynman (1972).***

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2} + d(Q^2)$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}$$

$$R(s) = \theta(s - 4m_\pi^2) \left[1 + \int_s^\infty \rho_{\mathbf{D}}(\sigma) \frac{d\sigma}{\sigma}\right]$$

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^\infty \frac{R(s)}{(s+Q^2)^2} ds$$

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2} \left[1 + \int_{4m_\pi^2}^\infty \rho_{\mathbf{D}}(\sigma) \frac{\sigma - 4m_\pi^2}{\sigma + Q^2} \frac{d\sigma}{\sigma}\right]$$

$$\rho_{\mathbf{D}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[D(-\sigma - i\varepsilon) - D(-\sigma + i\varepsilon)\Big] = -\frac{d\, R(\sigma)}{d\, \ln \sigma}$$

- ***A.V. Nesterenko, J. Papavassiliou, JPG32 (2006).***

Thus one arrives at the following integral representations:

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2}\left[1 + \int_{4m_\pi^2}^{\infty} \rho_{\mathbf{D}}(\sigma)\, \frac{\sigma - 4m_\pi^2}{\sigma + Q^2}\, \frac{d\sigma}{\sigma}\right]$$

$$R(s) = \theta(s - 4m_\pi^2)\left[1 + \int_{s}^{\infty} \rho_{\mathbf{D}}(\sigma) \frac{d\sigma}{\sigma}\right]$$

$$\rho_{\mathbf{D}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left[D_{\mathbf{exact}}(-\sigma - i\varepsilon) - D_{\mathbf{exact}}(-\sigma + i\varepsilon)\right] = -\frac{d\, R(\sigma)}{d\, \ln \sigma}$$

■ ***A.V. Nesterenko, J. Papavassiliou, JPG32 (2006).***

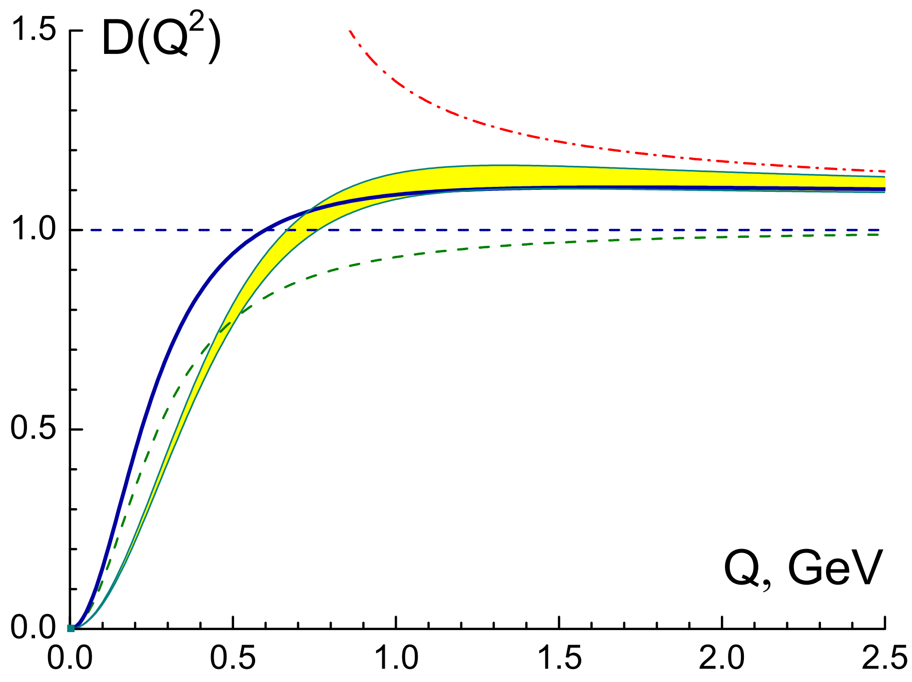
- **nonperturbative constraints on $D(Q^2)$ are satisfied**
- **congruent analysis of spacelike and timelike processes**

In the limit $m_\pi = 0$ the obtained expressions become identical to those of the Analytic perturbation theory:

■ ***D.V. Shirkov, I.L. Solovtsov, PRL79 (1997); PLB442 (1998); TMP150 (2007).***

There is no unique way to compute the corresponding spectral density by making use of perturbative $D_{\mathbf{pert}}(Q^2)$. In what follows the one-loop spectral function is adopted:

$$\rho^{(1)}(\sigma) = \left(1 + \frac{1}{\sigma}\right) \frac{1}{\ln^2 \sigma + \pi^2}$$



■ *A.V. Nesterenko, PRD62 (2000); PRD64 (2001).*

ADVANTAGES :

- **unphysical perturbative singularities are eliminated**
- **additional parameters are not introduced**
- **reasonable agreement with $D_{\mathbf{exp}}(Q^2)$ for all energies**

INCLUSIVE τ LEPTON DECAY

The inclusive semileptonic branching ratio:

$$R_\tau = \frac{\Gamma(\tau^- \to \mathrm{hadrons}^- \, \nu_\tau)}{\Gamma(\tau^- \to e^- \, \bar{\nu}_e \, \nu_\tau)} = R_{\tau,\mathrm{V}} + R_{\tau,\mathrm{A}} + R_{\tau,\mathrm{S}}.$$

Its nonstrange part associated with vector quark currents:

$$R_{\tau,\mathrm{V}} = \frac{N_\mathrm{c}}{2} \, |V_\mathrm{ud}|^2 S_\mathrm{EW} \left(\Delta_\mathrm{QCD} + \delta'_\mathrm{EW}\right) = 1.764 \pm 0.016$$

- *OPAL Collaboration, EPJC7 (1999).*

In this equation $N_\mathrm{c} = 3$, $|V_\mathrm{ud}| = 0.9738 \pm 0.0005$, $\delta'_\mathrm{EW} = 0.0010$, $S_\mathrm{EW} = 1.0194 \pm 0.0050$, $M_\tau = 1.777\,\mathrm{GeV}$, **and**

$$\Delta_\mathrm{QCD} = 2 \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\,\frac{s}{M_\tau^2}\right) R(s) \, \frac{ds}{M_\tau^2} \, .$$

Perturbative approach:

$$\Delta_{\mathbf{QCD}} = 1 + d_1\, \alpha_{\mathbf{s}}^{(1)}(M_\tau^2) \quad \longrightarrow \quad \Lambda = (678 \pm 55)\,\text{MeV}, \qquad n_{\text{f}} = 2$$

- *E. Braaten, S. Narison, A. Pich, NPB373 (1992).*

Current analysis:

$$\Delta_{\mathbf{QCD}} = 1 + d_1 \alpha_{\text{TL}}^{(1)}(M_\tau^2) - \delta_\Gamma + \frac{4}{\beta_0}\int_\chi^1 f(\xi)\rho^{(1)}\big(\xi M_\tau^2\big)d\xi - d_1 \delta_\Gamma \alpha_{\text{TL}}^{(1)}(m_\Gamma^2),$$

$$f(\xi) = \xi^3 - 2\xi^2 + 2, \qquad \chi = \frac{m_\Gamma^2}{M_\tau^2}, \qquad \delta_\Gamma = \chi\, f(\chi) \simeq 0.048, \qquad d_1 = \frac{1}{\pi},$$

$$\alpha_{\text{TL}}^{(1)}(s) = \frac{4\pi}{\beta_0}\theta(s - m_\Gamma^2)\int_s^\infty \rho^{(1)}(\sigma)\,\frac{d\sigma}{\sigma}, \qquad m_\Gamma = m_{\pi^0} + m_{\pi^-}$$

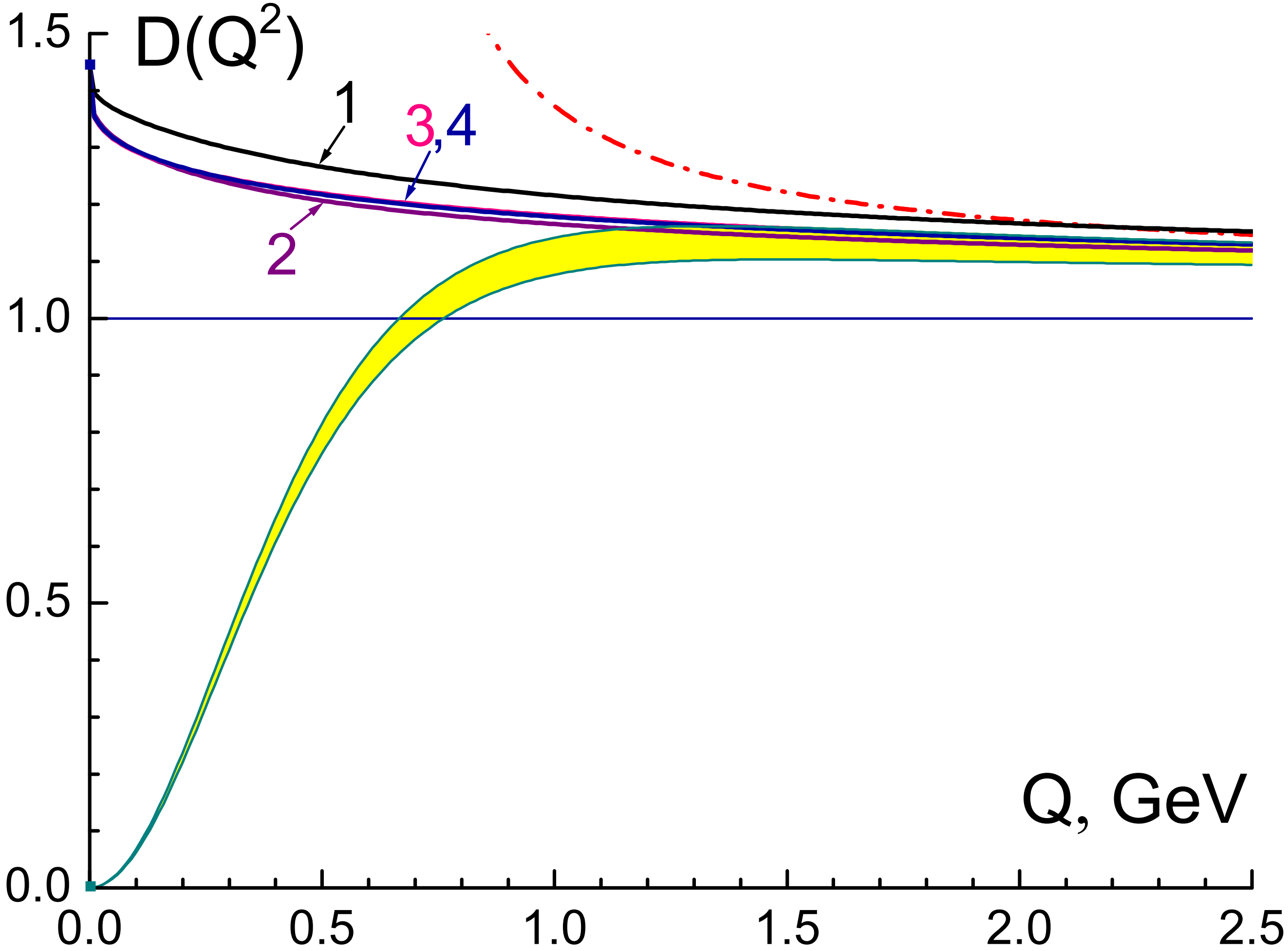
- **massive case:** $\Lambda = (941 \pm 86)\,\text{MeV}$
- **massless limit:** $\Lambda = (493 \pm 56)\,\text{MeV}$

SUMMARY

- **New integral representations for the Adler function and $R(s)$–ratio are derived**

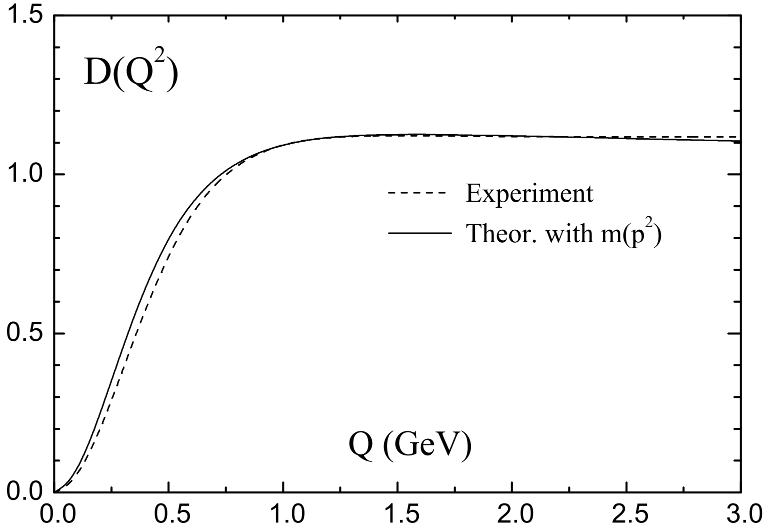
- **These representations possess appealing features:**
  - **unphysical perturbative singularities are eliminated**
  - **additional parameters are not introduced**
  - **the π^2–terms are automatically taken into account**
  - **reasonable description of $D(Q^2)$ in entire energy range**

- **The effects due to the pion mass play a substantial role in processing the data on the inclusive τ lepton decay**



$$\rho^{(\ell)}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[D^{(\ell)}_{\mathbf{pert}}(-\sigma - i\varepsilon) - D^{(\ell)}_{\mathbf{pert}}(-\sigma + i\varepsilon) \Big], \qquad m_\pi = 0$$

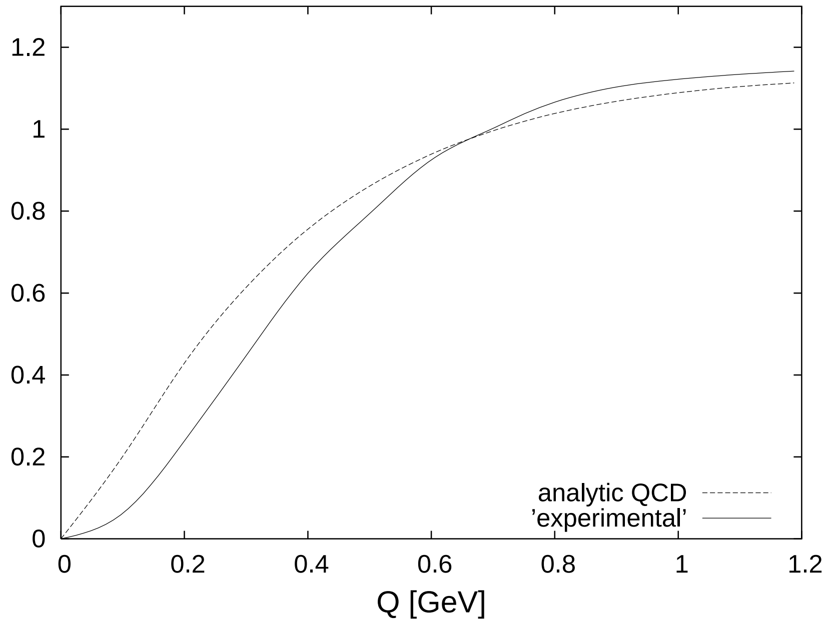
APT + relativistic quark mass threshold resummation:



[taken from MPLA21 (2006)]

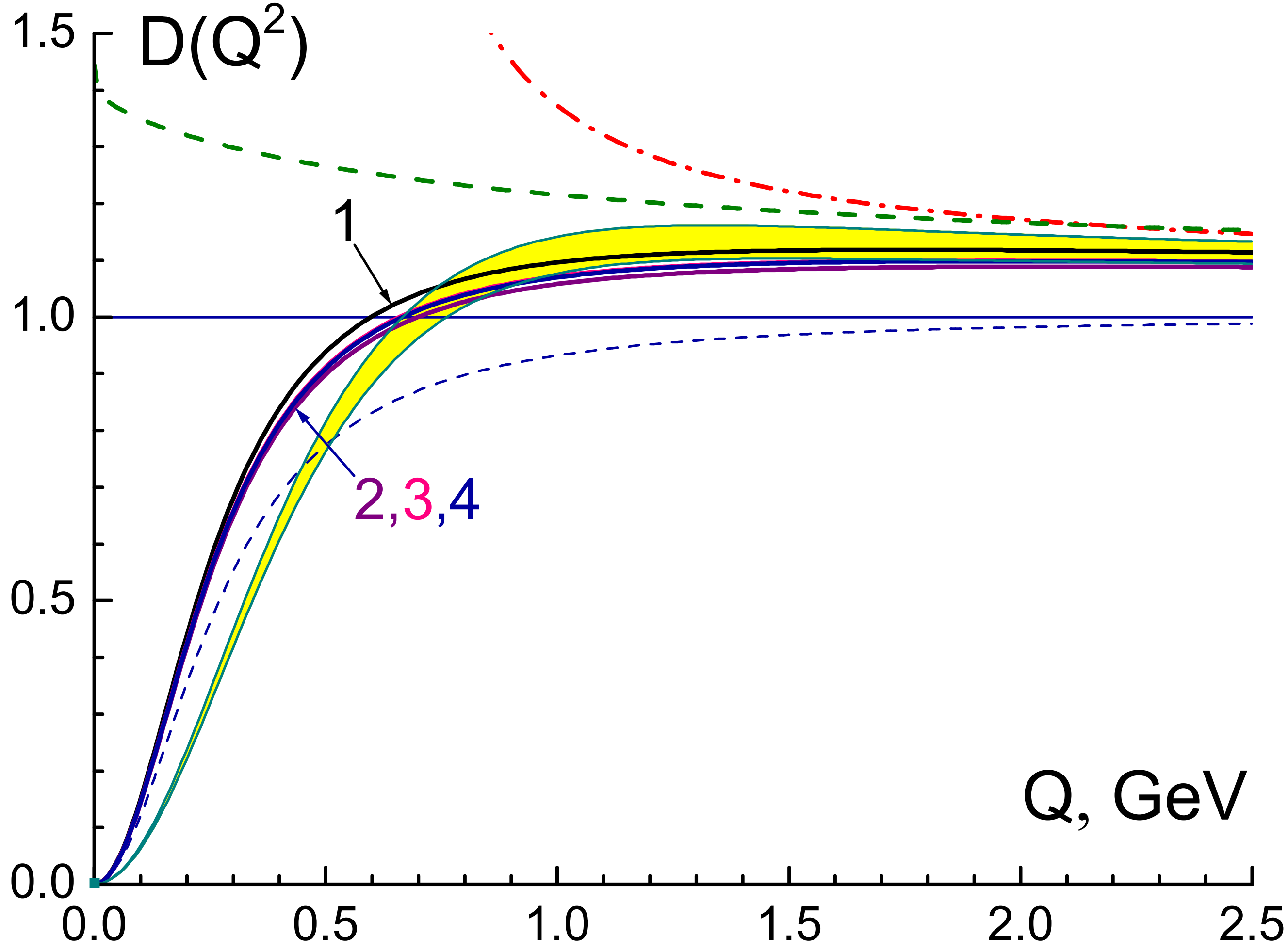
K.A. Milton, I.L. Solovtsov, O.P. Solovtsova (2001)–(2006)

APT + vector meson dominance assumption:



[taken from NPBPS164 (2007)]

G. Cvetic, C. Valenzuela, I. Schmidt (2005)–(2007)



$$\rho^{(\ell)}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[D^{(\ell)}_{\mathbf{pert}}(-\sigma - i\varepsilon) - D^{(\ell)}_{\mathbf{pert}}(-\sigma + i\varepsilon) \Big], \qquad m_\pi \neq 0$$