Instanton analysis in Kraichnan model with a frozen velocity field

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Introduction

- QFT $\rightarrow$ perturbation series (asymptotic)
- Usually only a few first terms of the series are known analytically
- Large order asymptotics (LOA) of perturbation series
- Borel-Leroy : some first terms + LOA $\rightarrow$ result
- Instanton analysis (Lipatov, 1977)
- Kraichnan model

$$\langle \mathbf{V}_i(\mathbf{x}_1, t_1)\mathbf{V}_j(\mathbf{x}_2, t_2)\rangle = \delta(t_2 - t_1) D_{ij}(\mathbf{x}_2 - \mathbf{x}_1)$$

Common statement and Kraichnan model

A wrong common statement

It is sufficiently to know the quantity of diagrams in the current order of perturbation series to get rough estimate of large order asymptotics

- Kraichnan model with a frozen velocity field

$$\langle \mathbf{V}_i(\mathbf{x}_1, t_1)\mathbf{V}_j(\mathbf{x}_2, t_2)\rangle = D_{ij}(\mathbf{x}_2 - \mathbf{x}_1)$$

- Exactly solvable case (for controlling results)

$$\langle \mathbf{V}_i(\mathbf{x}_1, t_1)\mathbf{V}_j(\mathbf{x}_2, t_2)\rangle = const$$

Kraichnan model with a frozen velocity field

- Stochastic equation

$$(\partial_t + g\nabla_i V_i - \nu\Delta)\varphi(x) = \xi(x), \qquad x \equiv \{t, \mathbf{x} \in \mathbb{R}^d\}$$

- The velocity field correlator in a momentum representation (Honkonen, 1988)

$$D^F_{ij}(\mathbf{q}) \equiv \lambda_T \left(\delta_{jk} - \frac{q_j q_k}{q^2}\right)\frac{1}{q^{2\alpha}} + \lambda_L \frac{q_j q_k}{q^2}\frac{1}{q^{2\alpha}}$$

- Response function is an object under consideration

$$G(t_2 - t_1, \mathbf{x}_2 - \mathbf{x}_1) = \left\langle \frac{\delta\varphi(\xi(t_2, \mathbf{x}_2))}{\delta\xi(t_1, \mathbf{x}_1)} \right\rangle_V$$

Exactly solvable model

- Stochastic equation

$$(\partial_t + g\nabla_i V_i - \nu\Delta)\varphi(x) = \xi(x)$$

- Velocity field correlator

$$\langle \mathbf{V}_i(\mathbf{x}_1, t_1)\mathbf{V}_j(\mathbf{x}_2, t_2)\rangle = const$$

- The Green function

$$G(t_2 - t_1, \mathbf{x}_2 - \mathbf{x}_1) = \frac{\Theta(T)(2\nu T)^{d/2}}{\sqrt{2\nu T + Dg^2 T^2}^{\,d}} e^{-\frac{\mathbf{x}^2}{4\nu T + 2Dg^2 T^2}}$$

Exactly solvable model

- The expansion of response function in g

$$G = \sum_{N=0}^{\infty} g^N G^{[N]}$$

- The N-th order of the Green function expansion

$$G^{[N]}(t_2 - t_1, \mathbf{x}_2 - \mathbf{x}_1) = \frac{\Theta(t-t')}{\sqrt{\pi}} \exp(1 - d/2) \exp\left(-\frac{\mathbf{x}^2}{8\nu T}\right) \times \left(\frac{\mathbf{x}^2}{4\nu T}\right)^{(1-d)/4} N^{(d-3)/4} \left(-\frac{DT}{2\nu}\right)^N \cos\left(\sqrt{\frac{N\mathbf{x}^2}{\nu T}} + \pi \frac{1-d}{4}\right)$$

Exactly solvable model

Conclusion

- The quantity of diagrams $\uparrow N!$
  BUT
  asymptotics has a power form

Martin-Siggia-Rose-formalism

- The response function

$$G(x_2 - x_1) = \int \mathcal{D}\mathbf{V}\mathcal{D}\varphi\mathcal{D}\varphi' \;\; \varphi(x_1)\varphi'(x_2)\exp(-S^{msr})$$

- The action in representation of QFT

$$S^{msr} = \frac{\varphi' D_\xi \varphi'}{2} + \frac{\mathbf{V}_i D^{-1}_{ij} \mathbf{V}_j}{2} + \varphi' \Big[\partial_t + g\mathbf{V}_i \nabla_i - \nu\Delta \Big] \varphi,$$

where φ' is an auxiliary field

Lagrangian variables

- There is no instanton in the context of MSR-formalism (E. Balkovsky et al., 1998; M.Nalimov et al., 2006)
- Lagrangian variables

$$\varphi(t,\mathbf{x}), \varphi'(t,\mathbf{x}) \to \mathbf{c}(\tau), \mathbf{c}'(\tau)$$

Representation of "fluid particles": $\mathbf{c}(\tau), \mathbf{c}'(\tau)$ are the coordinate and the momentum

Lagrangian variables

- Lagrangian variables

$$G(T,\mathbf{x}) = \Theta(T) \int\limits_{\mathbf{c}(0)=0}^{\mathbf{c}(T)=\mathbf{x}} \mathcal{D}\mathbf{c}\mathcal{D}\mathbf{c}'\mathcal{D}\mathbf{V}\exp(S^L), \qquad \begin{matrix} T \equiv t - t', \\ \mathbf{x} \equiv \mathbf{x}_2 - \mathbf{x}_1, \end{matrix}$$

where $$S^L = -\nu\mathbf{c}'^2 + i\mathbf{c}'[\partial_\tau \mathbf{c} - g\mathbf{V}(\mathbf{c})] - \frac{1}{2}\mathbf{V}_i D^{-1}_{ij}\mathbf{V}_j$$

- The action after the integration over the velocity field

$$\overline{S}^L(u) = -\nu\mathbf{c}'^2 + i\mathbf{c}'\partial_\tau\mathbf{c} - \frac{u}{2}\mathbf{c}'_i(\tau)D_{ij}(\mathbf{c}(\tau) - \mathbf{c}(\tau'))\mathbf{c}'_j(\tau'),$$

where $u \equiv g^2$

N-th order term extraction

- Cauchy formula

$$G(u) = \sum_{N=0}^{\infty} G^{[N]} u^N, \qquad G^{[N]} = \frac{1}{2\pi i} \oint_\gamma \frac{G(u)}{u^{N+1}} du,$$

$\gamma \ni 0$ is a closed contour in a complex plane

Instanton analysis

- Initial action

$$G^{[N]} \sim \oint \frac{du}{u} \int\limits_{\mathbf{c}(0)=0}^{\mathbf{c}(T)=\mathbf{x}} \mathcal{D}\mathbf{c}\mathcal{D}\mathbf{c}' \exp(\overline{S}^L - N \ln u)$$

- Scaling

$$G^{[N]} \sim \oint \frac{du}{u} \int\limits_{\bar{\mathbf{c}}(0)=0}^{\bar{\mathbf{c}}(T)=0} \mathcal{D}\mathbf{c}\mathcal{D}\mathbf{c}' e^{NS}, \qquad \mathbf{c}(\tau) = \bar{\mathbf{c}}(\tau) + \mathbf{x}\frac{\tau}{T},$$

$$S = -\nu\mathbf{c}'^2 + i\mathbf{c}'\partial_\tau\mathbf{c} - \frac{u}{2}\mathbf{c}'_i D_{ij}\mathbf{c}'_j - \ln u$$

Stationary equations and response function

- Stationary equations

$$\frac{\delta S}{\delta c} = 0, \frac{\delta S}{\delta c'} = 0, \frac{\partial S}{\partial u} = 0 \rightarrow$$

$$c_{st}, c'_{st}, u_{st} - \text{the main contribution in } G^{[N]}, \textit{ instanton}$$

- The response function large order asymptotics

$$G^{[N]} \sim \exp(NS(c_{st}, c'_{st}, u_{st}))\Phi(\Delta),$$

where $\Phi(\Delta)$ is a result of gaussian integration over fluctuations near the instanton point

An assumption

$\mathbf{c}$, $\mathbf{c}'$, $\mathbf{x}$ are collinear because of the initial symmetry of the model is brokendown by $\mathbf{x}$ only

$$D_{ij} \to D(\mathbf{x}) = \frac{D_0}{|\mathbf{x}|^{2\beta}}$$

The stationary equations system

$$\frac{\delta S}{\delta c'(\tau)} = 0 \quad \Rightarrow \quad -2\nu c'(\tau) + i\partial_\tau c(\tau) = u\varkappa(c(\tau))$$

$$\frac{\delta S}{\delta c(\tau)} = 0 \quad \Rightarrow \quad -i\partial_\tau c'(\tau) = uc'(\tau)\frac{\partial \varkappa(c(\tau))}{\partial c}$$

$$\frac{\delta S}{\delta u} = 0 \quad \Rightarrow \quad u = -\frac{2}{\int\limits_0^T d\tau c'(\tau)\varkappa(c(\tau))}$$

$$\varkappa_i(\mathbf{c}(\tau)) \equiv \int\limits_0^T d\tau' D_{ij}(\mathbf{c}(\tau) - \mathbf{c}(\tau'))\mathbf{c}'_j(\tau')$$

The conservation law

- The stationary equations system is a system of integro-differential equations
- The conservation law

$$-\nu c'^2 + ic'\partial_\tau c = iF, \qquad \text{where } F \text{ is an integral of motion}$$

- Different $F \Leftrightarrow$ different initial conditions on $\mathbf{c}$, that is different $\mathbf{x}$

Large order asymptotics for $G^{[N]}$ when $F = 0$

- Let $F = 0$

$$\mathbf{c}'(\tau) = \frac{i\partial_\tau \mathbf{c}(\tau)}{\nu} \qquad \text{– the conservation law}$$

$$\int_0^c \frac{dy}{\int_0^{|\mathbf{x}|} dz D(z-y)} = -\frac{u_{st}\tau}{\nu}, \qquad c \equiv c_{st}(\tau)$$

- $\mathbf{x}_0 \equiv \mathbf{x}\big|_{F=0}$

$$\mathbf{x}_0^2 = \frac{2(\beta-1)}{R(\beta)}\nu T, \qquad R(\beta) = \int_0^1 \frac{dh}{(1-h)^{(1-2\beta)} + h^{(1-2\beta)}}$$

Large order asymptotics for $G^{[N]}$ when $F = 0$

$$G^{[N]} \sim N^{N\beta} \frac{\exp(N)}{u_{st}^N}$$

Conclusion

So we constructed the analytical representation for large order asymptotics for response function of the Kraichnan model with a frozen velocity field when $\mathbf{x} = \mathbf{x}_0$

Instanton analysis in the exactly solvable model

- Instanton equations ($\mathbf{V} = const$)

$$\bar{\mathbf{c}} \equiv 0, \qquad c'_{st} = \pm\sqrt{\frac{N}{\nu T}} + \frac{i|\mathbf{x}|}{4\nu T} + O(N^{-1/2})$$

$$u_{st} = -\frac{2\nu}{DT} \pm \frac{i|\mathbf{x}|\sqrt{\nu T}}{DT^2}\frac{1}{\sqrt{N}} + \frac{|\mathbf{x}|^2}{4DT^2}\frac{1}{N} + O(N^{-3/2})$$

- The response function large order asymptotics

$$G^{[N]} \sim \Theta(T)\left(-\frac{DT}{2\nu}\right)^N N^{\frac{d-3}{4}} \cos\left(\sqrt{\frac{N\mathbf{x}^2}{\nu T}} + \pi\frac{1-d}{4}\right)$$

Summary

- The instanton family for the Kraichnan model with a frozen velocity field is found and one of them is represented. The corresponding perturbation theory series have finite or sometimes infinite radius of convergence.
- The properties of the series for the considered model may be used in resummation procedures to improve numerical results in the perturbation theory.
- The common statement that it is sufficient to determine the quantity of diagrams in a current order of perturbation series to get rough estimate of large order asymptotics is disproved.
- The exactly solvable model confirms our results.

Thank you.