# Turbulent Dynamo in Anisotropic Helical MHD

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RG-2008, Dubna

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# Introduction-Motivation

- The most interesting physical effect in helical MHD - turbulent dynamo
  - ⋆ H. K. Moffat, *Magnetic Field Generation in Electrically Conducting Fluids* (CUP, Cambridge, 1978)
  - ⋆ S. I. Vainstein, Ya. B. Zel'dovich, and A. A. Ruzmaikin, *The Turbulent Dynamo in Astrophysics* (Nauka, Moscow, 1980) [in Russian]
  - ⋆ S. I. Vainstein, *Magnetic Fields in Space*, (Nauka, Moscow, 1983) [in Russian]
- Why is the effect interesting and important?
  - ⋆ generation of magnetic fields in the Universe
  - ⋆ the origin of the magnetic field of the Earth

- A microscopic model of turbulent dynamo - field theory with spontaneous symmetry breaking
  - ⋆ the helical fluid with zero mean values of magnetic field ($\mathbf{B} = 0$) is unstable
  - ⋆ it is stabilized by the spontaneous appearance of the homogeneous magnetic field $\mathbf{B} \neq 0$
  - ⋆ random force for velocity field: L. Ts. Adzhemyan, A. N. Vasiliev, and M. Hnatich, Theor. Math. Phys. **72**, 940 (1987).
  - ⋆ general matrix of random forces: M. Hnatich, M. Jurcisin, and M. Stehlik, Magnetohydrodynamics **37**, 80 (2001).
  - ⋆ the absolute value $|\mathbf{B}|$ was found
- Open question: the direction of the generated magnetic field.
  - ⋆ the axis of the magnetic field of the Earth is near the direction of the axis of rotation
  - ⋆ rotation of the Earth: source of helicity as well as uniaxial anisotropy
- Helicity and the anisotropy of the kinetic energy pumping - absolute value and the direction of the generated magnetic field

# The Model



- The stochastic MHD equations:

$$\partial_t v_i = \nu_0 \Delta v_i - v_j \partial_j v_i + b_j \partial_j b_i + f_i, \quad (1)$$

$$\partial_t b_i = \nu_0 u_0 \Delta b_i - v_j \partial_j b_i + b_j \partial_j v_i + f_i^b, \quad (2)$$

- $v_i = v_i(t, \mathbf{x})$ - $i$-th component of the transverse (due to incompressibility) velocity field.

- $b_i = b_i(t, \mathbf{x})$ - $i$-th component of the transverse magnetic field: $b_i = \mathbb{B}_i/\sqrt{4\pi\rho}$ ($\mathbb{B}$ is magnetic induction, $\rho$ is constant density)

- $\nu_0$ - kinematic viscosity coefficient,

- $\nu_0 u_0 = c^2/(4\pi\sigma)$ - analog of viscosity: $\sigma$ is conductivity of the fluid, $c$ is speed of light.

- we suppose: $f_i^b$=0 and $\partial_i v_i = \partial_i b_i = \partial_i f_i = 0$

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$f_i \equiv f_i(x)$ - Gaussian random noise:

$$D_{ij}^f \equiv \langle f_i(x) f_j(x') \rangle = \delta(t-t') \int \frac{d\mathbf{k}}{(2\pi)^d} \mathbb{R}_{ij}(\mathbf{k}) d_f(k) e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}, \quad (3)$$

where

$$\mathbb{R}_{ij}(\mathbf{k}) = R_{ij}(\mathbf{k}) + H_{ij}(\mathbf{k}), \quad (4)$$

where

$$R_{ij}(\mathbf{k}) = \left(1 + \alpha_1 \frac{(\mathbf{n}\cdot\mathbf{k})^2}{k^2}\right) P_{ij} + \alpha_2 P_{is} n_s n_t P_{tj} \quad (5)$$

and

$$H_{ij}(\mathbf{k}) = i\rho\, \varepsilon_{ijl} k_l / k \quad (6)$$

with $P_{ij} = \delta_{ij} - k_i k_j / k^2$, $\rho$ ($|\rho| \in \langle 0, 1 \rangle$) - helicity parameter, and

$$d_f(k) = D_0 k^{4-d-2\varepsilon} \quad (7)$$

$\varepsilon = 0$ - logarithmic theory, $\varepsilon = 2$ - real value (gives Kolmogorov dimensions), $D_0 = g_0 \nu_0^3$ - amplitude factor

# Field Theoretic Formulation of The Model

- reformulation of the stochastic model into the equivalent field theoretic model with doubled set of fields $\Phi = \{\mathbf{v}, \mathbf{b}, \mathbf{v}', \mathbf{b}'\}$ yields

$$\begin{aligned} S(\Phi) &= \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \\ & v_i'(t_1,\mathbf{x_1}) D^f_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2}) v_j'(t_2,\mathbf{x_2}) \\ &+ \int dt\, d^d\mathbf{x}\; v_i' \left[-\partial_t v_i - v_j\partial_j v_i + b_j\partial_j b_i + \nu_0 \triangle v_i\right] \\ &+ \int dt\, d^d\mathbf{x}\; b_i' \left[-\partial_t b_i - v_j\partial_j b_i + b_j\partial_j v_i + \nu_0 u_0 \triangle b_i\right], \end{aligned} \tag{8}$$

where $D^f$ is a random force correlator (3).

- the stochastic averaging of random quantities is replaced with functional averages with weight $\exp(S(\Phi))$

- standard dimensional analysis $\longrightarrow$ UV divergent Green functions: $\langle v_i' v_j \rangle_{1-ir}$, $\langle b_i' b_j \rangle_{1-ir}$, and $\langle v_i' b_j b_l \rangle_{1-ir}$
  - ⋆ UV divergences are logarithmic (poles in $\varepsilon$) and linear (proportional to $\Lambda$)
  - ⋆ the action does not contain all needed structures to remove divergences multiplicatively
  - ⋆ linear divergences are present only in $\langle b_i' b_j \rangle_{1-ir}$
  - ⋆ needed counterterm for linear divergences: $\mathbf{b}' \cdot \mathrm{rot}\, \mathbf{b}$
- to have multiplicatively renormalized model in anisotropic case

$$
\begin{aligned}
S^A(\Phi) \;=\; & \int dt\, d^d\mathbf{x}\, \{ v_i' [\nu_0 \chi_{10} (\mathbf{n}\cdot\partial)^2 v_i + \nu_0 \chi_{20} n_i \triangle (\mathbf{n}\cdot\mathbf{v}) \\
+ \; & \nu_0 \chi_{30} n_i (\mathbf{n}\cdot\partial)^2 (\mathbf{n}\cdot\mathbf{v}) + \lambda_{10} b_i (\mathbf{n}\cdot\partial)(\mathbf{n}\cdot\mathbf{b}) \\
+ \; & \lambda_{20} n_i \mathbf{n}\cdot\partial) b^2 + \lambda_{30} n_i (\mathbf{b}\cdot\partial)(\mathbf{n}\cdot\mathbf{b}) \\
+ \; & \lambda_{40} n_i (\mathbf{n}\cdot\partial)(\mathbf{n}\cdot\mathbf{b})^2 ] \\
+ \; & b_i' [\nu_0 u_0 \tau_{10} (\mathbf{n}\cdot\partial)^2 b_i + \nu_0 u_0 \tau_{20} n_i \triangle (\mathbf{n}\cdot\mathbf{b}) \\
+ \; & \nu_0 u_0 \tau_{30} n_i (\mathbf{n}\cdot\partial)^2 (\mathbf{n}\cdot\mathbf{b}) ] \} \qquad (9)
\end{aligned}
$$

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- $\chi_{i0}, \lambda_{i0}, \tau_{i0}$ - bare parameters of the systems
- the propagators in weak anisotropy limit (frequency-wave-number representation)

$$\Delta_{ij}^{v'v} = \frac{1}{i\omega + \nu p^2} \times \left[ \left( 1 - \frac{\nu \chi_1 (\mathbf{n}\cdot\mathbf{p})^2}{i\omega + \nu p^2} \right) P_{ij} - \frac{\nu(\chi_2 p^2 + \chi_3 (\mathbf{n}\cdot\mathbf{p})^2)}{i\omega + \nu p^2} P_{is} n_s n_t P_{tj} \right]$$

$$\Delta_{ij}^{b'b} = \frac{1}{i\omega + \nu u p^2} \times \left[ \left( 1 - \frac{\nu u \tau_1 (\mathbf{n}\cdot\mathbf{p})^2}{i\omega + \nu u p^2} \right) P_{ij} - \frac{\nu u(\tau_2 p^2 + \tau_3 (\mathbf{n}\cdot\mathbf{p})^2)}{i\omega + \nu u p^2} P_{is} n_s n_t P_{tj} \right]$$

$$\Delta_{ij}^{vv} = \frac{g\nu^3 p^{4-d-2\varepsilon}}{(-i\omega + \nu p^2)(i\omega + \nu p^2)} \left( A P_{ij} + B P_{is} n_s n_t P_{tj} \right)$$

with

$$A = 1 - \nu \chi_1 (\mathbf{n}\cdot\mathbf{p})^2 \left( \frac{1}{i\omega + \nu p^2} + \frac{1}{-i\omega + \nu p^2} \right) + \alpha_1 \frac{(\mathbf{n}\cdot\mathbf{p})^2}{p^2}$$

$$B = \alpha_2 - \nu(\chi_2 p^2 + \chi_3(\mathbf{n}\cdot\mathbf{p})^2)\left(\frac{1}{i\omega + \nu p^2} + \frac{1}{-i\omega + \nu p^2}\right)$$

- interaction vertexes

$$-v_i'(v_j\partial_j)v_i = v_i' V_{ijl} v_j v_l/2, \quad -b_i'[(v_j\partial_j)b_i - (b_j\partial_j)v_i] = b_i'\widetilde{V}_{ijl} b_j v_l$$

and

$$v_i'(b_j\partial_j)b_i = v_i'\widetilde{\widetilde{V}}_{ijl} b_j b_l$$

with

$$V_{ijl} = i(p_j\delta_{il} + p_l\delta_{ij}), \quad \widetilde{V}_{ijl} = -i(p_j\delta_{il} - p_l\delta_{ij})$$

and

$$\begin{aligned}\widetilde{\widetilde{V}}_{ijl} \;=\; & -\frac{i}{2}(p_j\delta_{il} + p_l\delta_{ij}) - \frac{i}{2}\lambda_1(\delta_{ij}n_l + \delta_{il}n_j)(\mathbf{n}\cdot\mathbf{p}) - i\lambda_2 n_i\delta_{jl}(\mathbf{n}\cdot\mathbf{p}) \\ & -\frac{i}{2}\lambda_3 n_i(p_j n_l + p_l n_j) - i\lambda_4 n_i n_j n_l(\mathbf{n}\cdot\mathbf{p})\end{aligned}$$

# UV Renormalization and RG analysis



- All divergences can be removed multiplicatively by renormalization of bare parameters $g_0, \nu_0, u_0, \chi_{i0}, \tau_{i0}, \lambda_{j0}; i = 1,2,3; j = 1,2,3,4$ and fields $\mathbf{b}, \mathbf{b}'$:

$$\begin{aligned} g_0 &= g\mu^{2\varepsilon} Z_g, \quad \nu_0 = \nu Z_\nu, \quad u_0 = u Z_u, \quad \chi_{i0} = \chi_i Z_{\chi_i}, \\ \tau_{i0} &= \tau_i Z_{\tau_i}, \quad \lambda_{j0} = \lambda_j Z_{\lambda_j}, \quad Z_b = Z_{b'}^{-1} \end{aligned}$$

- Renormalized action is

$$\begin{aligned} S^R(\Phi) &= \int dt\, d^d\mathbf{x}\, \{v_i'[\nu Z_1 \triangle v_i + \nu\chi_1 Z_2 (\mathbf{n}\cdot\partial)^2 v_i \\ &+ \nu\chi_2 Z_3 n_i \triangle(\mathbf{n}\cdot\mathbf{v}) + \nu\chi_3 Z_4 n_i (\mathbf{n}\cdot\partial)^2(\mathbf{n}\cdot\mathbf{v}) \\ &+ Z_9 b_j \partial_j b_i + \lambda_1 Z_{10} b_i (\mathbf{n}\cdot\partial)(\mathbf{n}\cdot\mathbf{b}) + \lambda_2 Z_{11} n_i \mathbf{n}\cdot\partial) b^2 \\ &+ \lambda_3 Z_{12} n_i (\mathbf{b}\cdot\partial)(\mathbf{n}\cdot\mathbf{b}) + \lambda_4 Z_{13} n_i (\mathbf{n}\cdot\partial)(\mathbf{n}\cdot\mathbf{b})^2] \\ &+ b_i'[\nu u Z_5 \triangle b_i + \nu u \tau_1 Z_6 (\mathbf{n}\cdot\partial)^2 b_i + \nu u \tau_2 Z_7 n_i \triangle(\mathbf{n}\cdot\mathbf{b}) \\ &+ \nu u \tau_3 Z_8 n_i (\mathbf{n}\cdot\partial)^2(\mathbf{n}\cdot\mathbf{b})]\} \end{aligned} \qquad (10)$$

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- RG functions: $\beta$ and $\gamma$ functions:

$$\begin{aligned}
\beta_g &= g(-2\varepsilon + 3\gamma_1), \quad \beta_u = u(\gamma_1 - \gamma_5), \\
\beta_{\chi_1} &= \chi_1(\gamma_1 - \gamma_2), \quad \beta_{\chi_2} = \chi_2(\gamma_1 - \gamma_3), \quad \beta_{\chi_3} = \chi_3(\gamma_1 - \gamma_4), \\
\beta_{\tau_1} &= \tau_1(\gamma_5 - \gamma_6), \quad \beta_{\tau_2} = \tau_2(\gamma_5 - \gamma_7), \\
\beta_{\lambda_1} &= \lambda_1(\gamma_9 - \gamma_{10}), \quad \beta_{\lambda_2} = \lambda_2(\gamma_9 - \gamma_{11}), \\
\beta_{\lambda_3} &= \lambda_3(\gamma_9 - \gamma_{12}), \quad \beta_{\lambda_4} = \lambda_4(\gamma_9 - \gamma_{13})
\end{aligned}$$

and

$$\gamma_i = \mu\partial_\mu \ln Z_i$$

# Fixed Point and Scaling Regime



- Possible scaling regimes - the IR stable fixed points of RG-equations (defined by vanishing of $\beta-$functions)
- The asymptotic behavior of correlation functions $W$ is driven by the IR fixed point of RG:

$$\beta_{C_i}(C_j^*) = 0, \quad \Omega_{ij} = \partial_{C_i}\beta_{C_j}|_{C_i=C_i^*},$$

with exact value for $\gamma_\nu(C^*) = \gamma_1(C^*) = 2\varepsilon/3$ and $C_i = \{g, u, \chi_i, \tau_j, \lambda_l\}$

- one obtains for kinetic regime in our case:

$$\chi_3^* = \tau_2^* = 0 \tag{11}$$

and also we can take in kinetic regime (Adzhemyan, Hnatich, Honkonen, Stehlik (1995))

$$\lambda_i^* = 0 \tag{12}$$

- the IR stability condition is given by the eigenvalue:

$$\lambda = \frac{2\varepsilon(a + b\,\alpha_1 + c\,\alpha_2)}{3d(d-1)(d+6)(-56 + d(10 + d(-5 + d(4 + 3d))))}$$

with

$$\begin{aligned} a &= d(-26 + d(7 + d))(-56 + d(10 + d(-5 + d(4 + 3d)))), \\ b &= -2(-10 + d)(-32 + d(-8 + d(1 + d)(1 + 3d))), \\ c &= 2(-10 + d)(-16 + d(-52 + d(5 + d(-1 + 6d)))) \end{aligned}$$

- $\lambda > 0$ for $d = 3$, $\varepsilon > 0$, and $|\alpha_{1,2}| \ll 1$

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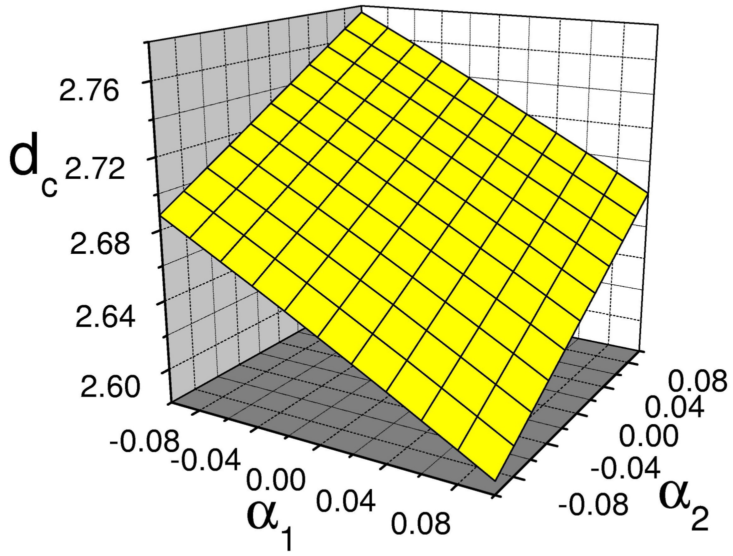


## Instabilities and Generation of the Homogeneous Field

- linear divergences of the solenoidal type - instabilities
- stabilization by spontaneous appearance of a homogeneous mean

magnetic field $\mathbf{B}$



- we make the shift in the action $\mathbf{b} \to \mathbf{b}+\mathbf{B}$, together with the results of the previous section

$$
\begin{aligned}
S(\Phi) \;=\; & \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \\
& v'_i(t_1,\mathbf{x_1}) D^f_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2}) v'_j(t_2,\mathbf{x_2}) \qquad (13)\\
+\; & \int dt\, d^d\mathbf{x}\ \{ v'_i[-\partial_t v_i - v_j\partial_j v_i + b_j\partial_j b_i + \nu_0 \triangle v_i \\
& +\nu_0\chi_{10}(\mathbf{n}\cdot\partial)^2 v_i + \nu_0\chi_{20} n_i \triangle(\mathbf{n}\cdot\mathbf{v})] \\
& +b'_i[-\partial_t b_i - v_j\partial_j b_i + b_j\partial_j v_i + \nu_0 u_0 \triangle b_i + \nu_0 u_0 \tau_{10}(\mathbf{n}\cdot\partial)^2 b_i] \\
& +v'_i(\mathbf{B}\cdot\partial) b_i + b'_i(\mathbf{B}\cdot\partial) v_i \}
\end{aligned}
$$

- it changes old propagators and also new propagators appear: $\Delta^{v'b}$, $\Delta^{b'v}$, $\Delta^{bv}$, and $\Delta^{bb}$
- the linear divergences are present only in Green function $\langle b'b \rangle$: given by four Feynman diagrams.

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- straightforward calculations leads to the fact that the necessary and sufficient condition to cancel the dangerous linear divergences is

$$\mathbf{B}||\mathbf{n}$$

with the result

$$\begin{aligned} \langle b_i' b_j \rangle \;\sim\; & i p_m \varepsilon_{iml} g \rho \\ & \times \left[ \nu \Lambda \delta_{jl} - |\mathbf{B}| C (\delta_{jl} + (1 + f(\alpha_1, \alpha_2)) n_j n_l) \right] \end{aligned}$$

where

$$C = \frac{5\pi(\tau_1 + 3\tau_1 u - 6(1+u) + \chi_1(5+3u))}{16\sqrt{u}(-5 + (-5+\tau_1)u + \chi_1(2+u))}$$

and

$$f(\alpha_1, \alpha_2)|_{\alpha_{1,2}=0} = 0 \tag{14}$$

and the absolute value of the field is given as

$$|\mathbf{B}| = \frac{16\nu\sqrt{u}(-5 + (-5+\tau_1)u + \chi_1(2+u))}{5\pi(\tau_1 + 3\tau_1 u - 6(1+u) + \chi_1(5+3u))}\Lambda$$

in the isotropic limit:

$$|\mathbf{B}| = \frac{8\nu\sqrt{u}}{3\pi}\Lambda$$

identification $l_D = \Lambda^{-1}$ and $l_D = \nu^{3/4}\bar{\varepsilon}^{-1/4}$ ($\bar{\varepsilon}$- energy dissipative rate)

- $$|\mathbf{B}| = \frac{16\sqrt{u}(-5+(-5+\tau_1)u+\chi_1(2+u))}{5\pi(\tau_1+3\tau_1 u-6(1+u)+\chi_1(5+3u))}(\nu\bar{\varepsilon})^{1/4}$$
- the "exotic" term $\sim n_j n_l$ results in the appearance of specific long-live pulses of Alfven waves which are orthogonal polarized with respect to spontaneous field $\mathbf{B}$ and due to the viscosity term have the form: $t\exp(-i\beta t)\exp(-\alpha t)$

# Conclusions

- Anisotropic turbulent dynamo was studied
- The renormalization of the model was done
- The relation between generated magnetic field and the axis of anisotropy was found
- Brief discussion of the "exotic" term was done