Low-energy constants in SU(2) and SU(3) ChPT

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RG'08, Shirkov's Fest

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Introduction

Effective field theory of QCD at low energies

$$\mathcal{L}_{\mathrm{QCD}} \xrightarrow{E<<M_\rho} \mathcal{L}_{\mathrm{eff}}$$

- Expressed in physical hadron fields
- Exploits the chiral symmetry of QCD for massless quarks
- Spontaneous symmetry breaking gives Goldstone bosons

  SU(2) $\Longrightarrow$ pions
  SU(3) $\Longrightarrow$ pions, kaons, eta

- Includes external currents
- Quantum field theory with $\mathcal{L}_{\mathrm{eff}}$ is non-renormalizable $\Longrightarrow$ requires the counterterms with highest derivatives
- The coefficients called Low Energy Constants LECs are not fixed by chiral symmetry

Introduction

- **Calculations with $\mathcal{L}_{\rm eff}$ give an expansion in quark masses and external momenta**

Chiral perturbation theory (ChPT)

Gasser, Leutwyler 1984,1985

Introduction

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Chiral perturbation theory (ChPT)

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- **ChPT exploits systematically quark mass dependence at low–energies**
- **Two options for strange quark**
  - Treat m_s on same footing as heavy quarks — 2 flavor ChPT
  - Treat $m_s \bar{s}s$ as perturbation — 3 flavor ChPT

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- Two options for strange quark
  - Treat m_s on same footing as heavy quarks — 2 flavor ChPT
  - Treat $m_s \bar{s}s$ as perturbation — 3 flavor ChPT
- The degrees of K and η freeze for

$$|p^2| \ll M_K^2 , \qquad m_u, m_d \ll m_s$$

- In this limit: relations among the 2 flavor vs. the 3 flavor low–energy constants of the effective Lagrangians.
- These relations give additional information on the values of the low–energy constants.

Introduction

- **The procedure of findings the relations between SU(2) and SU(3) LECs is called as "matching".**

- **The matching at one-loop level has been done by Gasser and Leutwyler in 1985.**

- **The aim of our research is to perform the matching at two-loop level.**

Effective Lagrangians

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$$

$U \in SU(n)$ contains the Goldstone fields.

LECs l_i, c_i and L_i, C_i are not fixed by chiral symmetry. Local monomials K_i, P_i and X_i, Y_i are known.

Gasser, Leutwyler 1984,1985; Bijnens, Colangelo,Ecker 1999

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$$\mathcal{L}_2^{SU_2} = \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + M^2(U + U^\dagger)\rangle\,, \qquad M^2 = (m_u + m_d)B,$$

$$\mathcal{L}_4^{SU_2} = \sum_{i=1}^{10} l_i K_i, \qquad \mathcal{L}_6^{SU_2} = \sum_{i=1}^{56} c_i P_i\,.$$

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$$\mathcal{L}_4^{SU_3} = \sum_{i=1}^{12} L_i X_i, \qquad \mathcal{L}_6^{SU_3} = \sum_{i=1}^{94} C_i Y_i \,.$$

$U \in SU(n)$ contains the Goldstone fields.
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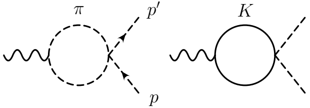
Example of matching: vector form factor

$$\langle \pi^+(p') \,|\, \tfrac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)|\pi^+(p)\rangle = (p+p')_\mu F_V(t) \; ; \; t = (p'-p)^2 \, ,$$

In the chiral limit $m_u = m_d = 0$:

2 flavours : $$F_{V,2}(t) = 1 + \frac{t}{F^2}\Phi(t,0;d) - \frac{\ell_6 t}{F^2}$$

3 flavours : $$F_{V,3}(t) = 1 + \frac{t}{F_0^2}\left[\Phi(t,0;d) + \tfrac{1}{2}\Phi(t,M_K;d)\right] + \frac{2L_9 t}{F_0^2}$$



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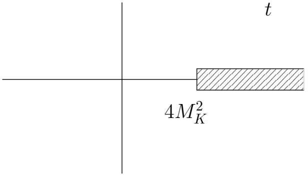

$$\Phi(t,M_K;d) = \sum_{n=0}^{\infty} \Phi_n(M_K,d)\left(\frac{t}{M_K^2}\right)^n$$

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$$\text{3 flavours :} \qquad F_{V,3}(t) = 1 + \frac{t}{F_0^2}\left[\Phi(t,0;d) + \tfrac{1}{2}\Phi(t,M_K;d)\right] + \frac{2L_9 t}{F_0^2}$$

Drop terms of order "t" and higher. It is seen that $F_{V,3}(t)$ reduces to $F_{V,2}(t)$ if we put

$$-\ell_6 = 2L_9 + \frac{1}{2}\Phi_0(M_K, d).$$

At $d = 4$, this equation gives the relation between renormalized LECs

$$\ell_6^r(\mu) = -2L_9^r(\mu) + \frac{1}{192\pi^2}(\ln B_0 m_s/\mu^2 + 1).$$

Gasser, Leutwyler (85)

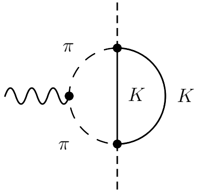
Matching at two loops

What about a matching at two–loop order? Some remarks:

- **For ℓ_6 one can extract its strange quark mass dependence at two–loops from the literature**

2 flavours:	$F_{V,2}(t)$,	Gasser, Leutwyler (84)
3 flavours:	$F_{V,3}(t)$	Bijnens, Talavera (02)

- **Despite literature, still an exhaustive work, because two–loop diagrams need to be known analytically in an expansion in $t/B_0 m_s$**
  [up to logarithms $\ln(-t/B_0 m_s)$]



Matching at two loops

- **One can get the matching for $\mathbf{F}, \mathbf{B}, \ell_1, \ldots, \ell_6$ from available two-loop calculations of the various matrix elements.**

- **But the matching at order $\mathbf{p}^6$ in this manner requires a tremendous amount of two-loop calculations in $\mathbf{ChPT_{2,3}}$.**

- **Therefore, we have developed a generic method based on the path integral formulation of ChPT.**

Generating functional

Loop expansion in a scalar field theory

- N scalar fields $\phi_1, \ldots, \phi_N$ and M external sources $j = j_1, \ldots, j_M$.
- Action

$$S[\phi, j] = S_2[\phi, j] + \hbar S_4[\phi, j] + \hbar^2 S_6[\phi, j] + \cdots$$

- The generating functional for connected Green's functions

$$\exp(-Z[j]/\hbar) = \mathcal{N}^{-1} \int [d\phi] \exp(-S[\phi, j]/\hbar) \,,$$

- The loop expansion is constructed as an expansion around the solution of the equation of motion:

$$\phi_k = \phi_{cl,k} + \xi_k \,, \qquad \left. \frac{\delta S_2[\phi, j]}{\delta \phi_k} \right|_{\phi_k = \phi_{cl,k}} = 0.$$

- Introduce the abbreviation $\bar{S}_{2n} = S_{2n}[\phi_{cl}, j]$.

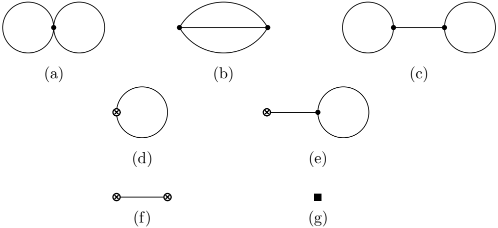
Generating functional

- **Functional integration over fluctuation field ξ gives**

$$\mathbf{Z} = \mathbf{Z_0} + \hbar \mathbf{Z_1} + \hbar^2 \mathbf{Z_2} + \mathbf{O}(\hbar^3),$$

$$\mathbf{Z_0} = \bar{\mathbf{S}}_2\,, \qquad \mathbf{Z_1} = \bar{\mathbf{S}}_4 + \tfrac{1}{2}\mathbf{Tr}\,\mathbf{ln}(\mathbf{D}/\mathbf{D}^0)\,,$$

$$\mathbf{Z_2} =$$



- **The propagator is the inverse of the differential operator $\mathbf{D}$ from quadratic term over ξ.**
- **Dotted vertices stem from $\bar{\mathbf{S}}_2$, crossed vertices from $\bar{\mathbf{S}}_4$.**
- **Diagram (g) represents the tree graphs of $\bar{\mathbf{S}}_6$.**

Generating functional

Two-flavor limit of the SU(3) generating functional.

- **The restricted framework of ChPT_3:**
  - massless light quarks $m_u = m_d = 0$
  - SU(2)-type external sources

$$s = p = 0\,, \qquad v_\mu = \sum_1^3 \lambda^a v_\mu^a\,, \qquad a_\mu = \sum_1^3 \lambda^a a_\mu^a\,.$$

  - external momenta are small $|p^2| << B_0 m_s$
  - tree level:

$$u = u^{(2)} e^{\frac{i}{2F_0}\eta\lambda_8}\,, \qquad u^{(2)} = \begin{pmatrix} \square & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}\,.$$

Inserting this ansatz into the EOM yields that also the solution of the η field is trivial, $\eta = 0$.

Generating functional

- **Loops at order $\hbar \Longrightarrow \frac{1}{2}\mathrm{Tr}\ln(D/D^0) = \frac{1}{2}\ln(\det D/\det D^0)$**

Generating functional

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Determinant:

- **Separation of heavy and light fields**

$$\ln\det D = \ln\det D_\pi + \ln\det D_\eta + \underbrace{\ln\det D_K}_{(1)} + \underbrace{\ln\det(1 - D_\pi^{-1} D_{\pi\eta} D_\eta^{-1} D_{\eta\pi})}_{(2)}$$

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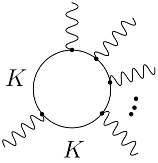
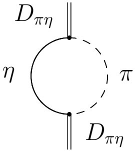


(1)



(2)

- **(1) Short distance expansion with heat–kernel $\Longrightarrow$ manifestly covariant at all steps**
- **(2) $\pi - \eta$ mixing**

[gives no headaches at this order]

Generating functional

- **Matching at one-loop order:**

$$\bar{S}^{(3)}_{\mathrm{tree}} + \frac{1}{2}\ln\frac{\det D}{\det D^0} = \bar{s}^{(2)}_{\mathrm{tree}} + \frac{1}{2}\ln\frac{\det d}{\det d^0}$$

- **E.g. for ℓ_6:**

$$\Big(-2L_9 \underbrace{-\frac{1}{12}\int\frac{dq}{(2\pi)^d}\frac{1}{[M_K^2+q^2]^2}}_{\text{from}\,\det D_K}\Big)\int dx\langle f_{+\mu\nu}[u_\mu,u_\nu]\rangle = \ell_6\int dx\,\underbrace{\langle f_{+\mu\nu}[u_\mu,u_\nu]\rangle}_{\text{chiral operator}}$$

Generating functional

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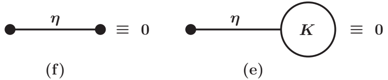
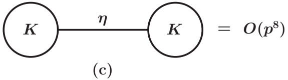
- **From which one verifies again**

$$-2L_9^r(\mu) + \frac{1}{192\pi^2}(\ln B_0 m_s/\mu^2 + 1) = \ell_6^r(\mu)$$

Generating functional

- **Loops at order $\hbar^2$**
- **The one-particle reducible diagrams with eta and kaons**



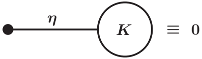


- **tadpole and butterfly diagrams**

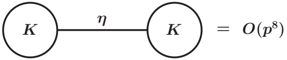
Generating functional

- **Loops at order $\hbar^2$**
- **The one-particle reducible diagrams with eta and kaons**

(f)



(e)



(c)

- **tadpole and butterfly diagrams**
- **sunset diagram is more difficult to evaluate**
  - **need to know the propagators with two covariant derivatives**
  - **need to expand the Seeley–coefficients around x = y up to 4th order**
  - **need to express the normal derivatives via the covariant ones (use fixed gauge)**
  - **need to evaluate the tensorial two-loop diagrams of the sunset topology analytically**

Generating functional

Results of matching at two–loops

$$p^2; \quad p^4 \quad \longrightarrow \quad F, B; \quad \ell_1, \ldots, \ell_7$$

Gasser, Haefeli, Ivanov, Schmidt PLB 652. (2007) 21

$$p^6 \quad \longrightarrow \quad c_1, \ldots, c_{56}$$

Gasser, Haefeli, Ivanov, Schmidt done, to be published

Two examples:

$$F = F_0 \left\{ 1 + z \left[8\, N\, L_4^r - \frac{1}{2} \ln \frac{\overline{M}_K^2}{\mu^2} \right] + z^2 \left[d_F - \frac{11}{12} \ln^2(\Xi_F^2/\overline{M}_K^2) \right] + \mathcal{O}(z^3) \right\} ,$$

$$l_6^r = \frac{1}{12\, N} \left(1 + \ln \frac{\overline{M}_K^2}{\mu^2} \right) - 2\, L_9^r + z \left[d_6 - \frac{1}{8\, N} \ln^2(\Xi_6^2/\overline{M}_K^2) \right] + \mathcal{O}(z^2) \, .$$

The expansion parameter

$$z = \frac{\overline{M}_K^2}{N F_0^2} , \qquad N = 16\pi^2 ,$$

where $\overline{M}_K$ stands for the kaon mass at next–to–leading order at $m_u = m_d = 0$, and F_0 denotes the pion decay constant at $m_u = m_d = m_s = 0$.

Generating functional

The dimensionless parameters d_F, d_6 and the logarithmic scales Ξ_F, Ξ_6 are given by

$$
\begin{aligned}
d_F \;=\;& -\frac{841}{1056} + \frac{2}{3}\,\rho_1 - \frac{1}{11}\,\ln\frac{4}{3} + \frac{1}{33}\left(\ln\frac{4}{3}\right)^2 \\
& + N\left[\left(\frac{5824}{99} + \frac{64}{11}\,\ln\frac{4}{3}\right) L_1^r + \left(\frac{884}{99} + \frac{16}{11}\,\ln\frac{4}{3}\right) L_2^r \right. \\
& \left. + \left(\frac{4651}{297} + \frac{12}{11}\,\ln\frac{4}{3}\right) L_3 - \left(\frac{952}{33} + \frac{72}{11}\,\ln\frac{4}{3}\right) L_4^r\right] \\
& + N^2\left[\frac{173056}{297}\,(L_1^r)^2 + \frac{10816}{297}\,(L_2^r)^2 + \frac{14884}{297}\,(L_3)^2 + \frac{7792}{33}\,(L_4^r)^2 \right. \\
& + \frac{86528}{297}\,L_1^r L_2^r + \frac{101504}{297}\,L_1^r L_3 - \frac{56576}{99}\,L_1^r L_4^r + \frac{25376}{297}\,L_2^r L_3 \\
& - \frac{14144}{99}\,L_2^r L_4^r - \frac{16592}{99}\,L_3 L_4^r + 64\,L_4^r L_5^r \\
& \left. - 256\,L_4^r L_6^r - 128\,L_4^r L_8^r + 32\,C_{16}^r\right] ,
\end{aligned}
$$

$$
\ln(\Xi_F^2/\overline{M}_K^2) \;=\; \frac{14}{11} - \frac{2}{11}\,\ln\frac{4}{3} + N\left[\frac{244}{33}\,L_3 + \frac{832}{33}\,L_1^r + \frac{208}{33}\,L_2^r - \frac{136}{11}\,L_4^r\right] - \ln\frac{\overline{M}_K^2}{\mu^2}
$$

Generating functional

$$d_6 = -\frac{1}{N}\left(\frac{163}{288} + \frac{1}{16}\,\rho_1 - \frac{1}{24}\,\ln\frac{4}{3}\right) + 2N\left(\frac{7}{24N} - 2\,L_3 - 2\,L_9^r\right)^2$$
$$+8N\left(4\,C_{13}^r + C_{64}^r\right)\,,$$

$$\ln(\Xi_6^2/\overline{M}_K^2) = -\frac{7}{6} - \ln\frac{\overline{M}_K^2}{\mu^2} + 8\,N\,(L_3 + L_9^r)\,.$$

Furthermore

$$\rho_1 = \sqrt{2}\,\mathrm{Cl}_2(\arccos(1/3)) \cong 1.41602\,,$$

$$\mathrm{Cl}_2(\theta) = -\frac{1}{2}\int_0^\theta d\phi\;\ln\left(4\sin^2\frac{\phi}{2}\right)\,.$$

Some applications

- The scale independent LEC

$$\bar{l}_2 = 3N l_2^r(\mu) - \ln \frac{M_\pi^2}{\mu^2}$$

- It has been determined from a dispersive analysis

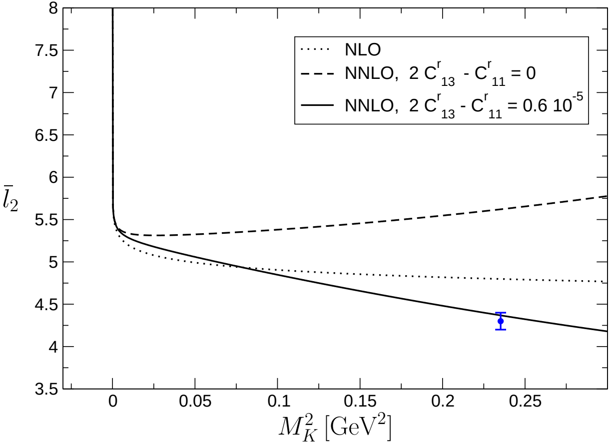
$$\bar{l}_2 = 4.3 \pm 0.1$$

- As follows from our formulae, $\bar{l}_2$ depend on L_2^r, L_3^r and the combination $2C_{13}^r - C_{11}^r$.

$$L_2^r = (+0.73 \pm 0.12)\,10^{-3}\,, \qquad L_3 = (-2.35 \pm 0.37)\,10^{-3}\,,$$

$$\mu = M_\rho = 770\,\mathrm{MeV}.$$

Generating functional



Summary

- **The degrees of K and η freeze for**

$$|p^2| \ll M_K^2\,, \qquad m_u, m_d \ll m_s$$

- **In this limit, one can establish relations among the 2 flavor vs. the 3 flavor low energy constants**

Results of matching at two–loops

chiral order	LECs	
p^2	F^2B	Moussallam (00)
p^2	B	Kaiser, Schweizer (06)
p^2, p^4	$F, B, \ell_1, \ldots, \ell_{10}$	Gasser, Haefeli, Ivanov, Schmid (07)
p^6	$c_1, \ldots, c_{56}$	Gasser,Haefeli, Ivanov, Schmid (08)