Renormalization group in stochastic hydrodynamics

Juha Honkonen
Outline

- Stochastic hydrodynamics
- Structure functions
- Functional representation of the stochastic problem
- Asymptotic analysis by RG and OPE
- Two-parameter expansion
- Improved $\varepsilon$ expansion
- Two-loop results
  - Kolmogorov constant
  - Prandtl number
- Conclusion
Randomly forced Navier-Stokes equation for incompressible fluid ($\nabla \cdot \mathbf{v} = 0$)

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu_0 \nabla^2 \mathbf{v} - \frac{\nabla p}{\rho} + \mathbf{f}.$$
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Isotropic pumping: gaussian distribution of random force with zero mean and the correlation function

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\langle f_m(t, \mathbf{k}) f_n(t', \mathbf{k}') \rangle = \left( \delta_{mn} - \frac{k_m k_n}{k^2} \right) (2\pi)^d \delta(t - t') \delta(\mathbf{k} + \mathbf{k}') df(k).
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$$

Transport of a passive scalar admixture (temperature, concentration): add advection-diffusion equation

$$
\partial_t \theta + \mathbf{v} \cdot \nabla \theta = \kappa_0 \nabla^2 \theta + f_{\theta}.
$$
Thermal fluctuations vs. random stirring

Thermal fluctuations described by the correlation function (UV cutoff implied)

\[ d_f(k) = D_{20}k^2, \quad D_{20} = 2\nu_0 T/\rho. \]
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\[ d_f(k) = D_{10}k^{4-d}(k^2 + m^2)^{-\varepsilon}, \quad m \sim \frac{1}{L}. \]
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This is a \(\delta\) sequence yielding \(\sim \delta(k)\) in the limit \(\epsilon \to 2, m \to 0\).

Field-theoretic RG initiated by De Dominicis & Martin (1979).
Statistical description of the turbulent flow by structure functions of the velocity field

\[ S_n(r) = \langle \left[ v_\parallel(t, x + r) - v_\parallel(t, x) \right]^n \rangle, \quad v_\parallel = \frac{v \cdot r}{r}. \]

Correlation functions with coinciding arguments: asymptotic analysis of composite operators needed.
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Kolmogorov scaling (1941) in the inertial range:

\[ S_n(r) \propto (\overline{\varepsilon} r)^{n/3}, \quad m \ll k \ll k_d. \]
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Kolmogorov constant \( C_K \) and \( \frac{4}{5} \) (at \( d = 3 \)) law

\[ S_2(r) \sim C_K (\overline{\varepsilon}r)^{2/3}, \quad S_3(r) \sim -\frac{12}{d(d+2)} \overline{\varepsilon}r. \]
Cast the Navier-Stokes problem into the field-theoretic form: De Dominicis-Janssen (or Martin-Siggia-Rose) action

\[
S_{NS}(v, v') = \frac{1}{2} v' Dv' - v' \left[ \partial_t v + (v \nabla) v - \nu_0 \nabla^2 v \right],
\]
Field-theoretic (MSR) representation

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where \((P_{mn} = \delta_{nm} - k_n k_m / k^2)\)

\[
D_{mn}(t, x + r, t', x) = \delta(t - t') \int dr \exp[i(k \cdot r)] P_{mn} d_f(k).
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Bare propagators for perturbation theory

\[ \langle v_m(t) v'_n(t') \rangle_0 = \theta(t - t') P_{mn} \exp \left[ -\nu_0 k^2 (t - t') \right], \]

\[ \langle v_m(t) v_n(t') \rangle_0 = \frac{d_f(k) P_{mn}}{2\nu_0 k^2} \exp \left[ -\nu_0 k^2 |t - t'| \right], \langle v'_m(t) v'_n(t') \rangle_0 = 0. \]
Renormalization in space dimension $d > 2$

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Only one renormalization constant for $d > 2$.

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\]

Connect to bare parameters introducing \( \mu \):

\[
\nu_0 = \nu Z_v, \quad g_{10} = D_{10} \nu_0^{-3} = g_1 \mu^{2\epsilon} Z_v^{-3}.
\]
Consider velocity pair correlation function $G(k)$:

$$
\int d\mathbf{r} \exp \left[ i (\mathbf{k} \cdot \mathbf{r}) \right] \langle v_n(t, \mathbf{x} + \mathbf{r}) v_m(t, \mathbf{x}) \rangle = \left( \delta_{nm} - \frac{k_n k_m}{k^2} \right) G(k).
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$$

Solution of the RG equation for the velocity correlator

$$
G(k) = \nu^2 k^{2-d} R \left( \frac{k}{\mu}, g_1, \frac{m}{\mu} \right) = \bar{\nu}^2 k^{2-d} R \left( 1, \bar{g}_1, \frac{m}{\bar{k}} \right).
$$

Invariant (running) parameters $\bar{\nu}, \bar{g}_1$ from

$$
g_{10} = \bar{g}_1 k^{2\varepsilon} Z_\nu^{-3} \left( \bar{g}_1, \frac{m}{\bar{k}} \right), \quad \bar{\nu} = \left( \frac{D_{10} k^{-2\varepsilon}}{\bar{g}_1} \right)^{1/3}.
$$
For $\varepsilon > 0 \exists$ an IR-stable fixed point: $\bar{g}_1 \rightarrow g_{1*} \propto \varepsilon$. Basic scaling dimensions exact:

$$\Delta_v = 1 - 2\varepsilon/3, \quad \Delta_\omega = 2 - 2\varepsilon/3.$$
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IR fixed point yields large-scale limit ($k \to 0$, $u = m/k = \text{const}$)

$$G(k) \sim (D_{10}/g_{1*})^{2/3} k^{2-d-4\varepsilon/3} R(1, g_{1*}, u), \quad R(1, g_{1*}, u) = \sum_{n=1}^{\infty} \varepsilon^n R_n(u).$$
Large-scale asymptotic behaviour

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Translate in traditional variables; trade $D_{10}$ for the mean energy injection rate $\bar{\mathcal{E}}$ ($2 > \varepsilon > 0$):

$$\bar{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d kd f(k) \Rightarrow D_{10} = \frac{4(2 - \varepsilon) \Lambda^{2\varepsilon-4\bar{\mathcal{E}}}}{\bar{\mathcal{S}}_d(d-1)}, \quad \Lambda = \left(\frac{\bar{\mathcal{E}}/\nu^3_0}{\nu^3_0}\right)^{1/4}.$$
Inertial-range scaling

Large-scale scaling in terms of $\bar{E}$ and $\nu_0$ for $2 > \varepsilon > 0$:

$$G(k) \sim \left[ \frac{4(2 - \varepsilon)}{\mathcal{S}_d(d - 1)g_1^*} \right]^{2/3} \nu_0^{2-\varepsilon} \bar{E}^{\varepsilon/3} k^{2-d-4\varepsilon/3} R(1, g_1^*, u).$$
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Freezing of scaling dimensions for $\varepsilon > 2$ [Adzhemyan, Antonov & Vasil’ev (1989)]: $D_{10}$ acquires scale dependence through

$$D_{10} = 4(\varepsilon - 2) m^{4-2\varepsilon} \bar{E} / S_d(d - 1), \quad m = 1/L.$$
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Yields independence of $\nu_0$, Kolmogorov exponents $\forall \varepsilon > 2$:

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The inertial-range limit $u = m/k \to 0$ tough. Use OPE.
The limit $u = m/k \to 0$ beyond RG. To collect terms $\varepsilon \ln u \sim 1$, use operator-product expansion for composite operators $F$:

$$F_1(t, x_1)F_2(t, x_2) = \sum_{\alpha} C_{\alpha}(x_1 - x_2) F_{\alpha} [(x_1 + x_2)/2, t].$$

$C_{\alpha}$ analytic in $(mr)^2$: singularities due to dangerous operators $\langle F_{\alpha}(x) \rangle \propto m^{\Delta_{F_{\alpha}}}$ with $\Delta_{F_{\alpha}} < 0$. 
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Dangerous operators not known for \( 0 < \varepsilon < 2 \): \( u \to 0 \) safe!
advection of passive scalar

- hydrodynamic fluctuations, momentum-shell RG: Forster, Nelson & Stephen (1976),
- LR correlated injection, field-theoretic RG: Adzhemyan, Vasil’ev & Pis’mak (1983),
- decaying scalar, hydrodynamic fluctuations, LR correlated injection, field-theoretic RG: Hnatich (1990, reflecting boundary), Hnatich, JH (2000, absorbing boundary);
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Ramifications of the Navier-Stokes problem

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- **anisotropic random forcing**
  - LR, momentum-shell RG, weak anisotropy: Rubinstein & Barton (1987),
  - LR, FTRG, weak anisotropy: Adzhemyan, Hnatich, Horvath & Stehlik (1995); Kim & Serdukov (1995);
Skewness factor in the inertial range

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Connect to experimental data through $(m = 0)$

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Use independent of $D_{10}$ quantity - the skewness factor  
[Adzhemyan, Antonov, Kompaniets & Vasil’ev (2003)]:

$$S = S_3/S_2^{3/2}.$$
For $\varepsilon \geq \frac{3}{2}$ the structure function $S_2(r) \sim \text{const}$, replace in $S$ by the function with powerlike asymptotics $r \partial_r S_2(r)$ and define:

$$Q(\varepsilon) \equiv \frac{r \partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r \partial_r S_2(r)}{[-S_3(r)]^{2/3}}.$$
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Calculate Kolmogorov constant and skewness factor unambiguously as

$$C_K = \left[ \frac{3Q(2)}{2} \right] \left[ \frac{12}{d(d+2)} \right]^{2/3}, \quad S = -\left[ \frac{3Q(2)}{2} \right]^{-3/2}.$$
Two-loop corrections to $C_K$ and $S$ large: $\approx 100\%$ change for $d = 3$ but rapidly decreasing with growing $d$.

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$$d_f(k) = D_{10}k^{4-d-2\varepsilon} + D_{20}k^2$$

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Yes, inverse energy cascade far from the linear extrapolation path.
Additional $UV$-renormalization near $d = 2$ required

$$S_R = \frac{1}{2} v' \left( D_1 k^{4-d-2\varepsilon} + D_2 Z_{D_2} k^2 \right) v' - v' \left[ \partial_t v + (v \nabla) v - \nu Z_{\nu} \nabla^2 v \right]$$

with $\nu_0 = \nu Z_{\nu}$ and

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Near $d = 2 \exists$ IR-stable fixed point giving rise to double expansion in $\varepsilon$ and $2\Delta = d - 2$. 
Minimal subtractions on rays

Two-parameter renormalization not entirely trivial; problems

- analytic renormalization: no MS scheme,
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These are two different subsequences of the double series

$$Q(\varepsilon, d) = \varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[2\varepsilon/(d - 2)\right]^k q_{kl} \left[(d - 2)/2\right]^l.$$
Improved $\varepsilon$ expansion

Combine the information from both expansions

$$Q_{eff}^{(n)} = \varepsilon^{1/3} \left[ \sum_{k=0}^{n-1} Q_k(d) \varepsilon^k + \sum_{k=0}^{n-1} \Psi_k \left( \frac{d - 2}{2\varepsilon} \right) \varepsilon^k - \sum_{k,l=0}^{n-1} \left( \frac{2\varepsilon}{d - 2} \right)^k q_{kl} \left( \frac{d - 2}{2} \right)^l \right].$$

Subtraction term to account for double counting in the overlap region.
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Improved two-loop Kolmogorov constant

Comparison of one-loop and two-loop results for $C_K$:

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Recommended experimental value: $C_K = 2.0$ (Sreenivasan, 1995).
Prandtl number for thermal conduction: $\Pr = \nu_0/\kappa_0 = 1/u$. 
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Define turbulent (effective) inverse Prandtl number:

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u_{eff} = \nu_*^{(0)} (1 - 0.0358 \varepsilon) + O(\varepsilon^2), \quad \nu_*^{(0)} = \frac{\sqrt{43/3} - 1}{2}, \quad d = 3.
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At \( \varepsilon = 2 \) the turbulent Prandtl number \( \text{Pr}_t \) close to accepted experimental value \( \text{Pr}_t \approx 0.81 \):

\[
    \text{Pr}_t^{(0)} \approx 0.72, \quad \text{Pr}_t \approx 0.77.
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