Renormalization group in stochastic hydrodynamics

Juha Honkonen

Outline

- Stochastic hydrodynamics
- Structure functions
- Functional representation of the stochastic problem
- Asymptotic analysis by RG and OPE
- Two-parameter expansion
- Improved ε expansion
- Two-loop results
    - Kolmogorov constant
    - Prandtl number
- Conclusion

Stochastic hydrodynamics

Randomly forced Navier-Stokes equation for incompressible fluid ($\nabla \cdot \mathbf{v} = 0$)

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu_0 \nabla^2 \mathbf{v} - \frac{\nabla p}{\rho} + \mathbf{f} \, .$$

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Isotropic pumping: gaussian distribution of random force with zero mean and the correlation function

$$\langle \, f_m(t,\mathbf{k}) f_n(t',\mathbf{k}') \, \rangle = \left(\delta_{mn} - \frac{k_m k_n}{k^2} \right) (2\pi)^d \delta(t-t') \delta(\mathbf{k}+\mathbf{k}') \, d_f(k) \, .$$

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Transport of a passive scalar admixture (temperature, concentration): add advection-diffusion equation

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta = \kappa_0 \nabla^2 \theta + f_\theta \,.$$

Thermal fluctuations vs. random stirring

Thermal fluctuations described by the correlation function (UV cutoff implied)

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This is a δ sequence yielding $\sim \delta(\mathbf{k})$ in the limit $\varepsilon \to 2$, $m \to 0$.
Field-theoretic RG initiated by De Dominicis & Martin (1979).

Kolmogorov scaling of structure functions

Statistical description of the turbulent flow by structure functions of the velocity field

$$S_n(r) = \left\langle \left[v_\parallel(t, \mathbf{x} + \mathbf{r}) - v_\parallel(t, \mathbf{x}) \right]^n \right\rangle , \quad v_\parallel = \frac{\mathbf{v} \cdot \mathbf{r}}{r} .$$

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Kolmogorov constant C_K and $\frac{4}{5}$ (at $d = 3$) law

$$S_2(r) \sim C_K (\overline{\varepsilon}\, r)^{2/3} , \quad S_3(r) \sim -\frac{12}{d(d+2)} \overline{\varepsilon}\, r .$$

Field-theoretic (MSR) representation

Cast the Navier-Stokes problem into the field-theoretic form: De Dominicis-Janssen (or Martin-Siggia-Rose) action

$$S_{\rm NS}(\mathbf{v},\mathbf{v}') = \frac{1}{2}\mathbf{v}' D \mathbf{v}' - \mathbf{v}' \left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu_0 \nabla^2 \mathbf{v}\right] ,$$

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where ($P_{mn} = \delta_{nm} - k_n k_m / k^2$)

$$D_{mn}(t, \mathbf{x}+\mathbf{r}, t', \mathbf{x}) = \delta(t-t') \int d\mathbf{r} \, \exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r})\right] P_{mn} \, d_f(k) \, .$$

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Bare propagators for perturbation theory

$$\langle v_m(t) v'_n(t') \rangle_0 = \theta(t-t') P_{mn} \exp\left[-\nu_0 k^2 (t-t')\right] ,$$

$$\langle v_m(t) v_n(t') \rangle_0 = \frac{d_f(k) P_{mn}}{2\nu_0 k^2} \exp\left[-\nu_0 k^2 |t-t'|\right] ,\ \langle v'_m(t) v'_n(t') \rangle_0 = 0 .$$

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Only one renormalization constant for $d > 2$.

$$S_{\mathrm{R}}(\mathbf{v},\mathbf{v}') = \frac{1}{2}\mathbf{v}' D \mathbf{v}' - \mathbf{v}' \left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu Z_\nu \nabla^2 \mathbf{v}\right] .$$

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Connect to bare parameters introducing μ:

$$\nu_0 = \nu Z_\nu , \qquad g_{10} = D_{10}\nu_0^{-3} = g_1 \mu^{2\varepsilon} Z_\nu^{-3} .$$

RG solution for the correlation function

Consider velocity pair correlation function $G(k)$:

$$\int d\mathbf{r}\,\exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r})\right]\langle v_n(t,\mathbf{x}+\mathbf{r})v_m(t,\mathbf{x})\rangle = \left(\delta_{nm} - \frac{k_n k_m}{k^2}\right) G(k)\,.$$

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Solution of the RG equation for the velocity correlator

$$G(k)=\nu^2 k^{2-d}R\left(\frac{k}{\mu},g_1,\frac{m}{\mu}\right)=\bar{\nu}^2 k^{2-d}R\left(1,\bar{g}_1,\frac{m}{k}\right)\,.$$

Invariant (running) parameters $\bar{\nu}$, $\bar{g}_1$ from

$$g_{10}=\bar{g}_1 k^{2\varepsilon}Z_\nu^{-3}\left(\bar{g}_1,\frac{m}{k}\right),\qquad \bar{\nu}=\left(\frac{D_{10}k^{-2\varepsilon}}{\bar{g}_1}\right)^{1/3}\,.$$

Large-scale asymptotic behaviour

For $\varepsilon > 0 \ \exists$ an IR-stable fixed point: $\bar{g}_1 \to g_{1*} \propto \varepsilon$. Basic scaling dimensions exact:

$$\Delta_v = 1 - 2\varepsilon/3\,, \quad \Delta_\omega = 2 - 2\varepsilon/3\,.$$

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IR fixed point yields large-scale limit ($k \to 0$, $u = m/k =$const)

$$G(k) \sim (D_{10}/g_{1*})^{2/3}\, k^{\,2-d-4\varepsilon/3} R(1, g_{1*}, u)\,,\; R(1, g_{1*}, u) = \sum_{n=1}^{\infty} \varepsilon^n R_n(u)$$

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Translate in traditional variables; trade D_{10} for the mean energy injection rate $\overline{\mathcal{E}}$ ($2 > \varepsilon > 0$):

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k) \;\Rightarrow\; D_{10} = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}\overline{\mathcal{E}}}{\overline{S}_d(d-1)}\,,\; \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}\,.$$

Inertial-range scaling

Large-scale scaling in terms of $\overline{\mathcal{E}}$ and ν_0 for $2 > \varepsilon > 0$:

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The inertial-range limit $u = m/k \to 0$ tough. Use OPE.

Operator-product expansion

The limit $u = m/k \to 0$ beyond RG. To collect terms $\varepsilon \ln u \sim 1$, use operator-product expansion for composite operators F:

$$F_1(t, \mathbf{x}_1) F_2(t, \mathbf{x}_2) = \sum_\alpha C_\alpha(\mathbf{x}_1 - \mathbf{x}_2) F_\alpha \left[(\mathbf{x}_1 + \mathbf{x}_2)/2, t \right] .$$

C_α analytic in $(mr)^2$: singularities due to dangerous operators $\langle F_\alpha(x) \rangle \propto m^{\Delta_{F_\alpha}}$ with $\Delta_{F_\alpha} < 0$.

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Dangerous operators not known for $0 < \varepsilon < 2$: $u \to 0$ safe!

Ramifications of the Navier-Stokes problem

- advection of passive scalar
  - hydrodynamic fluctuations, momentum-shell RG: Forster, Nelson & Stephen (1976),
  - LR correlated injection, field-theoretic RG: Adzhemyan, Vasil'ev & Pis'mak (1983),
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- anisotropic random forcing
  - LR, momentum-shell RG, weak anisotropy: Rubinstein & Barton (1987),
  - LR, FTRG, weak anisotropy: Adzhemyan, Hnatich, Horvath & Stehlik (1995); Kim & Serdukov (1995);
  - LR, FTRG, strong anisotropy: Buša, Hnatich, JH & Horvath (1997).

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Use independent of D_{10} quantity - the skewness factor [Adzhemyan, Antonov, Kompaniets & Vasil'ev (2003)]:

$$\mathcal{S} = S_3/S_2^{3/2}\,.$$

Unambiguous Kolmogorov constant

For $\varepsilon \geq \frac{3}{2}$ the structure function $S_2(r) \sim$ const, replace in $\mathcal{S}$ by the function with powerlike asymptotics $r\partial_r S_2(r)$ and define:

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{[-S_3(r)]^{2/3}}.$$

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Calculate Kolmogorov constant and skewness factor unambiguously as

$$C_K = \left[\frac{3Q(2)}{2}\right] \left[\frac{12}{d(d+2)}\right]^{2/3}, \quad \mathcal{S} = -\left[\frac{3Q(2)}{2}\right]^{-3/2}.$$

Effect of low-dimensional fluctuations

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Coarse-graining of finite band-width forcing always generates the local term (Forster, Nelson & Stephen, 1977).

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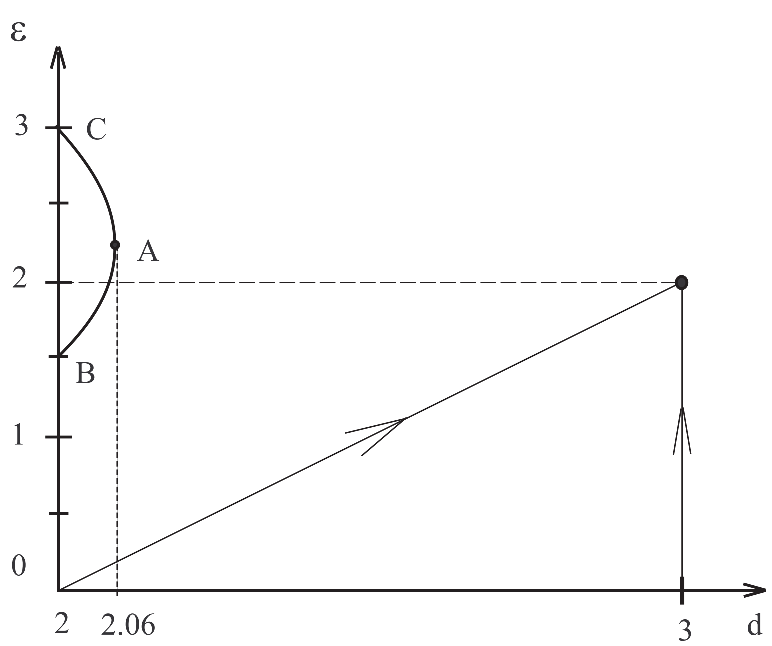

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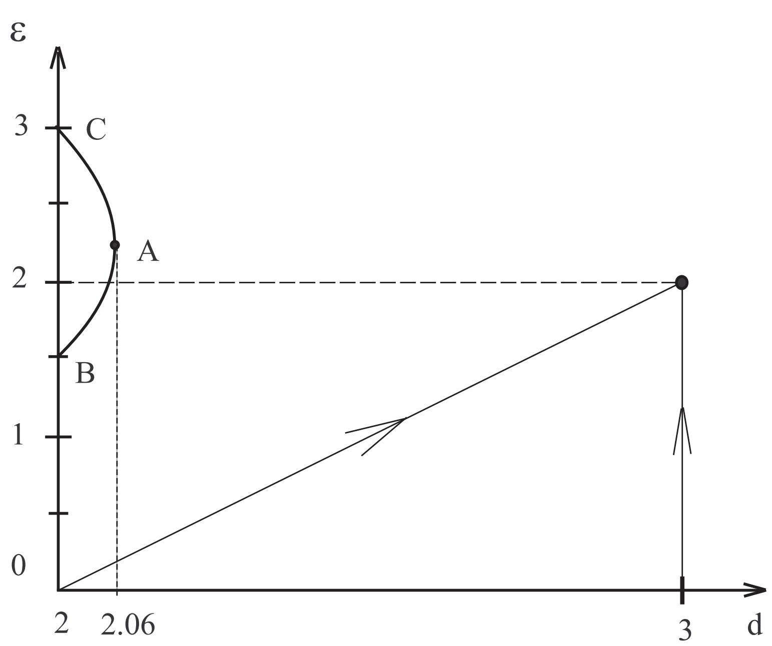

Yes, inverse energy cascade far from the linear extrapolation path.

Two-parameter expansion

Additional UV-renormalization near $d=2$ required

$$S_{\rm R}=\frac{1}{2}\mathbf{v}'\left(D_1k^{4-d-2\varepsilon}+D_2Z_{D_2}k^2\right)\mathbf{v}'-\mathbf{v}'\left[\partial_t\mathbf{v}+(\mathbf{v}\nabla)\mathbf{v}-\nu Z_\nu\nabla^2\mathbf{v}\right]$$

with $\nu_0=\nu Z_\nu$ and

$$g_{01}=D_{10}\nu_0^{-3}=g_1\mu^{2\varepsilon}Z_\nu^{-3}\,,\;g_{20}=D_{20}\nu_0^{-3}=g_2\mu^{2-d}Z_{D_2}Z_\nu^{-3}\,.$$

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The RG solution [$m=0$, UV cutoff Λ imposed]

$$G(k, g_{10}, g_{20}, \nu_0, \Lambda) = \left(D_{10}/\bar{g}_1\right)^{2/3} k^{2-d-4\varepsilon/3} R_\Lambda\left(1, \bar{g}_1, \bar{g}_2, \Lambda/k\right)\,.$$

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Near $d = 2$ $\exists$ IR-stable fixed point giving rise to double expansion in ε and $2\Delta = d - 2$.

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- the remainder is analytic continuation from $d < 2$.

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These are two different subsequences of the double series

$$Q(\varepsilon, d) = \varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} [2\varepsilon/(d-2)]^k \, q_{kl} \, [(d-2)/2]^l \,.$$

Improved ε expansion

Combine the information from both expansions

$$Q_{eff}^{(n)} = \varepsilon^{1/3}\left[\sum_{k=0}^{n-1} Q_k(d)\varepsilon^k + \sum_{k=0}^{n-1}\Psi_k\left(\frac{d-2}{2\varepsilon}\right)\varepsilon^k - \sum_{k,l=0}^{n-1}\left(\frac{2\varepsilon}{d-2}\right)^k q_{kl}\left(\frac{d-2}{2}\right)^l\right].$$

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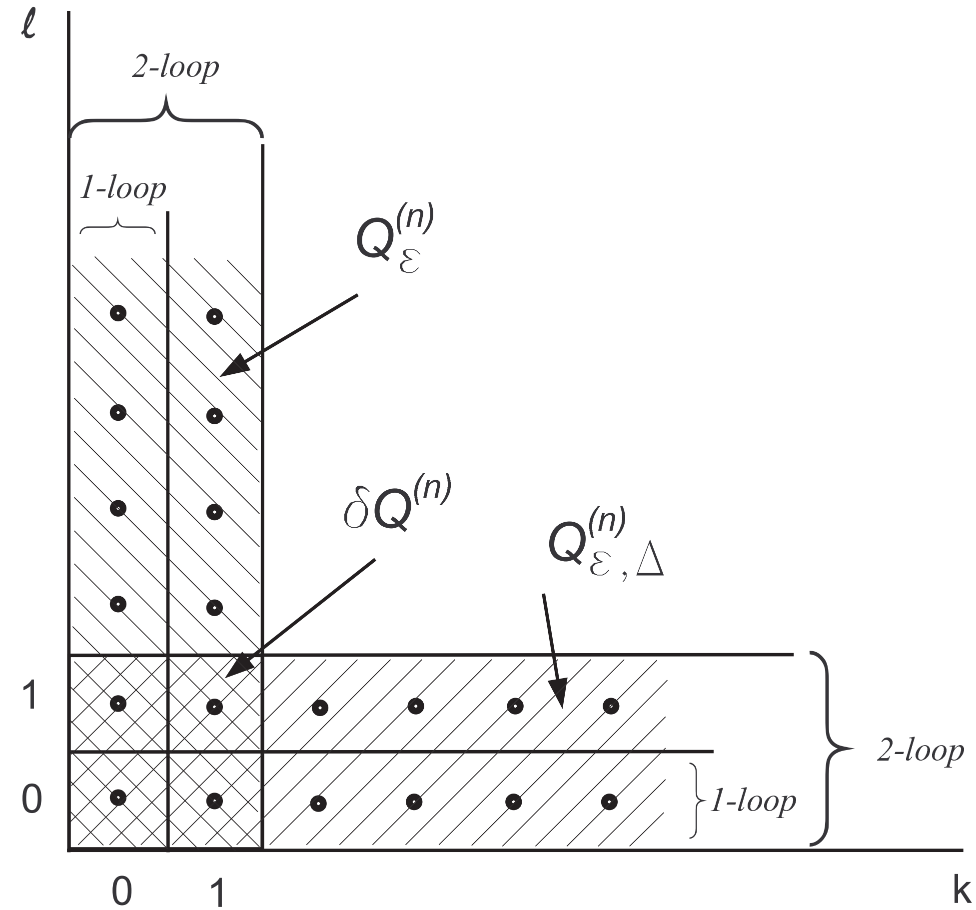

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Improved two-loop Kolmogorov constant

Comparison of one-loop and two-loop results for C_K:

n	C_ε	$C_{\varepsilon,\Delta}$	C_δ	C_{eff}
1	1.47	1.68	1.37	1.79
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Recommended experimental value: $C_K = 2.0$ (Sreenivasan, 1995).

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At $\varepsilon = 2$ the turbulent Prandtl number Pr$_t$ close to accepted experimental value Pr$_t \approx 0.81$:

$$\mathrm{Pr}_t^{(0)} \simeq 0.72\,, \qquad \mathrm{Pr}_t \simeq 0.77\,.$$

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- significant improvement of numerical results