General Computer Algebra Based Approach to Systems of PDEs

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Contents

RGS Method

The Renormgroup Symmetry (RGS) method invented in 90-s (Kovalev, Pustovalov, Shirkov) combines the classical Lie symmetry method and the Renormgroup (RG) method to investigate and solve PDEs (and also integro-differential equations) together with boundary conditions (BCs) of Cauchy type in mathematical physics. The RGS method:

- Specification of RG-manifold (PDEs + parameters in eqs. and BCs).
- Finding generators of Lie symmetry admitted by RG-manifold.
- Restriction of the symmetry to solutions of PDEs+BCs.
- Construction of invariant solutions.

In practice, except small problems, application of this approach is very hard computationally, and needs computer algebra assistance.

Possible Algorithmization

But what can we hope to do algorithmically in the general polynomially nonlinear case of differential equation systems?

- Check compatibility, i.e., consistency.
- Detect arbitrariness in general (analytical) solution.
- Eliminate a subset of variables.
- Check if an extra equation is a consequence of the initial equations.
- Find algebra of infinitesimal Lie or RG symmetries.
- Reduce the problem to (a finite set) of "smaller" problems.
- Formulate a well-posed initial value problem (PDEs).
- ....

Universal Algorithmic Approach

Is there a "universal" algorithmic tool for the listed subproblems?

If the system has polynomial nonlinearity in unknowns with "algorithmically computable" coefficients, then such a tool exists and based on transformation of the system into another set of equations with certain "nice" properties.

For linear PDEs such a form is canonical, i.e., uniquely defined by the initial systems and an order on the variables, and called reduced Gröbner basis (GB) (Buchberger, Winkler'98).

Another "nice" canonical form is called Involutive Basis (IB) (Gerdt, Blinkov'98). IB is also GB, although (in most cases) redundant as a Gröbner one.

Nonlinear PDEs can be split ((Thomas'37, Gerdt'98) into a finitely many involutive subsystems.

Cauchy-Kovalevskaya theorem

A normal system of PDEs

$$\frac{\partial^{m_j} u_j}{\partial x_1^{m_j}} = f_j\left(x, u, \ldots, \frac{\partial^{\mu_1+\cdots+\mu_n} u}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}}\right) \quad (1 \le j \le k)$$

$x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_k)$, $\sum_{i=1}^n \mu_i \le m_j$, $\mu_1 < m_j \ge 1$ which is analytic at

$$x_i = x_i^o,\ \ u_j = u_j^o,\ \ \frac{\partial^{\mu_1+\cdots+\mu_n} u_j}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} = p^o_{j;\mu_1\cdots\mu_n},\ \ (1 \le i \le n,\ \ 1 \le j \le k)$$

has a unique analytic solution at $(x_1^0, \ldots, x_n^o)$ satisfying the initial data

$$\begin{aligned}
& u_j = \phi_j(x_2, \ldots, x_n), \\
& \frac{\partial u_j}{\partial x_1} = \phi_j^{(1)}(x_2, \ldots, x_n), \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1 \le j \le k) \\
& \frac{\partial^{m_j-1} u_j}{\partial x_1^{m_j-1}} = \phi_j^{(m_j-1)}(x_2, \ldots, x_n),
\end{aligned}$$

for $x_1 = x_1^o$ with functions $\phi_j, \ldots, \phi_j^{(m_j-1)}$ analytic at $\{x_2^o, \ldots, x_n^o\}$.
Normal systems are particular cases of involutive systems.

Constructive Theory of Involution

Cartan (1899, 1901): Involutivity of Pfaff type equations. Kähler (1934): generalization to arbitrary exterior PDEs.
Riquier (1910), Janet (1920), Thomas (1937): Involutivity of PDEs.
Spencer (1965), Kuranishi (1967), Goldschmidt (1969), Pommaret (1978): Formal Theory of differential systems.
Reid (1991): Standard Form of linear PDEs.
Wu (1991): Relation of Riquier-Janet theory to Gröbner bases.
Zharkov, Blinkov (1993): Pommaret Bases of polynomial ideals.
Gerdt, Blinkov (1996): Involutive Separation / Monomial Division $\Longrightarrow$ Involutive Bases.
Reid, Wittkopf, Boulton (1996): Reduced Involutive Form of PDEs.
Gerdt (1999): Involutive Systems of Linear PDEs.
Seiler (2002): Combinatorial Aspects of Involutivity.
Chen, Gao (2002): Involutive Characteristic Sets for PDEs.
Gerdt, Blinkov (2005): Janet-like Monomial Division
Gerdt (2008): Involutive Nonlinear PDEs

Implementation

Arais, Shapeev, Yanenko (1974): Cartan algorithm in Auto-Analytik.
Schwarz (1984): Riquier-Janet theory in Reduce.
Hartley, Tucker (1991): Cartan algorithm in Reduce.
Schwarz (1992): Janet bases for linear PDEs in Reduce.
Reid, Wittkopf, Boulton (1993): Standard Form and Rif (2000) in Maple.
Seiler (1994): Formal theory in Axiom.
Zharkov, Blinkov (1993); Gerdt, Blinkov (1995): Pommaret bases in Reduce.
Kredel (1996): Pommaret bases in MAS.
Nischke (1996): Polynomial Janet and Pommaret bases in C++ (PoSSoLib).
Berth (1999): Polynomial and differential involutive bases in Mathematica.
Cid (2000)-Roberts (2002) Polynomial and linear differential Janet bases in Maple.
Gerdt, Blinkov, Yanovich (2000-02): Polynomial Janet bases in Reduce, C/C++.
Hausdorf, Seiler (2000-2002): Janet and Pommaret bases in MuPAD.
Chen, Gao (2002): Involutive Extended Characteristic Sets in Maple.
Hemmecke (2002): Sliced Involutive Algorithm in Aldor.
Blinkov (2005): Janet-like Bases in C++.
Robertz (2005) Janet-like polynomial, linear differential and difference bases in Maple.

Integrability Conditions

Let $\mathcal{R}_q$ be a system of PDEs of order q in n independent variables x_i $(1 \leq i \leq n)$ and m dependent variables u^α $(1 \leq \alpha \leq m)$

$$\mathcal{R}_q : \left\{ \Phi_j(x_i, u^\alpha, u^\alpha_\mu) = 0 \quad (1 \leq j \leq k) \right. \qquad \text{manifold in } \{\, u^\alpha_{|\mu|\leq q} \,\}$$

where $\mu = \{\mu_1, \ldots, \mu_n\}$ is multi-index, $|\mu| = \sum_{i=1}^n \mu_i \leq q$ and

$$u^\alpha_\mu = \frac{\partial^{|\mu|} u^\alpha}{\partial x^\mu} \equiv \frac{\partial^{\mu_1+\cdots+\mu_n} u^\alpha}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}}\,, \qquad u^\alpha_{|\mu|=0} = u^\alpha$$

Definition. An integrability condition for $\mathcal{R}_q$ is an equation of order $\leq q$ which is differential but not pure algebraic consequence of $\mathcal{R}_q$.

Example (Seiler'94)

$$\mathcal{R}_1 : \left\{ \begin{array}{l} u_z + y\,u_x = 0 \\ u_y = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} u_{yz} + y\,u_{xy} + u_x = 0 \\ u_{xy} = u_{yz} = 0 \end{array} \right. \Longrightarrow \boxed{u_x = 0}$$

$$\Longrightarrow \mathcal{R}_1 : \{u_x = u_y = u_z = 0$$

Geometric Constructions)

Definition. The 1st prolongation $\mathcal{R}_{q+1}$ of $\mathcal{R}_q$

$$\mathcal{R}_{q+1} : \begin{cases} f_j(x_i, u^\alpha, p^\alpha_\mu) = 0 \quad (1 \leq j \leq k) \\ D_i\Phi_j = 0 \quad (1 \leq i \leq n) \end{cases}$$

where D_i is the total derivative operator w.r.t. x_i. Similarly, $\mathcal{R}_{q+r}$ is obtained by r prolongations of $\mathcal{R}$.

Definition. $\mathcal{R}^{(1)}_q = \pi^{q+1}_q\left(\mathcal{R}_{q+1}\right)$ is the projection of $\mathcal{R}_{q+1}$ in $\{\, u^\alpha_{|\mu|\leq q} \,\}$. Similarly, $\mathcal{R}^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(\mathcal{R}_{q+r+s})$ is obtained from $\mathcal{R}_q$ by $r+s$ prolongations and s projections.

Generally,

$$\mathcal{R}^{(1)}_{q+r} \subseteq \mathcal{R}_{q+r} \Longrightarrow dim\, \mathcal{R}^{(1)}_{q+r} \leq dim\, \mathcal{R}_{q+r}$$

and the number of (algebraically independent) integrability conditions which arise at the $(r+1)$ prolongation step is

$$dim\, \mathcal{R}_{q+r} - dim\, \mathcal{R}^{(1)}_{q+r}$$

Formal Integrability and Involutivity

Definition. A formally integrable system $\mathcal{R}_q$ has all the integrability conditions incorporated in it, that is,

$$(\forall\, r, s)\ \ [\, \mathcal{R}^{(s)}_{q+r+s} = \mathcal{R}_{q+r}\,]$$

Involutive system $\mathcal{R}_q$ is a formally integrable one with the complete (involutive) set of the leading derivatives (symbol of $\mathcal{R}_q$).

Definition. Given a system $\mathcal{R}_q$, its transformation into an involutive form is called completion.

Theorem (Cartan-Kuranishi-Rashevsky) For every consistent differential system $\mathcal{R}_q$, under certain regularity requirements, there exist integers r, s such that $\mathcal{R}^{(s)}_{q+r}$ is involutive with the same solution space.

Ranking

Definition. A total ordering $\prec$ over the set of derivatives $\partial_\mu u^\alpha$ is a ranking if $\forall\,, i, \alpha, \beta, \mu, \nu$ it satisfies

1. $\partial_i \partial_\mu u^\alpha \succ \partial_\mu u^\alpha$
2. $\partial_\mu u^\alpha \succ \partial_\nu u^\beta \quad \Longleftrightarrow \quad \partial_i \partial_\mu u^\alpha \succ \partial_i \partial_\nu u^\beta$

If $\mu \succ \nu \Longrightarrow \partial_\mu u^\alpha \succ \partial_\nu u^\beta$ the ranking is orderly, and if $\alpha \succ \beta \Longrightarrow \partial_\mu u^\alpha \succ \partial_\nu u^\beta$ the ranking is elimination.

We shall use the association between derivatives and monomials

$$\partial_\mu u^\alpha \equiv \frac{\partial^{\mu_1 + \cdots + \mu_n} u^\alpha}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} \Longleftrightarrow [x_1^{\mu_1} \cdots x_n^{\mu_n}]_\alpha$$

such that monomials associated with the different dependent variables u^α belong to different monomial sets U_α.

Given a ranking and $\mathcal{R}_q$, we obtain the set of leading derivatives in $\mathcal{R}_q$ $\Longrightarrow$ the m finite subsets of ("leading") monomials.

Involutive Partition of Variables

Observation (*Janet'20*) For every equation $f \in \mathcal{R}_q$ one can partition the set of independent variables into two subsets called multiplicative and nonmultiplicative such that the integrability conditions are generated by prolongations of $\mathcal{R}_q$ w.r.t. nonmultiplicative variables. Multiplicative prolongations provide elimination of the highest order derivatives (projection).

Definition. (Janet partition) Let V be a set of monomials associated with the leading derivatives in $\mathcal{R}_q$ for some fixed dependent variable u^α. Arrange elements in V in groups as follows $(d_1 > d_2 > \cdots > d_k)$

$$\begin{array}{rcll} \sum_{v\in V} v & = & x_1^{d_1} \cdot (\ \dots\ ) & x_1 - \text{multiplicative} \\ & + & x_1^{d_2} \cdot (\ \dots\ ) & x_1 - \text{nonmultiplicative} \\ & \vdots & \dots\dots\dots\dots & x_1 - \text{nonmultiplicative} \\ & + & x_1^{d_k} \cdot (\ \dots\ ) & x_1 - \text{nonmultiplicative} \end{array}$$

For x_2 this rule is recursively applied to every bracket (...), etc.

General theory of algorithmically acceptable partition of variables together with completion algorithms was developed in (Gerdt,Blinkov'98).

Basic Definitions

Let system

$$F = \{\, f_j(x_i, u^\alpha, \ldots, u^\alpha_\mu) \mid 1 \le i \le n,\ 1 \le j \le k,\ 1 \le \alpha \le m \,\}$$

be a set of differential polynomials, i.e. polynomials in u^α and its derivatives, and $\succ$ be a ranking. Then every element $f \in F$ is a polynomial in its highest ranking partial derivative (leader) $\mathrm{ld}(f)$

$$f = a_0\, \mathrm{ld}(f)^d + a_1\, \mathrm{ld}(f)^{d-1} + \cdots + a_d$$

$0 \neq a_0$ is initial of f ($\mathrm{init}(f)$) and $\partial_{\mathrm{ld}(f)} f$ is separant of f ($\mathrm{sep}(f)$).

Remark. For well-posedness (correctness) of Cauchy problem for the system $\{f = 0 \mid f \in F\}$ the conditions $\mathrm{init}(f) \neq 0$ and $\mathrm{sep}(f) \neq 0$ must hold (on the solutions of the system) for every $f \in F$. By this reason we shall consider systems of equations and inequations.

Algebraically Simple Systems

Definition. Let P and Q be finite sets of differential polynomials such that $P \neq \emptyset$ and contains equations $(\forall p \in P \mid p = 0)$ whereas Q contains inequations $(\forall q \in Q \mid q \neq 0)$. Then the pair $\langle P, Q \rangle$ of sets P and Q is differential system.

Let $\mathscr{DZ}(P/Q)$ and $\mathscr{Z}(P/Q)$ be respectively the set of differential and algebraic (if we consider elements in P and Q as algebraic polynomials in $u^\alpha, \ldots, u^\alpha_\mu$ over the algebraically closed coefficient field) "roots" of P not annihilating elements $q \in Q$ and $F_{\prec r} := \{ f \in F \mid \mathrm{ld}(f) \prec \mathrm{ld}(r) \}$.

Definition. (Thomas'37) A differential system $\langle P, Q \rangle$ is algebraically simple if

1. $\forall\, r \in \langle P, Q \rangle, \forall\, \mathbf{x} \in \mathscr{Z}(P_{\prec r}/Q_{\prec r}) \mid \mathrm{init}(r)(\mathbf{x}) \neq 0$;
2. $\forall\, r \in \langle P, Q \rangle, \forall\, \mathbf{x} \in \mathscr{Z}(P_{\prec r}/Q_{\prec r}) \mid r(\mathrm{ld}(r), \mathbf{x})$ is a squarefree (no multiple roots) polynomial in $\mathrm{ld}(r)$;
3. elements in $\langle P, Q \rangle$ have pairwise different leaders.

Decomposition into Simple Subsystems

Theorem. (Thomas'37,62) Any differential system $\langle P, Q\rangle$ in finitely many steps can be decomposed into a set of algebraically simple subsystems $\langle P, Q\rangle$ such that

$$\mathscr{DZ}\,(P/Q) = \bigcup_i \mathscr{DZ}\,(P_i/Q_i), \quad \mathscr{DZ}\,(P_i/Q_i) \bigcap_{i\neq j} \mathscr{DZ}\,(P_j/Q_j) = \emptyset.$$

The decomposition is done fully algorithmically (Wang'98,Gerdt'08).

Remark Prolongation preserves the first two simplicity properties. Due to this fact one can algorithmically complete simple components to involution by doing further decomposition in the course of completion if necessary (Gerdt'08). As a result, any differential system can be fully algorithmically decomposed into algebraically simple and involutive subsystems.

Principal and Parametric Derivatives

Now we assume that differential system $\langle P, Q\rangle$ is algebraically simple, involutive for an orderly ranking $\succ$ and autoreduced, i.e. every $f \in \langle P, Q\rangle$ does not contain derivative of a leaders of equationin P.

Definition. Derivative u^α_μ of the dependent variable u^α (as well as u^α itself) will be called of class α. Derivative u^α_μ occurring in P as a leader $(\,\exists p \in P \mid u^\alpha_\mu = \mathrm{ld}(p)\,)$ is called principal and derivative u^β_ν that does not occur among leaders and is not a prolongation of a leader of class β is called parametric.

Denote by $M_J(p, P)$ and $NM_J(p, P)$ multiplicative and nonmultiplicative variables for $p \in P$ according to the Janet partition. For every parametric derivative $q := u^\beta_\nu$ define Janet partition of variables as

$$M_J(q) := M_J(q, q \cup P), \quad NM_J(q) := M_J(q, q \cup P)$$

Cauchy Data

Lemma. The set V_α of parametric derivatives of class α $(1 \leq \alpha \leq m)$ can be decomposed as the following disjoined union

$$V_\alpha = \bigcup_{v \in \mathcal{V}_\alpha} \bigcup_{D_v} D_v \circ v$$

where D_v is the set of all multiplicative prolongations (derivations) of v w.r.t. its variables Janet multiplicative variables and $\mathcal{V}_\alpha$ is a finite set.

Elements v in the decomposition are called generators of set V_α. They can be found algorithmically for every α.

Theorem (Finikov'48) An involutive and algebraically simple system has unique solution if generators with nonempty sets of multiplicative variables are arbitrary functions of these variables at the fixed values of the nonmultiplicative variables from the initial point $x_i = x_i^o$, and generators having no multiplicative variables take arbitrary constant values. The values of arbitrary functions at the initial point together with the constants must satisfy the system.

Lie Symmetries

Given a finite system of PDEs

$$f_k(x_i, y_j, \ldots, \partial_\alpha y_j) = 0, \ \ (1 \leq k \leq r)$$

one looks for one-parameter infinitesimal transformations

$$\left\{ \begin{array}{l} \tilde{x}_i(\lambda) = x_i + \xi_i(x_i, y_j)\lambda + O(\lambda^2), \\ \tilde{y}_j(\lambda) = y_j + \eta_j(x_i, y_j)\lambda + O(\lambda^2), \end{array} \right.$$

that preserve the form of the system.

The invariance conditions are

$$\left\{ \begin{array}{l} \mathcal{K}^{(\alpha)} f_k(x_i, y_j, \ldots, \partial_\alpha y_j)|_{f_k=0} = 0, \Longrightarrow \text{Determining Linear PDEs in } \xi_i, \eta_j \\ \mathcal{K}^{(\alpha)} = \xi_i \partial_{x_i} + \eta_j \partial_{y_j} + \zeta_{j;i} \partial_{y_{j;i}} + \cdots + \zeta_{j;\alpha} \partial_{y_{j;\alpha}} \end{array} \right.$$

Here $\partial_i y_j$ denoted by $y_{j;i}$, etc.

Example

Example. The Harry Dym equation ($n = 2, m = 1$)

$$\partial_t y - y^3 \partial_{xxx} y = 0$$

The symmetry operator is now determined by the system

$$\begin{aligned}
&\partial_y \xi_1 = 0, \quad \partial_x \xi_1 = 0,\ \partial_y \xi_2 = 0,\ \partial_{yy}\eta = 0,\\
&\partial_{xy}\eta - \partial_{xx}\xi_2 = 0,\ \partial_t \eta - y^3 \partial_{xxx}\eta = 0,\\
&3\,y^3 \partial_{xxy}\eta + \partial_t \xi_2 - y^3 \partial_{xxx}\xi_2 = 0,\ y\,\partial_t \xi_1 - 3\,y\,\partial_x \xi_2 + 3\,\eta = 0.
\end{aligned}$$

Its Janet involutive form for the orderly ranking with $\partial_y \succ \partial_x \succ \partial_t$, $\xi_1 \succ \xi_2 \succ \eta$ is

$$\begin{aligned}
&\partial_{xx}\eta = 0,\ \ \partial_{xt}\eta = 0,\ \ \partial_y \eta - \frac{1}{y}\,\eta = 0,\ \ \partial_t \eta = 0, \quad \partial_y \xi_2 = 0,\\
&\partial_t \xi_2 = 0,\ \partial_{tt}\xi_1 = 0,\ \partial_y \xi_1 = 0,\ \partial_x \xi_1 = 0,\ \partial_x \xi_2 - \frac{1}{3}\,\partial_t \xi_1 - \frac{1}{y}\,\eta = 0.
\end{aligned}$$

Example (cont.)

There are five generators of parametric derivatives $\xi_1, \partial_t\xi_1, \xi_2, \eta, \partial_x\eta$. All of them have no multipliers $\Longrightarrow$ the general solution depends on five arbitrary constants $\Longrightarrow$ the five-dimensional Lie symmetry group.

The involutive determining system in this example is also easy to integrate:

$$\xi_1 = c_1 + c_2 t, \quad \xi_2 = c_3 + c_4 x + c_5 x^2, \quad \eta = (c_4 - \frac{1}{3}\, c_2 + 2\, c_5 x)\, y$$

This gives the Lie symmetry operators

$$Z_1 = \partial_t,\ Z_2 = t\partial_t - \frac{1}{3} y\partial_y,\ Z_3 = \partial_x,\ Z_4 = x\partial_x + y\partial_y,\ Z_5 = x^2\partial_x + 2xy\partial_y$$

generating Lie algebra

$$[Z_1, Z_2] = Z_1, \quad [Z_3, Z_4] = Z_4, \quad [Z_3, Z_5] = 2\, Z_4, \quad [Z_4, Z_5] = Z_5.$$

Navier-Stokes Equations in R^2

$$\begin{cases} u_t + u\,u_x + v\,u_y = -\frac{1}{\rho}p_x + \nu(u_{xx} + u_{yy}), \\ v_t + u\,v_x + v\,v_y = -\frac{1}{\rho}p_y + \nu(v_{xx} + v_{yy}), \\ u_x + v_y = 0. \end{cases}$$

Here (u, v) is the velocity field, p is the pressure, $\rho > 0$ is the constant density (incompressible fluid) and $\nu > 0$ is the constant kinematic viscosity. For the orderly ranking with $t \succ x \succ y$, and $u \succ v \succ p$ the Janet involutive form is given by

$$\begin{cases} \nu\,\underline{v_{xx}} + \nu\,v_{yy} - v_t - u\,v_x - v\,v_y - \frac{1}{\rho}\,p_y = 0, \\ \nu\,\underline{v_{xy}} - \nu\,u_{yy} + u_t - u\,v_y - v\,u_y + \frac{1}{\rho}\,p_x = 0, \\ \frac{1}{\rho}\,\underline{p_{xx}} + \frac{1}{\rho}\,p_{yy} + 2\,v_x u_y + v_y^2 = 0, \\ \underline{u_x} + v_y = 0. \end{cases}$$

The 3rd equation is an integrability condition and is the well-known Poisson equation for the pressure. This equation plays an important role in numerical analysis of the Navier-Stokes equations.

Cauchy Conditions

Given a Janet basis for a system of PDEs analytic at some point with $x_i = x_i^o$, one can formulate initial value problem providing existence and uniqueness of an analytic solution much like the Cauchy-Kovalevskaya theorem.

For the Navier-Stokes equations it yields the initial conditions

Function	Generators	Multiplicative variables	Initial data
u	u	y, t	$u\|_{x=x_o} = \phi_1(y, t)$
v	v	y, t	$v\|_{x=x_o} = \phi_2(y, t)$
	v_x	t	$\partial_x v\|_{x=x_o, y=y_o} = \phi_3(t)$
p	p	y, t	$p\|_{x=x_o} = \phi_4(y, t)$
	p_x	y, t	$\partial_x p\|_{x=x_o} = \phi_5(y, t)$

with 5 arbitrary functions: 4 functions of two variables and 1 function of one variable.

Decomposition of Nonlinear Systems: Example

$$\left\langle \begin{array}{l} (u_y + v)u_x + 4v\,u_y - 2v^2 \\ (u_y + 2v)u_x + 5v\,u_y - 2v^2 \end{array}, \emptyset \right\rangle$$

$$\Downarrow$$

algebraically simple subsystems

$$\left\langle \begin{array}{l} (u_y + v)u_x + 4v\,u_y - 2v^2 \\ u_y^2 - 3u_y + 2v^2 \end{array}, v \right\rangle \bigcup \left\langle \begin{array}{l} u_x \\ v \end{array}, u_y \right\rangle \bigcup \left\langle \begin{array}{l} u_y \\ v \end{array}, \emptyset \right\rangle$$

$$\Downarrow$$

involutive and algebraically simple subsystems

$$\left\langle \begin{array}{l} (u_y + v)u_x + 4v\,u_y - 2v^2 \\ u_y^2 - 3u_y + 2v^2 \\ v_x + v_y \end{array}, v \right\rangle \bigcup \left\langle \begin{array}{l} u_x \\ v \end{array}, u_y \right\rangle \bigcup \left\langle \begin{array}{l} u_y \\ v \end{array}, \emptyset \right\rangle$$

$$\Downarrow$$

Cauchy conditions

$$\left\{ \begin{array}{l} u(x_o, y_o) = C \\ v(x_o, y) = \phi(y) \neq 0 \end{array} \right\} \{\, u(x_0, y) = \psi(y),\ \psi'_y \neq 0 \,\} \{\, u(x, y_o) = \xi(x) \,\}$$

Implementations of GB/IB/Decomposition

Software	Commutative algebra	PDE	Language
Maple	+	diffalg	Maple
		Rif	Maple
	Gb		C
	FGb		C
Mathematica	+	–	C
Reduce	+	–	Lisp
Epsilon	Zero Decom.	ODE	Maple
OreModules	–	LPDE	Maple
Janet	–	LPDE	Maple
LDA	–	–	Maple
GINV	+		Python/C++
JB	+	–	C

Conclusions

- Completion of differential systems to involution is the most general and universal technique for study their algebraic properties. In particular, to pose Cauchy problem and to integrate determining systems for infinitesimal Lie and RG symmetries.
- Linear PDEs admit algorithmic completion to involution whereas nonlinear PDEs admit algorithmic spliting into involutive subsystems.
- Involutive systems have all their integrability conditions incorporated in them that makes easier their qualitative and quantitative analysis.
- Special algorithms for completion to involution have been designed and (partially) implemented.