On the correlation numbers in Minimal Gravity and Matrix Models

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Two approaches to 2D quantum geometry

Continuous approach | Discret approach

$\downarrow$ | $\downarrow$

"Liouville Gravity" | "Matrix Models"

Impressive body of evidence that the two describe the same reality:

- Operators O_k^{LG} and O_k^{MM} have identical scale dimensions
- Some correlation numbers coincide:

$$\langle\, O_1^{LG} ... O_n^{LG}\,\rangle = \langle\, O_1^{MM} ... O_n^{MM}\,\rangle$$

But with "naive" identification many correlation numbers are not in agreement.

Resolution [*Moore, Seiberg, Staudacher, 1991*]: **Resonance relations**:

$$[O_k] \ = [\tau_{k_1}][O_{k_2}]$$

$$\downarrow$$

$$\text{Umbiguity} \quad O_k^{MM} = O_k^{LG} + B_k^{k_1 k_2}\, \tau_{k_1}\, O_{k_2}^{LG}$$

- In many cases the disagreement can be fixed by adjusting the parameters (e.g. $B_k^{k_1 k_2}$ above).

• This work: Trying to find exact map for special class of models:

$$\text{"Minimal Gravity"}\ \mathcal{MG}_{2/2p+1} \leftrightarrow \begin{array}{c}\text{"p} - \text{criticality" in}\\ \text{One} - \text{Matrix Model}\end{array}$$

∘ The problem is rather "rigid" (more constraints then the parameters).

∘ Nonetheless, the map exists up to the level of four point corr. numbers.

∘ The resulting 1-, 2-, 3-, and 4-point correlation numbers are in perfect agreement.

1. Minimal Gravity

1.1. Quantum Geometry

$$\sum_{\text{topologies}} \int D[g]\, D[\phi]\, e^{-S[g,\phi]}$$

$g(x)$ - Riemannian metric on 2D manifold $\mathbb{M}$ (assume sphere), ϕ - "matter" fields

Invariant correlation functions ("correlation numbers"):

$$\langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_N} \,\rangle = Z^{-1} \int \tilde{O}_{k_1} ... \tilde{O}_{k_N}\, e^{-S[g,\phi]}\, D[g,\phi]$$

with

$$\tilde{O}_k = \int_{\mathbb{M}} O_k(x)\, d\mu_g(x)$$

$O_k(x)$ - local fields (built from ϕ and g).

Generating function: $\{\tau\} = \{\tau_1, ..., \tau_n\}$

$$W(\{\tau\}) = Z(\{\tau\})/Z(\{0\}),\ Z(\{\tau\}) = \int D[g,\phi]\, e^{-S_\tau[g,\phi]}\,,$$

$$S_\tau[g,\phi] = S_0[g,\phi] + \sum_k \tau_k\, \tilde{O}_k$$

so that

$$\langle \tilde{O}_{k_1} ... \tilde{O}_{k_N} \rangle = \left. \frac{\partial^N W(\{\tau\})}{\partial \tau_{k_1} ... \partial \tau_{k_N}} \right|_{\tau=0}$$

The parameters $\{\tau\}$ may be regarded as the coordinates in the "theory space" Σ.

1.2. Conformal Matter, and Liouville Gravity

$$g^{\mu\nu}\, T_{\mu\nu}^{\text{matter}} = -\frac{c}{12}\, R$$

Conformal Gauge $g_{\mu\nu} = e^{2b\varphi}\,\widehat{g}_{\mu\nu}$: $\quad\Rightarrow\quad$ Decoupling

$$S[g,\phi] \;\to\; S_{\text{L}}[\varphi] + S_{\text{Ghost}}[B,C] + S_{\text{Matter}}[\phi]$$

with

$$S_L[\phi] = \frac{1}{4\pi}\int \sqrt{\widehat{g}}\,\left[\widehat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\,\widehat{R}\,\varphi + 4\pi\mu\, e^{2b\,\varphi}\right] d^2x\,,$$

$$S_{\text{Ghost}}[B,C] = \frac{1}{2\pi}\int \sqrt{\widehat{g}}\; B_{\mu\nu}\,\nabla^{\mu}C^{\nu}\, d^2x\,,$$

$\Big(B_{\mu\nu} = B_{\nu\mu}\,, \quad \widehat{g}^{\mu\nu}B_{\mu\nu} = 0\Big)$,

$$26 - c = 1 + 6\,Q^2 \qquad\qquad Q = b + 1/b\,.$$

($S_{\text{Matter}}[\phi]$ is conformally invariant, with the central charge c).

Correlation numbers $\langle \tilde{O}_{k_1} ... \tilde{O}_{k_N} \rangle$ with

$$\tilde{O}_k = \int V_k(x) \, \Phi_k(x) \, d^2 x$$

$\Phi_k(x)$ - (spinless) primary fields of the matter CFT, with the conformal dimensions (Δ_k, Δ_k) $V_k(x)$ - "gravitational dressings",

$$V_k(x) = e^{2a_k \varphi(x)} \,, \qquad a_k(Q - a_k) + \Delta_k = 1$$

Gravitational dimensions of $\tilde{O}_k$ control the scale dependence of the corr. functions:

$$\tilde{O}_k \;\sim\; \mu^{\delta_k} \,, \qquad \delta_k = -\frac{a_k}{b}$$

1.3. Correlation numbers

$$\langle \tilde{O}_{k_1} ... \tilde{O}_{k_n} \rangle = |(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)|^2 \times$$

$$\int d^2 x_4 ... d^2 x_n \underbrace{\langle O_{k_1}(x_1) O_{k_2}(x_2) O_{k_3}(x_3) O_{k_4}(x_4) ... O_{k_n}(x_n) \rangle}$$

$$\downarrow$$

$$\langle V_{k_1}(x_1) ... V_{k_n}(x_n) \rangle_{\text{Liouville}} \; \langle \Phi_{k_1}(x_1) ... \Phi_{k_n}(x_n) \rangle_{\text{Matter}}$$

• The Liouville correlation functions are expressed in terms of the "Conformal Blocks", e.g.

$$\langle V_{k_1}(x_1)...V_{k_4}(x_4)\rangle_{\text{Liouville}} =$$

$$\int \frac{dP}{4\pi} C_L\Big(a_{k_1}, a_{k_2}, Q/2 + iP\Big) C_L\Big(Q/2 - iP, a_{k_3}, a_{k_4}\Big) \times |\mathcal{F}_{\Delta(P)}(1-\Delta_i|x_i)|^2$$

with $\Delta(P) = Q^2/4 + P^2$, and the "Liouville Structure Constants"

$$C_L(a_1, a_2, a_3) = \Big(\pi\mu\,\gamma(b^2)\Big)^{(Q-a)/b} \frac{\Upsilon_b(b)}{\Upsilon_b(a-Q)} \prod_{i=1}^{3} \frac{\Upsilon_b(2a_i)}{\Upsilon_b(a-a_i)}$$

where $a = a_1 + a_2 + a_3$,

$$\log\Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{(Q-2x)^2}{4}\, e^{-2t} - \frac{\sinh^2((Q/2-x)t)}{\sinh(bt)\sinh(t/b)}\right]$$

• Integration over the moduli $x_4, ..., x_n$ is to be performed.

1.4. Matter CFT: "Minimal Models"

$$\mathcal{M}_{p/q} \qquad c = 1 - 6\,\frac{(p-q)^2}{pq}$$

Finite number of primary fields

$$\Phi_{(n,m)} \quad (n = 1, ..., p-1\,, \quad m = 1, ..., q-1\,, \quad n \leq m)\,,$$

with (in principle) computable correlation functions, e.g.

$$\langle\, \Phi_{(n_1,m_1)}(x_1)...\Phi_{(n_4,m_4)}(x_4)\,\rangle_{MM} =$$

$$\sum_{(n,m)} \mathbb{C}^{(n,m)}_{(n_1,m_1)(n_2,m_2)}\,\mathbb{C}^{(n,m)}_{(n_3,m_3)(n_4,m_4)} |\mathcal{F}_{(n,m)}(\Delta_i|x)|^2$$

Fusion rules:

$$\Phi_{(n_1,m_1)}\Phi_{(n_2,m_2)} = \sum_{n=|n_1-n_2|+1}^{N}\ \sum_{m=|m_1-m_2|+1}^{M} [\Phi_{(n,m)}]\,,$$

with

$$N = \min(n_1 + n_2 - 1, 2p - n_1 - n_2 - 1)\,,$$

$$M = \min(m_1 + m_2 - 1, 2q - m_1 - m_2 - 1)$$

1.5. "Yang-Lee series" of the Minimal Models $\mathcal{M}_{2/2p+1}$

- $\mathcal{M}_{2/2p+1}$ has p primary fields

$$\Phi_k \equiv \Phi_{(1,k+1)}, \qquad k = 0, 1, ..., p-1 \quad (p, p+1, ..., 2p-1)$$

Fusion rules

$$[\Phi_{k_1}][\Phi_{k_2}] = \sum_{k=|k_1-k_2|\,:\,2}^{k_1+k_2} [\Phi_k], \qquad [\Phi_k] = [\Phi_{2p-k-1}]$$

$$\text{"Parity"}: \qquad \Phi_k \qquad \begin{cases} + \text{ for even } k \\ - \text{ for odd } k \end{cases}$$

$$\Phi_k = \Phi_{2p-k-1} \quad \rightarrow \quad \text{"Parity violation"}$$

- Correlation functions:

$$\langle\, \Phi_k \,\rangle = \delta_{k,0}, \qquad \langle\, \Phi_k \Phi_{k'} \,\rangle \ \sim \ \delta_{k,k'}$$

$$\langle\, \Phi_{k_1}\Phi_{k_2}\Phi_{k_3} \,\rangle = 0$$

$$\text{if} \quad \begin{cases} k_1 + k_2 < k_3, \ \text{etc}, & \text{for} \quad k_1+k_2+k_3 \ \text{ even} \\ k_1+k_2+k_3 < 2p-1 & \text{for} \quad k_1+k_2+k_3 \ \text{ odd} \end{cases}$$

$$\langle \Phi_{k_1} ... \Phi_{k_n} \rangle = 0$$

$$\text{if} \quad \begin{cases} k_1 + ... + k_{n-1} < k_n, & \text{for} \quad k_1 + ... + k_n \ \text{ even} \\ k_1 + ... + k_n < 2p - 1 & \text{for} \quad k_1 + ... + k_n \ \text{ odd} \end{cases}$$

- Interpretations:

$\mathcal{M}_{2/3}$ - "empty" theory (has only identity operator)
$\mathcal{M}_{2/5}$ - Yang-Lee edge criticality [*Cardy, 1985*]
$\mathcal{M}_{2/2p+1}$ - Yang-Lee multi-criticality?

1.6. Minimal gravity $\mathcal{MG}_{p/q}$:

$$\mathcal{M}_{p/q} \quad \text{coupled to the Liouville Gravity}$$

• Early computations of the correlation numbers: [*Goulian & Li, 1991; Di Francesco & Kutasov, 1991; ...*]

• Systematic approach [*Alexei Zamolodchikov, 2004; Belavin &Al.Zamolodchikov, 2006*]:

"Higher Liouville Equations of Motion"

$$\int_{\text{moduli}} [...] = \int_{\text{moduli}} [\text{total derivative}] \ \rightarrow \ \text{Boundary terms}$$

• Results for $\mathcal{MG}_{2/2p+1}$:

$$\tilde{O}_k = \int V_k(x)\, \Phi_k(x)\, d^2x\,, \qquad V_k(x) = e^{(k+2)b\,\varphi(x)}$$

with

$$b = \sqrt{2/(2p+1)}$$

⋆ One-point correlation numbers

$$\langle \mathcal{O}_k \rangle = 0\,,$$

⋆⋆ Two-point numbers

$$\langle \tilde{O}_k \tilde{O}_{k'} \rangle = \frac{\delta_{kk'}}{Z_p}\, \frac{1}{2p-2k-1}\, \mathrm{Leg}_L^2(k)\,,$$

with

$$Z_p = [(2p-1)(2p+1)(2p+3)]^{-1}$$

and

$$\mathrm{Leg}_L(k) = \frac{\left[\pi\mu\,\gamma\left(\frac{2}{2p+1}\right)\right]^{-\frac{k+2}{2}}}{2p-1}\left[\frac{\pi^2\,\gamma\left(\frac{2}{2p+1}\right)\gamma\left(\frac{2p+1}{2}\right)}{\gamma\left(\frac{2p-2k-1}{2p+1}\right)\gamma\left(\frac{2p-2k-1}{2}\right)}\right]^{1/2}$$

⋆ ⋆ ⋆ Three-point correlation numbers:

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3} \rangle = \frac{N_{k_1k_2k_3}}{Z_p}\, \prod_{i=1}^{3} \mathrm{Leg}_L(k_i)$$

where $N_{k_1k_2k_3}$ enforces the "fusion rules"

$$N_{k_1k_2k_3} = \begin{cases} 1 & \text{if the fusion rules of } \mathcal{M}_{2/2p+1} \text{ are satisfied} \\ 0 & \text{otherwise} \end{cases}$$

$\star\star\star\star$ Four-point correlation numbers:

$$\langle \tilde{O}_{k_1}\tilde{O}_{k_2}\tilde{O}_{k_3}\tilde{O}_{k_4}\rangle = \frac{\Sigma_{k_1k_2k_3k_4}}{Z_p}\prod_{i=1}^{4}\mathrm{Leg}_L(k_i)$$

$$\Sigma_{k_1\ldots k_4} = (k_1+1)(p+k_1+3/2) - \sum_{i=2}^{4}\ \sum_{s=-k_1:2}^{k_1} |p-1/2-k_i-s|$$

Applies when the number of conformal blocks in $\langle\Phi_{k_1}\ldots\Phi_{k_4}\rangle$ is exactly k_1. This holds for instance if

$$k_1 \le k_2 \le k_3 \le k_4\,, \qquad \text{and} \quad k_1+k_4 \le k_2+k_3\,.$$

$$\sum_{s=-k_1:2}^{k_1}\left|p-\frac{1}{2}-k_i-s\right| = (k_1+1)\Big(-2p+\frac{3}{2}+\sum_{i=1}^{4}k_i\Big)+\sum_{i=2}^{4}\tilde{F}_p(k_1+k_i)\,,$$

$$\tilde{F}_p(k) = \frac{(p-k-1)(p-k-2)}{2}\,\Theta(k-p)\,,\qquad \Theta(k) = \begin{cases}1 & \text{for} \quad k\ge 0\\ 0 & \text{for} \quad k<0\end{cases}$$

$\star\ldots\star$ Higher-point functions are (in principle) computable [*Belavin, Al.Zamolodchikov, unpublished*]

• Generating function: $\{\tau\} = \{\tau_1, \tau_2, ..., \tau_{p-1}\}$

$$W_{\mathcal{MG}}(\mu, \{\tau\}) = \left\langle \; \exp\left\{ -\sum_{i=1}^{p-1} \tau_i \, \tilde{O}_i \right\} \right\rangle_{\mathcal{MG}_{2/2p+1}}$$

The cosmological constant μ may be treated as $\mu = \tau_0$

$$S[\mathcal{MG}] = ... + \mu \underbrace{\int e^{2b\varphi(x)} \, d^2x}\, + ...$$

$$\tilde{O}_0 = \int V_0(x) \, \Phi_0(x) \, d^2x \,, \quad \Phi_0 = I$$

Dimensions:

$$\tau_k \;\sim\; \mu^{\frac{k+2}{2}} \,, \quad k = 0, 1, ..., p-1$$

By the definition

$$\langle \, \tilde{O}_{k_1} ... \tilde{O}_{k_n} \, \rangle = \left. \frac{\partial^n W_{\mathcal{MG}}(\mu, \{\tau_i\})}{\partial \tau_{k_1} ... \partial \tau_{k_n}} \right|_{\{\tau_i\}=0} , \qquad \{\tau_i\} = \{\tau_1, ..., \tau_n\}$$

2. Matrix Models

$$\underbrace{\text{Continuous (scaling) limit of the ensemble of planar graphs}}$$

$$\downarrow$$

Quantum Geometry

2.1. One-matrix Model The planar graphs = Feynmann diagrams associated with the perturbative evaluation of the matrix integral

$$Z = \log \int dM \; e^{-N \,\mathrm{tr}\left(\frac{1}{2} M^2 - \sum_{n=3} \frac{\alpha_n}{n!} M^n\right)}$$

M- Hermitian $N \times N$ matrix, N being the device for sorting out the topologies

$$Z = N^2 Z_0 + Z_1 + ... + N^{2-2g} Z_g + ...$$

Each term Z_g generates discretized surfaces, of the topology g, made of triangles and higher polygons, with the weights determined by α_i.

• We concentrate on $g = 0$ (sphere) Σ -space of the "potentials" $V(M) = \sum_{n=3} \frac{\alpha_n}{n!} M^n$.

• The sum of the planar graphs exhibits *critical behavior*, in the vicinities of certain critical hyper-surfaces in Σ:

$$... \subset \Sigma_p \subset ... \subset \Sigma_2 \subset \Sigma_1 \subset \Sigma$$

$$\uparrow$$

"p-criticality"

2.2. Solution of the One-matrix Model ($g = 0$)

• Result [*Brezin&Kazakov,1990;Douglas&Shenker,1990; Gross&Migdal, 1990*]: Near p-critical surface

$$u_* = u_*(t_0, ..., t_{p-1}) = \frac{\partial^2 Z(t_0, ..., t_{p-1})}{\partial t_{p-1}^2}$$

with u_* being certain solution of

$$Q(u) \equiv u^{p+1} - t_0\, u^{p-1} - ... - t_k\, u^{p-k-1} - ... - t_{p-1} = 0$$

$\{t_0, t_1, ..., t_{p-1}\}$ - deviations from Σ_p.
More convenient expression for Z:

$$Z = \frac{1}{2} \int_0^{u_*} Q^2(u)\, du\,.$$

• Interpretation [*Staudacher, 1990; Bresin&Douglas& &Kazakov&Shenker,1990; Gross&Migdal,1990*]: Take

$$t_0 = \mu \quad -\text{''cosmological constant''}$$

Then

$$[u] = [\mu^{\frac{1}{2}}]\,, \quad [t_k] = [\mu^{\frac{k+2}{2}}]\,, \qquad [Z] = [\mu^{\frac{2p+3}{2}}]\,,$$

exactly the gravitational dimensions of $\mathcal{MG}_{2/2p+1}$,

$$t_k \;\sim\; \tau_k\,, \quad k = 0, 1, 2, ..., p-1.$$

Convenient to separate $t_0 = \tilde{\mu}$ and $\{t_i\} = \{t_1, t_2, ..., t_{p-1}\}$ Matrix Model correlation numbers:

$$\langle\, \mathcal{O}_{k_1} ... \mathcal{O}_{k_n} \,\rangle_{MM} \equiv \frac{\partial^n W_{MM}(\mu, \{t_i\})}{\partial t_{k_1} ... \partial_{k_n}} \bigg|_{\{t_i\}=0}, \qquad \{t_i\} = \{t_1, ..., t_n\}$$

with

$$W_{MM}(\mu, \{t_i\}) = \frac{Z(t_0 = \mu, t_1, ..., t_n)}{Z(t_0 = \mu, 0, ..., 0)}$$

With the (naive) identification

$$t_k \ \sim \ \tau_k$$

one would expect

$$\langle\, \mathcal{O}_{k_1} ... \mathcal{O}_{k_n} \,\rangle_{MM} = \langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_n} \,\rangle_{\mathcal{MG}} \ \times \text{ [Leg factors]}$$

<u>This expectation fails</u>.

Since

$$Q(u) = u^{p+1} - \mu\, u^{p-1} - \sum_{k=1}^{p-1} t_k\, u^{p-k-1}\,, \qquad Z = \frac{1}{2} \int_0^{u_*} Q^2(u)\, du$$

we have $u_*(\mu, 0, ..., 0) = \sqrt{\mu}$, and

$$\left.\frac{\partial Z}{\partial t_k}\right|_{\{t=0\}} = \int_0^{u_*} Q(u)\, \frac{\partial Q(u)}{\partial t_k}\, du \left.\vphantom{\frac{\partial Q}{\partial t}}\right|_{\{t=0\}} = -\frac{2\,\mu^{\frac{2p-k+1}{2}}}{(2p-k-1)(2p-k+1)}$$

$$\left.\frac{\partial^2 Z}{\partial t_k \partial t_{k'}}\right|_{\{t=0\}} = \int_0^{u_*} \frac{\partial Q(u)}{\partial t_k}\, \frac{\partial Q(u)}{\partial t_{k'}}\, du \left.\vphantom{\frac{\partial Q}{\partial t}}\right|_{\{t=0\}} = \frac{\mu^{\frac{2p-k-k'-1}{2}}}{2p-k-k'-1}$$

etc

in sharp contrast with

$$\langle \tilde{O}_k \rangle_{\mathcal{MG}} \ = \ 0\,, \quad k = 1, 2, ..., p-1 \quad (\text{since } \ \langle \Phi_k \rangle_{CFT} = 0)$$

$$\langle \tilde{O}_k \tilde{O}_{k'} \rangle_{\mathcal{MG}} \ \sim \ \delta_{kk'}\,, \qquad (\text{since } \langle \Phi_k \Phi_{k'} \rangle_{CFT} \sim \delta_{kk'})$$

etc

Resolution [*Moore, Seiberg, Staudacher, 1991*]:
$\underline{Resonances}$ between the operators $\tilde{O}_k$.

2.3. Resonance transformations

$$[t_k] = [\mu^{\frac{k+2}{2}}], \qquad [\tau_k] = [\mu^{\frac{k+2}{2}}]$$

It is possible to have, e.g.

$$[t_k] = [\tau_{k_1}][\tau_{k_2}] \qquad (k = k_1 + k_2 + 2 \geq 2)$$

$(k = 0, 1, 2, ..., p-1)$. I.e.

$$t_k \;=\; \tau_k \;+\; \sum_{\substack{k_1,k_2=0 \\ k_1+k_2=k+2}}^{p-1} c_k^{k_1 k_2}\, \tau_{k_1}\tau_{k_2} \;+\; \text{higher order terms}$$

Thus

$$t_0 = \tau_0 = \mu\,,$$

$$t_1 = \tau_1\;, \qquad ([t_1] = [\mu^{3/2}])$$

$$t_2 = \tau_2 + A_2\,\mu^2\;, \qquad ([t_2] = [\mu^2])$$

$$t_3 = \tau_3 + B_3\,\mu\,\tau_1\;, \qquad ([t_3] = [\mu][t_1])$$

$$t_4 = \tau_4 + A_4\,\mu^3 + B_4\,\mu\,\tau_2 + C_4\,\tau_1^2$$

etc

generally

$$t_k = \tau_k + \underbrace{A_k\, \mu^{\frac{k+2}{2}}}_{} + \sum_{n=0}^{n\le k/2} \underbrace{B_k^{k-2n}\, \mu^n\, \tau_{k-2n}}_{} +$$

$\uparrow$ k – even, $k \geq 2$

$\uparrow$ $k \geq 3$

$$\frac{1}{2} \sum_{n=0} \sum_{k_1+k_2=k-2-2n} \underbrace{C_k^{k_1,k_2}\, \mu^n\, \tau_{k_1}\tau_{k_1}}_{} + \ldots$$

$\uparrow$ $k \geq 4$

$$W_{MM}(\{t\}) \;\rightarrow\; \tilde{W}_{MM}(\{\tau\}) \equiv W_{MM}(\{t(\tau)\})$$

The right thing to expect is

$$\frac{\partial^N \tilde{W}_{MM}(\{\tau\})}{\partial\tau_{k_1}\ldots\partial\tau_{k_N}} = \langle\, \tilde{O}_{k_1}\ldots\tilde{O}_{k_n}\, \rangle_{\mathcal{MG}}$$

under special choice of the "Liouville coordinates" $\{\tau_1, ..., \tau_n\}$.

$$\tau_k = 0\,, \qquad k = 1, 2, ..., p-1$$

$$\downarrow$$

$$t_k = A_k\,\mu^{\frac{k+2}{2}} \qquad - \text{ "Liouville background"}$$

Problem: Finding the "Liouville coordinates" $\{\tau\}$, such that

- One-point numbers:

$$\langle\, \tilde{O}_k\,\rangle_{MM} = \frac{\partial \tilde{W}(\mu, \{\tau\})}{\partial \tau_k}\bigg|_{\{\tau\}=0} = 0 \qquad \text{for} \quad k = 1, 2, ..., p-1$$

- Two-point numbers:

$$\langle\, \tilde{O}_k \tilde{O}_{k'}\,\rangle_{MM} = \frac{\partial^2 \tilde{W}(\mu, \{\tau\})}{\partial \tau_k \partial \tau_{k'}}\bigg|_{\{\tau\}=0} \sim \delta_{kk'}$$

- Three-point numbers:

$$\langle\, \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3}\,\rangle_{MM} = \frac{\partial^3 \tilde{W}(\mu, \{\tau\})}{\partial \tau_{k_1} \partial \tau_{k_2} \partial \tau_{k_3}}\bigg|_{\{\tau\}=0} = 0$$

obey the fusion rules.

• Multi-point numbers obey fusion rules, e.g. For even $k_1 + ... + k_n$

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} ... \tilde{O}_{k_n} \rangle_{MM} = 0 \qquad \text{if} \quad k_n > k_1 + k_2 + ... + k_{n-1}$$

For odd $k_1 + ... + k_n$

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} ... \tilde{O}_{k_n} \rangle_{MM} = 0 \qquad \text{if} \quad k_1 + k_2 + ... + k_n < 2p - 1$$

Building the Liouville coordinates order by order in $\{\tau\}$:

∘ The resonance transforms do not affect odd parity correlation functions.

∘ Starting from $n = 4$ there are not enough parameters to exterminate the "wrong" correlation numbers:

$$[\tau_k] = [\mu^{\frac{k+2}{2}}] \;\rightarrow\; [\tau_{k_1+k_2}] = [\tau_{k_1}][\tau_{k_2}][\mu^2]$$

2.4. Resonance terms in Z

$$[Z] = [\mu^{\frac{2p+3}{2}}]$$

Defined up to regular terms:

$$Z\{\tau\}) \to Z(\{\tau\}) + \sum_n \sum_{k_1 \dots k_n} z^{k_1 \dots k_n} \underbrace{\tau_{k_1} \dots \tau_{k_n}}$$

$$\uparrow$$

$$[\tau_{k_1}][\tau_{k_2}]\dots[\tau_{k_n}] = [Z]$$

$$Z_{\text{reg}} = z_0 \, \mu^{\frac{2p+3}{2}} + z_k \, \mu^{\frac{2p-k+1}{2}} \, \tau_k + z_{k_1 k_2} \, \mu^{\frac{2p-k_1-k_2-1}{2}} \, \tau_{k_1} \tau_{k_2} + \dots$$

where only integer powers of μ are admitted. The resonance terms affect negative parity correlation functions

$$\langle \, \tilde{O}_{k_1} \dots \tilde{O}_{k_n} \, \rangle_{MM} \qquad (\sum_i k_i \;\; \text{odd})$$

with

$$k_1 + \dots + k_n \leq 2p + 3 - 2n$$

The fusion rules requires vanishing of the odd corr. numbers for

$$k_1 + \dots + k_n \leq 2p - 3$$

Again, starting from $n = 4$, there is not enough resonance terms to adjust.

3. Finding the Liouville coordinates

$$Q(u) = Q(u|\{\tau\}) = u^{p+1} - \mu\, u^{p-1} - \sum_{k=1}^{p-1} t_k(\{\tau\})\, u^{p-k-1}$$

Expansion in $\{\tau\}$:

$$Q(u|\{\tau\}) = Q_0(u) + \sum_{k=1}^{p-1} \tau_k\, Q_k(u) + \frac{1}{2} \sum_{k_1,k_2=1}^{p-1} \tau_{k_1}\tau_{k_2}\, Q_{k_1k_2}(u) + ...$$

with

$$Q_0(u) = u^{p+1} - \sum_{l=1} A_l\, \mu^l\, u^{p+1-2l}\,,$$

$$Q_k(u) = u^{p-k-1} - B_k^{(1)}\, \mu\, u^{p-k-3} - ... - B_k^{(l)}\ \mu^l\, u^{p-k-2l-1} - ...$$

$$Q_{k_1k_2}(u) = -C_{k_1k_2}^{(0)}\, u^{p-k_1-k_2-3} - C_{k_1k_2}^{(1)}\, \mu\, u^{p-k_1-k_2-5} - ...$$

etc

Parity (even or odd)

$$Q_0(-u) = (-)^{p+1}\, Q_0(u)\,,$$

$$Q_k(-u) = (-)^{p+1-k}\, Q_k(u)\,,$$

$$Q_{k_1k_2}(-u) = (-)^{p+1-k_1-k_2}\, Q_{k_1k_2}(u)\,,$$

etc

$$Q_0(u) = Q(u)\Big|_{\{\tau=0\}}\,,$$

$$Q_k(u) = \frac{\partial Q(u)}{\partial \tau_k}\bigg|_{\{\tau=0\}}\,,$$

$$Q_{kk'}(u) = \frac{\partial^2 Q(u)}{\partial \tau_k \partial \tau_{k'}}\bigg|_{\{\tau=0\}}\,,$$

3.1. Partition function

$$Z(\mu, \{\tau\}) = \frac{1}{2} \int_0^{u_*} Q^2(u)\, du$$

where $u_* = u_*(\{\tau\})$ is some root of $Q(u)$

$$Q(u_*) = 0\,.$$

(All roots are real, and u_* is the maximal root)

3.2. One- and Two-point correlation numbers

$$\left.\frac{\partial Z}{\partial \tau_k}\right|_{\tau=0} = \int_0^{u_0} Q(u)\, Q_k(u)\, du\,,$$

$$\left.\frac{\partial^2 Z}{\partial \tau_k \partial_{k'}}\right|_{\tau=0} = \int_0^{u_0} Q_k(u)\, Q_{k'}(u)\, du\,,$$

where

$$Q_0(u_0) = 0\,.$$

For the corr. numbers with positive parity

$$\int_0^{u_0} \;\to\; \frac{1}{2}\int_{-u_*}^{u_*}$$

$$\int_{-u_0}^{u_0} Q(u)\,Q_k(u)\,du = 0\,, \qquad k = 1, ..., p-1$$

$$\int_{-u_0}^{u_0} \Big[Q_k(u)\,Q_{k'}(u) + Q_0(u)Q_{kk'}(u)\Big]\,du \;\sim\; \delta_{kk'}\,,$$

$$Q_0(u) = u_0^{p+1}\,q(u/u_0)\,, \qquad\qquad q_0(1) = 0\,,$$

$$Q_k(u) = u_0^{p-k-1}\,q(u/u_0)\,, \qquad\qquad ...$$

$$\int_{-1}^{1} q_0(x)q_k(x)\,dx = 0\,, \qquad\qquad k = 1, ..., p-1\,,$$

$$\int_{-1}^{1} [q_k(x)q_{k'}(x) + q_0(x)q_{kk'}(x)]\,dx \sim \delta_{kk'}$$

Solution: the Legendre polynomials

$$q_k(x) = g_k\, P_{p-k-1}(x)\,, \qquad g_k = \frac{(p-k-1)!}{(2p-2k-3)!!}$$

$$q_0(x) = g_0\,[P_{p+1}(x) - P_{p-1}(x)]\,, \qquad g_0 = \frac{(p+1)!}{(2p+1)!!}$$

$$\downarrow$$

$$Z(\mu,\{\tau=0\}) = \frac{1}{4}\int_{-u_0}^{u_0} Q_0^2(u)\,du = g_0^2\, u_0^{2p+3}\,\frac{2p+1}{(2p-1)(2p+3)}\,,$$

$$\langle\,\tilde{O}_k\,\rangle_{MM} = \frac{1}{2}\int_{-u_0}^{u_0} Q_0(u)Q_k(u)\,du = 0\,, \qquad k = 1,2,...,p-1$$

$$\langle\,\tilde{O}_k\tilde{O}_{k'}\,\rangle_{MM} = \frac{1}{2}\int_{-u_0}^{u_0} Q_k(u)Q_{k'}(u)\,du =$$

$$\frac{\delta_{kk'}}{Z_p}\,\frac{1}{2p-2k-1}\,[\mathrm{Leg}(k)]^2$$

with $Z_p = [(2p-1)(2p+1)(2p+3)]^{-1}$,

$$\mathrm{Leg}(k) = \frac{u_0^{-k-2}}{2p+1}\,\frac{g_k}{g_0}$$

3.3. Three-point correlation number

$$Z = \int_0^{u_*} Q^2(u)\, du$$

$$\left.\frac{\partial^3 Z}{\partial\tau_{k_1}\partial\tau_{k_2}\partial\tau_{k_3}}\right|_{\tau=0} = -\frac{Q_{k_1}(u_0)Q_{k_2}(u_0)Q_{k_3}(u_0)}{Q_0'(u_0)} +$$

$$\int_0^{u_0} [Q_{k_1}(u)Q_{k_2k_3}(u)\, du + \text{two permutations}]\, du$$

The integral cancels the "wrong" part of the first term iff

$$Q_{k_1k_2}(u) = \frac{u_0^{-k_1-k_2-4}}{2p+1}\,\frac{g_{k_1}g_{k_2}}{g_0} \sum_{k=k_1+k_2+2:2}^{k\le p-1} u_0^{k+2}\,\frac{2p-2k-1}{g_k}\, Q_k(u)$$

With this

$$\langle\, \tilde{O}_{k_1}\tilde{O}_{k_2}\tilde{O}_{k_3}\,\rangle_{MM} = \frac{N_{k_1k_2k_3}}{Z_p}\prod_{i-1}^{3} \mathrm{Leg}(k_i)\,.$$

3.4. Non-trivial test: four-point correlation numbers

$$\frac{\partial^4 Z}{\partial \tau_{k_1} \ldots \partial \tau_{k_4}} = - \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} \; Q''}{(Q')^3} \Big|_{u=u_0}$$

$$+ \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q'_{k_4} + 3 \text{ permutations}}{(Q')^2} \Big|_{u=u_0}$$

$$- \frac{Q_{k_1} Q_{k_2} Q_{k_3 k_4} + 5 \text{ permutations}}{Q'} \Big|_{u=u_0}$$

$$+ \int_0^{u_0} \left[Q_{k_1 k_2}(u) Q_{k_3 k_4}(u) + 2 \text{ permutations} \right] du$$

$$+ \underbrace{\int_0^{u_0} \left[Q_{k_1}(u) Q_{k_2 k_3 k_4}(u) + \text{permutations} \right] du}$$

$$\uparrow$$

" Counterterm"

Evaluates to $\prod_{i=1}^{4}$ Leg$(k_i) \times$

$$\left[\sum_{i=1}^{4} \frac{(p-k_i)(p-k_i-1)}{2} - \frac{p(p+1)}{2} - F_p(k_{12|34}) - F_p(k_{13|24}) - F_p(k_{14|23})\right]$$

with

$$F_p(k) = \frac{(p-k-1)(p-k-2)}{2}\,\Theta(p-k-1)$$

and

$$k_{ij|nm} = \min(k_i+k_j, k_n+k_m)$$

• For $k_4 \leq k_1+k_2+k_3$ - reproduces exactly the four-point number of the Minimal Gravity $\mathcal{MG}_{2/2p+1}$

• At $k_4 > k_1+k_2+k_3$

$$\frac{(k_1+k_2+k_3+2-k_4)(2p-3-k_1-k_2-k_3-k_4)}{2} \prod_{i=1}^{4} \text{Leg}(k_i)$$

Vanishes exactly at the "dangerous" configurations.

Conclusion • The problem of finding the ”Liouville coordinates” is rather rigid (leads to over-determined constraints on the coefficients of the resonance transformation and regular terms).

• At the level of 4-point numbers the solution exists.

• The resulting two, three, and four point numbers exactly reproduce the results of the Minimal Gravity.