Effects of turbulent mixing on the nonequilibrium critical behaviour

N V Antonov, V I Iglovikov and A S Kapustin

Department of Theoretical Physics
Saint Petersburg State University

Renormalization Group and Related Topics
Dubna 2008

1 The problem

Spreading processes in physical, chemical, biological, ecological and sociological systems: autocatalytic reactions, percolation in porous media, forest fires, epidemic diseases, and so on.

Typical model: Random walk of two species on a lattice plus reaction:

Infection: $A + B \to B$
Healing: $B \to A$

Absorbing state: No infected individuals, $\rho_B \equiv 0$.

Fluctuating state: $\rho_B = \rho(t, \mathbf{x})$ is a random quantity; $\langle \rho(t, \mathbf{x}) \rangle \neq 0$.

Continuous (second-order) phase transition between these nonequilibrium steady states.

Universal scaling behaviour; critical exponents; new universality classes.

Reference: Hinrichsen H 2000 *Adv. Phys.* **49** 815

2 The model

Directed bond percolation process = simple epidemic process with recovery = Gribov's process = stochastic first Schlögl reaction

Continuous model: stochastic PDE

$$\partial_t \psi(t, \mathbf{x}) = \lambda_0 \left\{ (-\tau_0 + \partial^2)\psi(t, \mathbf{x}) - g_0 \psi^2(t, \mathbf{x})/2 \right\} + \zeta(t, \mathbf{x}), \qquad (1)$$

$\psi(t, \mathbf{x}) > 0$ — the agent's density
∂^2 — Laplace operator
λ_0 and g_0 — positive parameters
$\tau_0 \propto (T - T_c)$ deviation of the "temperature" from its critical value
d — the dimension of the $\mathbf{x}$ space
$\zeta(t, \mathbf{x})$ — Gaussian noise with correlation function

$$\langle \zeta(t, \mathbf{x}) \zeta(t', \mathbf{x}') \rangle = g_0 \lambda_0 \, \psi(t, \mathbf{x}) \, \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}'). \qquad (2)$$

3 Field theoretic formulation

Stochastic problem (1), (2) is equivalent to the "Reggeon field theory" with the action functional

$$\mathcal{S}(\psi,\psi^{\dagger}) = \psi^{\dagger}(-\partial_t + \lambda_0\partial^2 - \lambda_0\tau_0)\psi + \frac{g_0\lambda_0}{2}\left((\psi^{\dagger})^2\psi - \psi^{\dagger}\psi^2\right), \qquad (3)$$

the integrations are implied:

$$\psi^{\dagger}\partial_t\psi = \int dt \int d\mathbf{x}\psi^{\dagger}(t,\mathbf{x})\partial_t\psi(t,\mathbf{x}).$$

$\psi^{\dagger}(x) = \psi^{\dagger}(t,\mathbf{x})$ is the auxiliary "response field."

Correlation functions of the stochastic problem = functional averages with weight $\exp\mathcal{S}$.

The linear response function of the problem (1), (2) is given by the Green function

$$G = \langle\psi^{\dagger}(x)\psi(x')\rangle = \int \mathcal{D}\psi^{\dagger} \int \mathcal{D}\psi \;\; \psi^{\dagger}(x)\psi(x') \, \exp\mathcal{S}(\psi,\psi^{\dagger}).$$

Feynman rules: the bare propagator $G_0 = \langle\psi\psi^\dagger\rangle_0$:

$$G_0(t,k) = \theta(t)\exp\left\{-\lambda_0(k^2+\tau_0)\right\} \leftrightarrow G_0(\omega,k) = \frac{1}{-\mathrm{i}\omega + \lambda_0\,(k^2+\tau_0)} \tag{4}$$

and the two triple vertices $\sim (\psi^\dagger)^2\psi$, $\psi^\dagger\psi^2$.

Absorbing phase:

$$\langle\psi\ldots\psi\rangle = 0, \quad \langle\psi^\dagger\ldots\psi^\dagger\rangle = 0$$

Anomalous phase:

$$\langle\psi\ldots\psi\rangle \neq 0$$

Phase transition = breakdown of the symmetry:

$$\psi(t,\mathbf{x}) \to \psi^\dagger(-t,-\mathbf{x}), \quad \psi^\dagger(t,\mathbf{x}) \to \psi(-t,-\mathbf{x}), \quad g_0 \to -g_0. \tag{5}$$

Critical exponents η, ν, z are known to ε^2, where $\varepsilon = d-4$.

Reference: Janssen H-K and Täuber U C 2004 *Ann. Phys.* (NY) **315** 147.

4 Turbulent mixing

Inclusion of the velocity field $\mathbf{v} = \{v_i(t, \mathbf{x})\}$:

$$\partial_t \to \nabla_t = \partial_t + v_i \partial_i, \quad \partial_i = \partial / \partial x_i. \tag{6}$$

Incompressibility: $\partial_i v_i = 0$.

Obukhov–Kraichnan's rapid-change model: Gaussian distribution with the correlation function:

$$\langle v_i(t, \mathbf{x}) v_j(t', \mathbf{x}') \rangle = \delta(t - t')\, D_{ij}(\mathbf{r}), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'$$

$$D_{ij}(\mathbf{r}) = D_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d}\, P_{ij}(\mathbf{k})\, \frac{1}{k^{d+\xi}}\, \exp(\mathrm{i}\mathbf{k}\mathbf{r}), \quad k \equiv |\mathbf{k}|; \tag{7}$$

$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ — transverse projector
$D_0 > 0$
$0 < \xi < 2$ — free parameter (Hölder exponent)
the realistic ("Kolmogorov") value $\xi = 4/3$
the IR cutoff at $k = m \equiv 1/\mathcal{L}$
$\mathcal{L}$ — the integral turbulence scale.

Field theoretic model of the three fields $\Phi = \{\psi, \psi^\dagger, \mathbf{v}\}$ with the action

$$\mathcal{S}(\Phi) = \psi^\dagger(-\nabla_t + \lambda_0\partial^2 - \lambda_0\tau_0)\psi + \frac{\lambda_0 g_0}{2}\left((\psi^\dagger)^2\psi - \psi^\dagger\psi^2\right) + \mathcal{S}(\mathbf{v}), \quad (8)$$

$$\mathcal{S}(\mathbf{v}) = -\frac{1}{2}\int dt \int d\mathbf{x} \int d\mathbf{x}'\ v_i(t,\mathbf{x}) D^{-1}_{ij}(\mathbf{r}) v_j(t,\mathbf{x}'), \quad (9)$$

where

$$D^{-1}(\mathbf{r}) \propto D_0^{-1} r^{-2d-\xi}$$

— the kernel of the inverse linear operation for the function $D_{ij}(\mathbf{r})$ in (7).

Feynman rules involve the new propagator $\langle vv\rangle_0$ and the new vertex $-\psi^\dagger(v\partial)\psi$.

The coupling constants:

$$u_0 = g_0^2 \sim \Lambda^{4-d}, \qquad w_0 = D_0/\lambda_0 \sim \Lambda^{\xi}, \quad (10)$$

Λ — UV momentum scale.

5 UV divergences and the renormalization

The coupling constants:

$$u_0 = g_0^2 \sim \Lambda^{4-d}, \qquad w_0 = D_0/\lambda_0 \sim \Lambda^{\xi}, \tag{11}$$

Λ — UV momentum scale.

The model is logarithmic (the both coupling constants g_0 and w_0 are simultaneously dimensionless) at $d = 4$ and $\xi = 0$.

The UV divergences = singularities at $\varepsilon = (4 - d) \to 0, \ \xi \to 0$.

Dimensional analysis ("power counting"): superficial UV divergences can be present in the 1-irreducible functions

$$\langle\psi^{\dagger}\psi\rangle \quad \text{with the counterterms} \quad \psi^{\dagger}\partial_t\psi,\ \psi^{\dagger}\partial^2\psi,\ \psi^{\dagger}\psi,$$

$$\langle\psi^{\dagger}\psi\psi\rangle \quad \text{with the counterterm} \quad \psi^{\dagger}\psi^2,$$

$$\langle\psi^{\dagger}\psi^{\dagger}\psi\rangle \quad \text{with the counterterm} \quad (\psi^{\dagger})^2\psi,$$

$$\langle\psi^{\dagger}\psi v\rangle \quad \text{with the counterterm} \quad \psi^{\dagger}(v\partial)\psi$$

Galilean symmetry: divergence in the function

$$\langle\psi^{\dagger}\psi vv\rangle \quad \text{with the counterterm} \quad \psi^{\dagger}\psi v^2$$

is forbidden;

the counterterms $\psi^{\dagger}\partial_t\psi$ and $\psi^{\dagger}(v\partial)\psi$ appear in the combination $\psi^{\dagger}\nabla_t\psi$.

Symmetry (5): trilinear counterterms enter the renormalized action as the combination $(\psi^{\dagger})^2\psi - \psi^{\dagger}\psi^2$.

All these terms are present in the action (8), so the model is multiplicatively renormalizable.

The renormalized action:

$$\begin{aligned} \mathcal{S}_R(\Phi) &= \psi^\dagger \left(-Z_1\nabla_t + Z_2\lambda\partial^2 - Z_3\lambda\tau\right)\psi + \\ &+ Z_4\frac{\lambda g}{2}\left((\psi^\dagger)^2\psi - \psi^\dagger\psi^2\right) + \mathcal{S}(\mathbf{v}). \end{aligned} \tag{12}$$

λ, τ, g — are renormalized analogs of the bare parameters,
μ is the reference mass in the MS scheme,

$\mathcal{S}(\mathbf{v})$ is not renormalized:

$$D_0 = w_0\lambda_0 = w\lambda\mu^\xi. \tag{13}$$

Multiplicative renormalization of the fields

$$\psi \to \psi Z_{\psi}, \quad \psi^{\dagger} \to \psi^{\dagger} Z_{\psi^{\dagger}}, \quad v \to v Z_{v}$$

and the parameters:

$$\lambda_0 = \lambda Z_{\lambda}, \quad \tau_0 = \tau Z_{\tau}, \quad g_0 = g\mu^{\varepsilon/2} Z_g, \quad w_0 = w\mu^{\xi} Z_w. \tag{14}$$

The constants in Eqs. (12) and (14) are related as follows:

$$\begin{aligned} Z_1 = Z_{\psi} Z_{\psi^{\dagger}} &= Z_v Z_{\psi} Z_{\psi^{\dagger}} \quad Z_2 = Z_{\psi} Z_{\psi^{\dagger}} Z_{\lambda}, \quad Z_3 = Z_{\psi} Z_{\psi^{\dagger}} Z_{\lambda} Z_{\tau}, \\ Z_4 &= Z_{\psi} Z^2_{\psi^{\dagger}} Z_g Z_{\lambda} = Z^2_{\psi} Z_{\psi^{\dagger}} Z_g Z_{\lambda}, \quad 1 = Z_w Z_{\lambda}. \end{aligned} \tag{15}$$

There are exact relations between them due to the symmetries:

$$Z_{\psi} = Z_{\psi^{\dagger}}, \quad Z_v = 1, \quad Z_w = Z_{\lambda}^{-1}. \tag{16}$$

The constants Z_1–Z_4 are calculated directly from the diagrams, then the constants in (14) are found from (15).

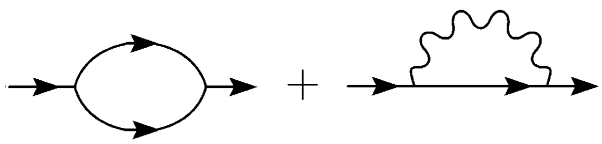
The one-loop results read:

$$Z_1 = 1 + \frac{u}{4\varepsilon}, \quad Z_2 = 1 + \frac{u}{8\varepsilon} - \frac{3w}{4\xi}, \quad Z_3 = 1 + \frac{u}{2\varepsilon}, \quad Z_4 = 1 + \frac{u}{\varepsilon}, \tag{17}$$

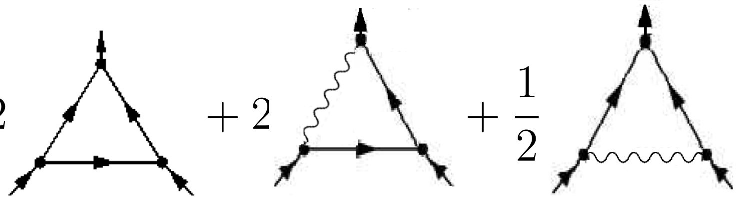
where we passed to the new couplings,

$$u \to u/16\pi^2, \quad w \to w/16\pi^2. \tag{18}$$

$$\langle\psi^\dagger\psi\rangle = -\left\{-\mathrm{i}\omega Z_1 + \lambda p^2 Z_2 + \lambda\tau Z_3\right\} + \frac{1}{2}$$



$$\langle\psi^\dagger\psi^\dagger\psi\rangle = gZ_4 + 2$$



$$\langle\psi^\dagger\psi\mathbf{v}\rangle = -\mathrm{i}\mathbf{p}Z_1 +$$

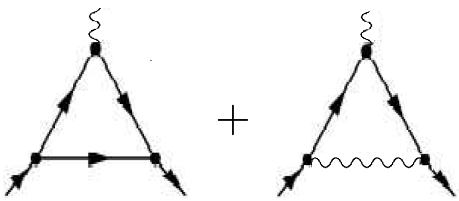


Figure 1: The one-loop approximation of the relevant 1-irreducible Green functions

6 RG functions and RG equations

The action functionals are related as

$$\mathcal{S}_R(\Phi, e, \mu) = \mathcal{S}(\Phi, e_0)$$

so that the Green functions are related as

$$G(e_0, \ldots) = Z_{\psi}^{N_\psi} Z_{\psi^\dagger}^{N_{\psi^\dagger}} G_R(e, \mu, \ldots). \tag{19}$$

Here: N_ψ and $N_{\psi^\dagger}$ — the numbers of corresponding fields
$e_0 = \{\lambda_0, \tau_0, u_0, w_0\}$ — the full set of bare parameters
$e = \{\lambda, \tau, u, w\}$ — their renormalized counterparts.

Let $\widetilde{\mathcal{D}}_\mu$ be the differential operation $\mu\partial_\mu$ for fixed e_0; operate on both sides of the equation (19) with it. This gives the basic RG equation:

$$\left\{ \mathcal{D}_{RG} + N_\psi \gamma_\psi + N_{\psi^\dagger} \gamma_{\psi^\dagger} \right\} G_R(e, \mu, \ldots) = 0, \tag{20}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_u \partial_u + \beta_w \partial_w - \gamma_\lambda \mathcal{D}_\lambda - \gamma_\tau \mathcal{D}_\tau. \tag{21}$$

Here $\mathcal{D}_x \equiv x\partial_x$ for any variable x, the anomalous dimensions γ are defined as

$$\gamma_F \equiv \widetilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any quantity } F, \tag{22}$$

and the β functions for the couplings u and w are

$$\beta_u \equiv \widetilde{\mathcal{D}}_\mu u = u\,[-\varepsilon - \gamma_u], \quad \beta_w \equiv \widetilde{\mathcal{D}}_\mu w = w\,[-\xi - \gamma_w]. \tag{23}$$

One-loop results:

$$\begin{aligned} \gamma_\psi = \gamma_{\psi^\dagger} = -\frac{u}{8}, \quad \gamma_\lambda = -\gamma_w = \frac{u}{8} + \frac{3w}{4}, \\ \gamma_\tau = -\frac{3u}{8} - \frac{3w}{4}, \quad \gamma_u = -\frac{3u}{2} - \frac{3w}{2}, \end{aligned} \tag{24}$$

with corrections of order u^2, w^2, uw and higher.

7 Fixed points and IR scaling regimes

Long-time large-distance asymptotic behaviour is determined by the IR attractive fixed points of the RG equations:

$$\beta_u(u_*, w_*) = 0, \quad \beta_w(u_*, w_*) = 0. \tag{25}$$

The fixed point is IR attractive if the matrix

$$\Omega = \{\Omega_{ij} = \partial\beta_i/\partial g_j\}, \tag{26}$$

is positive (eigenvalues have positive real parts).

The one-loop expressions:

$$\beta_u = u\,(-\varepsilon + 3u/2 + 3w/2), \quad \beta_w = w\,(-\xi + u/8 + 3w/4). \tag{27}$$

There are four different fixed points.

1. Gaussian (free) fixed point:
$u_* = w_* = 0; \quad \Omega_u = -\varepsilon, \quad \Omega_w = -\xi$
(all these expressions are exact).

2. $w_* = 0$ (exact result to all orders), $u_* = 2\varepsilon/3$; $\Omega_u = \varepsilon$, $\Omega_w = -\xi + \varepsilon/12$.

Effects of turbulent mixing are irrelevant; the basic critical exponents are independent on ξ and coincide to all orders with their counterparts for the "pure" DP class.

3. $u_* = 0$, $w_* = 4\xi/3$ (exact); $\Omega_u = -\varepsilon + 2\xi$, $\Omega_w = \xi$ (exact).

The nonlinearity $(\psi^{\dagger})^2\psi - \psi^{\dagger}\psi^2$ of the DP model is irrelevant, and we arrive at the rapid-change model of a passively advected scalar field ψ. For that model, the β function is given exactly by the one-loop approximation, hence the exact results for w_* and Ω_w.

4. $u_* = 4(\varepsilon - 2\xi)/5$, $w_* = 2(12\xi - \varepsilon)/15$. The eigenvalues:

$$\lambda^{\pm} = \frac{1}{20}\left(11\varepsilon - 12\xi \pm \sqrt{161\varepsilon^2 - 824\varepsilon\xi + 1104\xi^2}\right) \tag{28}$$

are both real for all ε and ξ and positive for $\varepsilon/12 < \xi < \varepsilon/2$.

This fixed point corresponds to a new nontrivial IR scaling regime (universality class), in which the nonlinearity of the DP model (3) and the turbulent mixing are simultaneously important; the corresponding critical exponents depend on the both RG expansion parameters ε and ξ and are calculated as double series in these parameters.

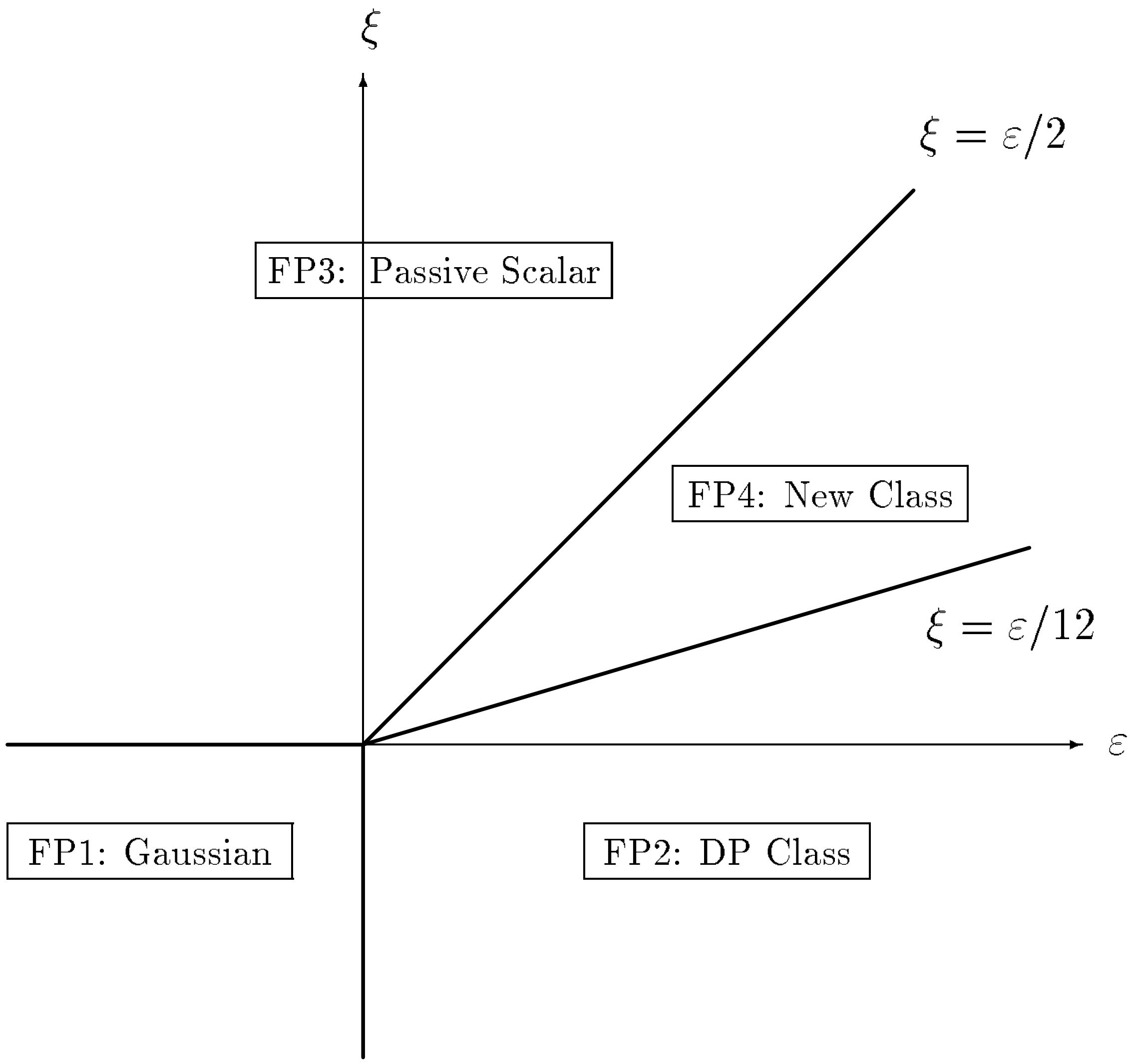


Figure 2: Regions of stability of the fixed points in the model (8).

8 Critical dimensions

Four fixed points of the model (3) correspond to four possible IR scaling (self-similar) regimes; for given ε and ξ only one of them is IR attractive and governs the IR behaviour. The Green functions have scaling form.

The linear response function has the form

$$G(t,\mathbf{x}) = x^{-2\Delta_\psi}\, F\left(\frac{x}{t^{1/\Delta_\omega}}, \frac{\tau}{t^{\Delta_\tau/\Delta_\omega}}\right), \quad x = |\mathbf{x}| \tag{29}$$

with some scaling function F.

For a given point, the critical dimensions Δ_f of the IR relevant quantities f are given by the relations

$$\begin{aligned} \Delta_\psi &= \Delta_{\psi^\dagger} = d/2 + \gamma^*_\psi, \\ \Delta_\tau &= 2 + \gamma^*_\tau, \quad \Delta_\omega = 2 - \gamma^*_\lambda \\ 2\Delta_v &= \Delta_\omega - \xi \end{aligned} \tag{30}$$

with $\gamma^*_f = \gamma_f(u_*, w_*)$.

From the explicit one-loop expressions (24) we find:

1. Gaussian (free) fixed point; all the expressions are exact:

$$\Delta_\psi = d/2, \quad \Delta_\tau = \Delta_\omega = 2. \tag{31}$$

2. Directed percolation (DP) regime; mixing irrelevant:

$$\Delta_\psi = 2 - 7\varepsilon/12, \quad \Delta_\tau = 2 - \varepsilon/4, \quad \Delta_\omega = 2 - \varepsilon/12. \tag{32}$$

The conventional critical exponents are related to (32) as

$$z = \Delta_\omega, \quad 1/\nu = \Delta_\tau, \quad d + \eta = 2\Delta_\psi.$$

The $O(\varepsilon^3)$ calculation is in progress.

3. Obukhov–Kraichnan exactly soluble regime; all results exact:

$$\Delta_{\omega} = \Delta_{\tau} = 2 - \xi, \quad \Delta_{\psi} = d/2. \tag{33}$$

4. New universality class (both mixing and DP interaction are relevant):

$$\Delta_{\psi} = 2 + (\xi - 3\varepsilon)/5, \; \Delta_{\tau} = 2 - (\varepsilon + 3\xi)/5, \; \Delta_{\omega} = 2 - \xi \text{ (exact)}. \tag{34}$$

The first two dimensions have nontrivial corrections in ε and ξ.

9 Spreading of a cloud

The mean-square radius $R(t)$ at time $t > 0$ of a cloud of "infected" particles, which started from the origin $\mathbf{x}' = 0$ at time $t' = 0$:

$$R^2(t) = \int d\mathbf{x}\; x^2\, G(t,\mathbf{x}), \quad G(t,\mathbf{x}) = \langle \psi(t,\mathbf{x})\psi^\dagger(0,\mathbf{0})\rangle, \quad x = |\mathbf{x}|. \tag{35}$$

Substituting the scaling form of the response function

$$G(t,\mathbf{x}) = x^{-2\Delta_\psi}\, F\left(\frac{x}{t^{1/\Delta_\omega}}, \frac{\tau}{t^{\Delta_\tau/\Delta_\omega}}\right)$$

gives

$$R^2(t) = t^{(d+2-2\Delta_\psi)/\Delta_\omega}\, f\left(\frac{\tau}{t^{\Delta_\tau/\Delta_\omega}}\right), \tag{36}$$

where the scaling functions f and F are related as follows:

$$f(z) = \int d\mathbf{x}\, x^{2-2\Delta_\psi}\, F(x,z).$$

At the critical point ($\tau = 0$) the power law holds:

$$R^2(t) \propto t^{(d+2-2\Delta_\psi)/\Delta_\omega} = t^{(2-2\gamma_\psi^*)/(2-\gamma_\lambda^*)}; \tag{37}$$

The Gaussian fixed point: the usual "1/2 law" $R(t) \propto t^{1/2}$ for the ordinary random walk is recovered.

The passive-scalar fixed point: the exact result $R(t) \propto t^{1/(2-\xi)}$.

For the most Kolmogorov value $\xi = 4/3$ this gives $R(t) \propto t^{3/2}$ in agreement with Richardson's "4/3 law" $dR^2/dt \propto R^{4/3}$.

For the other two fixed points the exponents in (36), (37) are given by infinite series in ε (point 2) or ε and ξ (point 4).

10 Conclusion

— four critical regimes, associated with four fixed points of the RG equations:

— Gaussian fixed point (ordinary diffusion or random walk);

— DP process, advection irrelevant;

— passively advected scalar field (infection processes irrelevant); the real cases $d = 2$ or 3 and $\xi = 4/3$ belongs to this regime;

— new nonequilibrium universality class, in which both the reaction and the turbulent mixing are relevant; the critical exponents are double series in ξ and $\varepsilon = 4 - d$.

— its region of IR stability $\varepsilon/12 < \xi < \varepsilon/2$ differs from naive expectation $\xi > 0$ and $\varepsilon > 0$.

Further investigation (in progress):

— anisotropy of the experimental set-up,

— compressibility, non-Gaussian character and finite correlation time of the advecting velocity field,

— effects of immunization (memory);

— interaction of the order parameter with other relevant degrees of freedom (mode-mode coupling),

— feedback of the reactants on the dynamics of the velocity (forest fires, chemical reactions).

REFERENCES:

Antonov N V, Hnatich M and Honkonen J 2006 *J. Phys. A: Math. Gen.* **39** 7867; *E-print LANL* cond-mat/06044434

Antonov N V and Ignatieva A A 2006 *J. Phys. A: Math. Gen.* **39** 13593; *E-print LANL* cond-mat/0607019

N V Antonov, V I Iglovikov and A S Kapustin 2008 *E-print LANL* cond-mat/0808.0076

Thank you for your attention!