Matching heavy-quark fields in QCD and HQET

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HQET = non-abelian Bloch–Nordsieck

Electron + soft photons (real and virtual)
Bloch, Nordsieck (1937)
see Bogoliubov, Shirkov §46

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(real and virtual): HQET (1990)

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QED electron propagator in the IR ≈ Bloch–Nordsieck electron propagator, see Bogoliubov, Shirkov §50.3

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Here we shall discuss this relation in detail, including corrections

HQET

$$p = mv + k$$

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QCD: Q HQET: $Q_v = \not{v} Q_v$

$$L = \bar{Q}_v iv \cdot D Q_v + \frac{1}{2m} \left(O_k + C_m(\mu) O_m(\mu) \right) + \mathcal{O}\left(\frac{1}{m^2} \right)$$

Matching S-matrix elements

Reparametrization invariance $v \to v + \delta v$

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QCD operator

$$O(\mu) = C(\mu)\tilde{O}(\mu) + \frac{1}{2m}\sum_i B_i(\mu)\tilde{O}_i(\mu) + \mathcal{O}\left(\frac{1}{m^2}\right)$$

Matching on-shell matrix elements

Q via Q_v, ...

Here we shall consider Q

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Lattice

- QCD $a \ll 1/m$
- HQET $a \ll 1/\Lambda_{\overline{\text{MS}}}$

HQET heavy-quark propagator in the Landau gauge
$\Rightarrow$ QCD propagator

Tree level

$$Q(x) = e^{-imv\cdot x}\left(1 + \frac{i\not{D}_\perp}{2m} + \cdots\right) Q_v(x)$$

C.L.Y. Lee (1991)
Körner, Thompson (1991)
Mannel, Roberts, Ryzak (1992)

Matching

Matching

$$<0|Q_0|Q(p)> = \left(Z_Q^{\mathrm{os}}\right)^{1/2} u(p)$$
$$<0|Q_{v0}|Q(p)> = \left(\tilde{Z}_Q^{\mathrm{os}}\right)^{1/2} u_v(k)$$

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Foldy–Wouthuysen transformation

$$u(mv+k) = \left[1 + \frac{\not{k}}{2m} + \mathcal{O}\left(\frac{k^2}{m^2}\right)\right] u_v(k)$$

Matching

$$Q_0(x) = e^{-imv\cdot x}\left[z_0^{1/2}\left(1+\frac{i\not{D}_\perp}{2m}\right)Q_{v0}(x)+\mathcal{O}\left(\frac{1}{m^2}\right)\right]$$

$$z_0 = \frac{Z_Q^{\mathrm{os}}(g_0^{(n_l+1)}, a_0^{(n_l+1)})}{\tilde{Z}_Q^{\mathrm{os}}(g_0^{(n_l)}, a_0^{(n_l)})}$$

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Reparametrization invariance Luke, Manohar (1992)

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Reparametrization invariance Luke, Manohar (1992)
Renormalized decoupling

$$z(\mu) = \frac{\tilde{Z}_Q(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))}{Z_Q(\alpha_s^{(n_l+1)}(\mu), a^{(n_l+1)}(\mu))} z_0$$

$m_c = 0$

- $\tilde{Z}_Q^{\text{os}} = 1$
- Z_Q^{os} (single scale m): Melnikov, van Ritbergen (2000)
- $\tilde{\gamma}_Q$: Melnikov, van Ritbergen (2000);
  Chetyrkin, Grozin (2003)
- γ_Q: Tarasov (1982);
  Larin, Vermaseren (1993)

Decoupling: Chetyrkin, Kniehl, Steinhauser (1998)

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Melnikov, van Ritbergen obtained $\tilde{Z}_Q$ from Z_Q^{os}
and finiteness of $z(\mu)$

Result

$$z(\mu) = 1 - (3L + 4)C_F \frac{\alpha_s^{(n_l)}(\mu)}{4\pi}$$
$$+ \left(z_{22}L^2 + z_{21}L + z_{20}\right) C_F \left(\frac{\alpha_s^{(n_l)}(\mu)}{4\pi}\right)^2$$
$$+ \left(z_{33}L^3 + z_{32}L^2 + z_{31}L + z_{30}\right) C_F \left(\frac{\alpha_s^{(n_l)}(\mu)}{4\pi}\right)^3 + \cdots$$

Depends on $a(\mu)$ starting from 3 loops

Large β_0

$$z(\mu) = 1 + \int_0^{\beta} \frac{d\beta}{\beta} \left(\frac{\gamma(\beta)}{2\beta} - \frac{\gamma_0}{2\beta_0} \right) + \frac{1}{\beta_0} \int_0^{\infty} du\, e^{-u/\beta} S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2} \right)$$

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$$\gamma(\beta) = \gamma_Q - \tilde{\gamma}_Q = -2\frac{\beta}{\beta_0} F(-\beta, 0) = 2C_F \frac{\beta}{\beta_0} \frac{(1+\beta)(1+\frac{2}{3}\beta)}{B(2+\beta, 2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)}$$

$\gamma_Q - \tilde{\gamma}_Q$ is gauge invariant at $1/\beta_0$

Borel image

$$
\begin{aligned}
S(u) &= \frac{F(0,u) - F(0,0)}{u} \\
&= -6C_F \left[e^{(L+5/3)u} \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} (1-u^2) - \frac{1}{2u} \right]
\end{aligned}
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\Delta z(\mu) &= \frac{3}{2} \frac{\Delta\bar{\Lambda}}{m}
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$z(\mu)$ is gauge invariant at $1/\beta_0$
Expand and integrate

Numerically

$$
\begin{aligned}
z(m) = 1 &- \frac{4}{3}\frac{\alpha_s^{(4)}(m)}{\pi} - (16.6629 - 4.5421)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^2 \\
&- (153.4076 + 42.6271 - 61.5397)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^3 \\
&- (1953.4013 + \cdots)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^4 + \cdots \\
= 1 &- \frac{4}{3}\frac{\alpha_s^{(4)}(m)}{\pi} - 12.1208\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^2 - 134.4950\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^3 \\
&- (1953.4013 + \cdots)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^4 + \cdots
\end{aligned}
$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$S(x) = S_L(x)$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x)\, e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}$$

$$\tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

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Landau, Khalatnikov (1955)
Fradkin (1955)
Zumino (1960)
Fukuda, Kubo, Yokoyama (1980)
Bogoliubov, Shirkov §45.5

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

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$$S(x) = S_0(x) e^{\sigma(x)}$$

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon)$$

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\sigma(x) &= \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon) \\
&= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)
\end{aligned}
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\log Z_\psi(\alpha, a) &= \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon} \\
\gamma_\psi(\alpha, a) &= 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)
\end{aligned}
$$

$d \log(a(\mu)\alpha(\mu))/d \log \mu = -2\varepsilon$ exactly

$\gamma_L(\alpha)$ starts from α^2

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Some Russian paper in the second half of the 1950s ?

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4 loops: Chetyrkin, Rétey (2000)

Gauge independence of $z(\mu)$ in QED

- $z_0 = Z_\psi^{\text{os}}$ gauge invariant
- $\log \tilde{Z}_\psi = (3 - a^{(0)}) \frac{\alpha^{(0)}}{4\pi\varepsilon}$
  Yennie, Frautschi, Suura (1961) exponentiation
  vanishes in the Yennie gauge
  $\alpha^{(0)} = \alpha_{\text{os}} \approx 1/137$
- $\log Z_\psi = -a^{(1)}(\mu) \frac{\alpha^{(1)}(\mu)}{4\pi\varepsilon} + (\text{gauge invariant})$
- Decoupling $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$
  Gauge dependence cancels in $\log(\tilde{Z}_\psi / Z_\psi)$

Result

$$z(m) = 1 - 4\frac{\alpha}{4\pi} + \left(16\pi^2 \log 2 - 24\zeta_3 - \frac{55}{3}\pi^2 + \frac{5957}{72}\right)\left(\frac{\alpha}{4\pi}\right)^2$$
$$- \left[1024a_4 + \frac{128}{3}\log^4 2 - 64\pi^2 \log^2 2 - \frac{11792}{9}\pi^2 \log 2\right.$$
$$\left. + 20\zeta_5 - 8\pi^2\zeta_3 + \frac{9494}{9}\zeta_3 + \frac{104}{45}\pi^4 + \frac{259133}{405}\pi^2 + \frac{249887}{324}\right]\left(\frac{\alpha}{4\pi}\right)^3$$

$$a_4 = \mathrm{Li}_4\left(\frac{1}{2}\right)$$

Conclusion

- Q via HQET operators: leading order and $1/m$ — one coefficient $z(\mu)$
- Calculated up to 3 loops;
  gauge dependent starting from the 3-rd loop
- Large β_0: all-order result at $1/\beta_0$ (gauge invariant)
- QED: $\gamma_\psi(\alpha, a) = 2a\dfrac{\alpha}{4\pi} + \gamma_L(\alpha)$
- QED: $z(\mu)$ is gauge invariant to all orders;
  calculated up to 3 loops