

# Shock wave in the Friedmann–Robertson-Walker space-time

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based on work by I. Ya. Aref'eva, E.O. Pozdeeva and A.A. Bagrov

- According to 't Hooft <sup>1</sup> shock waves <sup>2</sup> in the Minkowski space-time can be used to describe ultrarelativistic particles collisions.
- The shock gravitational waves are also known in (A)dS background. They are ultrarelativistic limits of Schwarzschild-(A)dS metrics <sup>3</sup>

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<sup>1</sup>G. 't Hooft, *Phys. Lett. B.* **198**, 61, 1987.

<sup>2</sup>P.C. Aichelburg and R.U. Sexl, *Gen. Relat. and Grav.*, V.24, 1971, 303.

<sup>3</sup>M. Hotta, M. Tanaka, *Clas. Quan. Grav.*, **10** (1993) 307–314.

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I.Ya. Aref'eva, A.A. Bagrov and L.V. Joukovskaya, *Algebra and analysis* **22(3)**, 3, 2010.

- Shock waves in AdS and in dS can be used to describe ultra relativistic particles collisions too <sup>4</sup>

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<sup>4</sup> S. S. Gubser, S. S. Pufu, A. Yarom, *Phys.Rev.D*, **78**, 2008, 066014  
I.Ya. Aref'eva, A.A. Bagrov and E.A. Guseva, *JHEP*, **0912**,009, 2009.

- In this talk the generalization of this construction for the ultra-relativistic particles in the Friedmann-Robertson-Walker space-time is presented.

- McVittie metric <sup>5</sup> in cosmological coordinates is

$$dS^2 = -\frac{\left(1 - \frac{m}{2a(t)\rho}\right)^2}{\left(1 + \frac{m}{2a(t)\rho}\right)^2} dt^2 + a(t)^2 \left(1 + \frac{m}{2a(t)\rho}\right)^4 (\rho^2 d\Omega^2 + d\rho^2),$$

$$d\Omega^2 = \sin^2 \theta d\phi^2 + d\theta^2,$$

where  $a(t)$  is arbitrary function of  $t$ .

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<sup>5</sup>G. C. McVittie, Mon. Not. R. Astron. Soc. 93, 325 (1933).

N. Kaloper, M. Klebanz and D. Martiny, McVittie's Legacy: Black Holes in an Expanding Universe, arXiv:1003.4777.

Some interesting cases of function  $a(t)$  corresponds to the following types of universes expansion:

- for  $a(t) = 1$ , the Hubble parameter  $H = 0$ , reduces McVittie metric to the Schwarzschild black hole of mass  $m$ ,
- for  $a(t) = e^{Ht}$ , the Hubble parameter  $H = \text{const}$ , reduces McVittie metric to de Sitter-Schwarzschild black hole of mass  $m$ ,
- for  $a(t) = k_2 t^n$ , the Hubble parameter  $H = \frac{\dot{a}}{a} = \frac{n}{t}$ .

## Shock wave in Minkowski space-time

The Schwarzschild black hole metric in Minkowski space-time:

$$ds_4^2 = -\frac{(1 - A^2)}{1 + A^2} dt^2 + (1 + A)^4(dx^2 + dy^2 + dz^2), \quad (1)$$
$$A = \frac{m}{2r}, \quad r^2 = x^2 + y^2 + z^2.$$

The first order small mass approximation

$$ds_1^2 = ds_{4M}^2 + 4A(ds_{4M}^2 + 2dt^2), \quad ds_{4M} = ds_4|_{A=0}.$$

## Shock wave in Minkowski space-time

The Lorenz transformation is

$$t = \gamma(\bar{t} - v\bar{x}), \quad x = \gamma(\bar{t} - v\bar{x}), \quad \gamma = \frac{1}{\sqrt{1 - v^2}}.$$

In terms of  $\bar{t}$ ,  $\bar{x}$  the function  $A$  is

$$A = \frac{p(1 - v^2)}{2\sqrt{(\bar{x} - v\bar{t})^2 + (1 - v^2)(\bar{y}^2 + \bar{z}^2)}}, \quad \text{where } p = m\gamma$$

and

$$dt^2 = \frac{(d\bar{t} - v d\bar{x})^2}{1 - v^2}.$$

Shock wave in Minkowski space-time

$$ds_\gamma^2 = ds_{4M}^2 + 4p \left( \frac{1}{|\bar{t} - \bar{x}|} - 2 \ln(\bar{y}^2 + \bar{z}^2)^{1/2} \delta(\bar{t}^2 - \bar{x}^2) \right) (d(\bar{t} - \bar{x}))^2,$$

is obtained by the ultra relativistic limit  $\gamma \rightarrow \infty$ .



## Shock wave in dS space-time

The Schwarzschild black hole metric in dS space-time:

$$dS^2 = - \left( 1 - \frac{2m}{R} - \frac{R^2}{b^2} \right) dt^2 + \frac{dR^2}{\left( 1 - \frac{2m}{R} - \frac{R^2}{b^2} \right)} + R^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

The first order small mass approximation of Schwarzschild black hole metric in dS

$$ds^2 = ds_{dS}^2 + \frac{2m}{R} dt^2 + \frac{2m}{R} \frac{dR^2}{\left( 1 - \frac{R^2}{b^2} \right)^2}, \quad ds_{dS}^2 = dS^2|_{m=0}$$

### Shock wave in dS space-time

- In the plane coordinates representation the metric is:

$$ds^2 = ds_{5M}^2 + ds_p^2, \text{ where } ds_{5M}^2 = -dZ_0^2 + \sum_{i=1}^4 dZ_i^2,$$

$$ds_p^2 = \frac{2mb^2}{(Z_4^2 - Z_0^2)^2 (b^2 + Z_0^2 - Z_4^2)^{3/2}} \times \\ ((b^2(Z_4^2 + Z_0^2) + Z_0^2 Z_4^2 - Z_4^4) dZ_0^2 - \\ -2(2b^2 + Z_0^2 - Z_4^2) dZ_0 dZ_4 + (b^2(Z_4^2 + Z_0^2) + Z_0^4 - Z_0^2 Z_4^2) dZ_4^2).$$

- The 4D hyperboloid condition to the coordinates in dS:

$$-Z_0^2 + \sum_{i=1}^4 Z_i^2 = b^2.$$

## Shock wave in dS space-time

- The Lorenz transformation along  $Z_1$  coordinate:

$$Z_0 = \gamma(Y_0 + vY_1), \quad Z_1 = \gamma(vY_0 + Y_1).$$

is applied to first order small mass approximation of Schwarzschild black hole in dS with mass rescaling  $m = p/\gamma$ .

- Shock wave in Minkowski space-time is

$$ds_\gamma^2 = -dY_0^2 + \sum_{i=1}^4 dY_i^2 + \\ + 4p \left( -2 + \frac{Y_4}{b} \ln \left( \frac{b + Y_4}{b - Y_4} \right) \right) \delta(Y_0 + Y_1) (d(Y_0 + Y_4))^2.$$

# Shock wave in Friedmann-Robertson-Walker space-time

## Coordinates relations

- For description ultrarelativistic particles movement by boost in plane coordinates representation can use the relation of 5D Minkowski space-time coordinates with 4D FRW coordinates.

## Shock wave in Friedmann-Robertson-Walker space-time Coordinates relations

- Connection between four-dimensional spatially flat cosmology and five-dimensional Minkowski space-time (see, for example, <sup>6</sup>).
- ◇ Consider the 5D Minkowski metric and 4D FRW metric:

$$dS_{5M}^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2, \quad M_5, \quad D=5,$$

$$ds_{FRW}^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad \text{FRW}, \quad D=4.$$

- ◇ If  $a(t)$  is arbitrary function of  $t$ , then the hyperboloid condition becomes non-stationary:

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = b^2(t)$$

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<sup>6</sup> M. N. Smolyakov Class.Quant.Grav.25:238003,2008

# Shock wave in Friedmann-Robertson-Walker space-time Coordinates relations

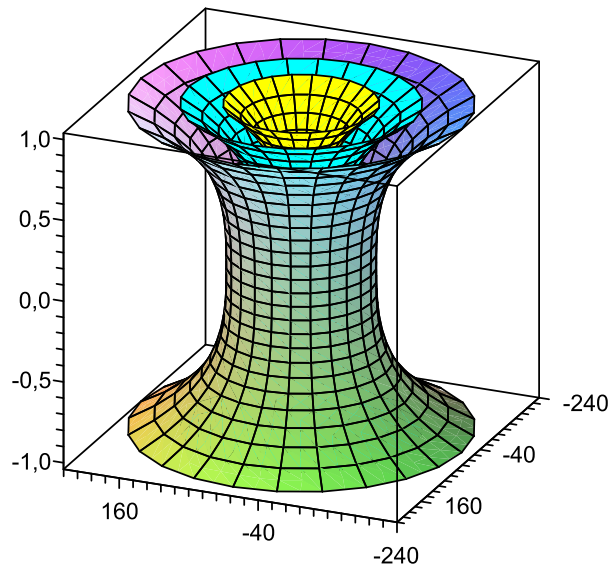


Figure 1: Hyperboloid for different  $t$ .

## Shock wave in Friedmann-Robertson-Walker space-time Coordinates relations

- The surface is defined by:

$$Z_0 = \frac{1}{2}\kappa_1 a(t) - \frac{1}{2} \frac{b^2(t)}{\kappa_1 a(t)} + \frac{1}{2} \frac{a(t)(x^2 + y^2 + z^2)}{\kappa_1},$$

$$Z_4 = \frac{1}{2}\kappa_1 a(t) + \frac{1}{2} \frac{b^2(t)}{\kappa_1 a(t)} - \frac{1}{2} \frac{a(t)(x^2 + y^2 + z^2)}{\kappa_1},$$

$$Z_1 = a(t)x, \quad Z_2 = a(t)y, \quad Z_3 = a(t)z.$$

- The metric in 5D Minkowski space-time is equal to metric in 4D FRW, if the following condition relates  $a(t)$  with  $b(t)$ ):

$$- \left( \frac{da(t)}{dt} \frac{b(t)}{a(t)} \right)^2 + 2 \frac{da(t)}{dt} \frac{db(t)}{dt} \frac{b(t)}{a(t)} + 1 = 0.$$

- In the case  $a(t) = \kappa_2 t^n$ , we get  $b(t) = \pm \frac{t}{\sqrt{n(n-2)}}$ .

## McVittie metric in small mass approximation

- McVittie metric

$$ds^2 = -\frac{(1-\mu)^2}{(1+\mu)^2} dt^2 + a^2(t) (1+\mu)^4 (dx^2 + dy^2 + dz^2),$$
$$\mu = \frac{m}{2a(t)\rho}.$$

- First order approximation ( $m^2 \sim 0$ ),

$$\frac{(1-\mu)^2}{(1+\mu)^2} \approx 1 - 4\mu, \quad (1+\mu)^4 \approx 1 + 4\mu,$$

to McVittie's metric is

$$ds_1^2 = ds_{FRW}^2 + 4\mu(ds_{FRW}^2 + 2dt^2).$$



## Shock wave in Friedmann-Robertson-Walker space-time

### McVittie metric in small mass approximation

- For  $a(t) = k_2 t^n$  the metric can be written in plane coordinates:

$$ds^2 = ds_{5M}^2 + \frac{2m}{\sqrt{Z_i^2}} \left( ds_{5M}^2 + \frac{2d(Z_0 + Z_4)^2}{n^2 \kappa_1^2 \kappa_2^2 (n(n-2)b^2(t))^{n-1}} \right),$$

where  $b^2(t) = -Z_0^2 + Z_i^2 + Z_4^2$ ,  $i = \overline{1, 3}$ .

## Lorentz transformation

- Boost in the 5-dimensional Minkowski space-time:

$$Z_0 = \gamma(\tilde{Z}_0 + v\tilde{Z}_1), \quad Z_1 = \gamma(\tilde{Z}_1 + v\tilde{Z}_0), \quad \gamma = \frac{1}{\sqrt{1 - v^2}}.$$

- We apply the Lorentz transformation to the McVittie metric in the first order small mass approximation:

$$ds_\gamma^2 = ds_{5M}^2 + \frac{2\tilde{m} \left( ds_{5M}^2 + 2 \frac{d(\gamma(\tilde{Z}_0 + v\tilde{Z}_1) + \tilde{Z}_4)^2}{p^2 \kappa_1^2 \kappa_2^2 (p(p-2)b^2(t))^{p-1}} \right)}{\gamma \sqrt{\gamma^2(v\tilde{Z}_0 + \tilde{Z}_1)^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2}}, \quad \tilde{m} = m\gamma$$

## Shock wave in Friedmann-Robertson-Walker space-time

### Lorentz transformation

or

$$ds_\gamma^2 = ds_{5M}^2 + \frac{2\tilde{m} \left( ds_{5M}^2 + 2 \frac{d(\gamma(\tilde{Z}_0 + v\tilde{Z}_1) + \tilde{Z}_4)^2}{p^2 \kappa_1^2 \kappa_2^2 t^{2(p-1)}} \right)}{\gamma \sqrt{\gamma^2 (v\tilde{Z}_0 + \tilde{Z}_1)^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2}}.$$

- For  $\gamma \rightarrow \infty$ , it is evidently that:

$$ds^2 \Big|_{v \rightarrow 1} \rightarrow ds_{5M}^2 + \frac{4\tilde{m}\gamma}{\sqrt{\gamma^2 (\tilde{Z}_0 + \tilde{Z}_1)^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2}} \left( \frac{d(\tilde{Z}_0 + \tilde{Z}_1)^2}{p^2 \kappa_1^2 \kappa_2^2 t^{2(p-1)}} \right)$$

## Limiting process $\gamma \rightarrow \infty$

- Limiting process  $\gamma \rightarrow \infty$  in generalized function meaning:

$$\int_{-\infty}^{\infty} \frac{\gamma}{\sqrt{\gamma^2 U^2 + X^2}} f(U) dU = f(0) \ln \frac{4\gamma^2}{X^2} + \int_{-\infty}^{\infty} \left( \frac{1}{|U|} \right)_{reg} f(U) dU$$

where

$$\begin{aligned} \int_{-\infty}^{\infty} \left( \frac{1}{|U|} \right)_{reg} f(U) dU &\equiv \\ &\equiv \int_{-1}^1 \frac{f(U) - f(0)}{|U|} dU + \int_{-\infty}^{-1} \frac{1}{|U|} f(U) dU + \int_1^{\infty} \frac{1}{|U|} f(U) dU. \end{aligned}$$

## Shock wave in Friedmann-Robertson-Walker space-time

Limiting process  $\gamma \rightarrow \infty$

The result can be presented by the Dirac-delta function

$$\lim_{\gamma \rightarrow \infty} \left[ \frac{\gamma}{\sqrt{\gamma^2 U^2 + X^2}} - \delta(U) \ln \gamma^2 \right] = -\delta(U) \ln \frac{X^2}{4} + \left( \frac{1}{|U|} \right)_{reg}.$$

## Lorentz transformations in the ultrarelativistic limit the McVittie metric

- After the regularization we have the gravitational waves metric

$$ds_\gamma^2 = ds_{5M}^2 + \frac{4\bar{m}}{p^2 \kappa_1^2 \kappa_2^2 (t)^{2(p-1)}} \delta(U) d(U)^2, \quad \bar{m} = \tilde{m} \ln \gamma^2, \quad U = Z_0 + Z_1,$$

where

$$t = \left( \frac{Z_0 + Z_4}{k_1 k_2} \right)^{1/n}, \quad t^2 = n(n-2)(-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2)$$

## Shock wave in Friedmann-Robertson-Walker space-time

### Lorentz transformations in the ultrarelativistic limit the McVittie metric

- The obtained metric can be presented with cosmological coordinates:

$$ds_\gamma^2 = ds_{FRW}^2 + \frac{4\bar{m}}{n^2 \kappa_1^2 \kappa_2^2 (t)^{2(n-1)}} \delta(U) d(U)^2,$$

$$U = \frac{1}{2} k_1 k_2 t^n - \frac{1}{2n(n-2)k_1 k_2 t^n} t^2 + \frac{1}{2} \frac{k_2 t^n (x^2 + y^2 + z^2)}{k_1} + k_2 t^n x$$

The most interesting for us case  $U = 0$ . Shock wave profile  $F(U)$

for  $U = 0$  is proportional to  $\frac{1}{(V/2 + Z_4)^{2/n}}$  :

$$F(U)|_{u \sim 0} \sim \frac{1}{(V/2 + Z_4)^{2/n}}. \quad (2)$$

## Conclusion

- It is proposed to use the boosted McVittie metric such as model of ultrarelativistic particle in the Friedmann-Robertson-Walker space-time with  $a(t) = kt^n$ .
- The shock wave corresponding ultrarelativistic particle in the Friedmann-Robertson-Walker space-time is constructed.



**Thank you for attention!**