ON THE PROBLEM OF UNIQUENESS AND HERMITICITY OF HAMILTONIANS FOR DIRAC PARTICLES IN GRAVITATIONAL FIELDS

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According to the paper of M.V.Gorbatenko, V.P.Neznamov
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The authors prove that the dynamics of spin 1/2 particles in stationary gravitational fields can be described using an approach, which builds upon the formalism of pseudo-Hermitian Hamiltonians.

The proof consists in the analysis of three expressions for Hamiltonians, which are derived from the Dirac equation and describe the dynamics of spin 1/2 particles in the gravitational field of the Kerr solution. The Hamiltonians correspond to different choices of tetrad vectors and differ from each other. The differences between the Hamiltonians confirm the conclusion known from many studies that the Hamiltonians derived from the Dirac equation are non-unique. Application of standard pseudo-Hermitian quantum mechanics rules to each of these Hamiltonians produces the same Hermitian Hamiltonian.

The eigenvalue spectrum of the resulting Hamiltonian is the same as that of the Hamiltonians derived from the Dirac equation with any chosen system of tetrad vectors.

For description of the dynamics of spin 1/2 particles in stationary gravitational fields can be used not only the formalism of pseudo-Hermitian Hamiltonians, but also an alternative approach, which employs the Parker scalar product. The authors show that the alternative approach is equivalent to the formalism of pseudo-Hermitian Hamiltonians.

The Dirac equation and reducing of him to the form of the Schrödinger equation

$$\gamma^{\alpha}\left(\frac{\partial\psi}{\partial x^{\alpha}}+\Phi_{\alpha}\psi\right)-m\psi=0$$

$$\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}=2g^{\alpha\beta}E$$

$$H_{\underline{\alpha}}^{\mu}H_{\underline{\beta}}^{\nu}g_{\mu\nu}=\eta_{\underline{\alpha}\underline{\beta}} \qquad \eta_{\underline{\alpha}\underline{\beta}}=\mathrm{diag}\left[-1,1,1,1\right]$$

$$\gamma^{\underline{\alpha}}\gamma^{\underline{\beta}}+\gamma^{\underline{\beta}}\gamma^{\underline{\alpha}}=2\eta^{\underline{\alpha}\underline{\beta}}E$$

$$\Phi_{\alpha}=-\frac{1}{4}H_{\mu}^{\underline{\varepsilon}}H_{\nu\underline{\varepsilon};\alpha}S^{\mu\nu} \qquad S^{\mu\nu}=\frac{1}{2}\left(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}\right) \qquad \gamma^{\alpha}=H_{\underline{\beta}}^{\alpha}\gamma^{\underline{\beta}}$$

$i\dfrac{\partial\Psi}{\partial t}=\hat{H}\Psi,$ t **- time of infinitive distant observer**

$$\hat{H}=-\frac{im}{\left(-g^{00}\right)}\gamma^{0}+\frac{i}{\left(-g^{00}\right)}\gamma^{0}\gamma^{k}\frac{\partial}{\partial x^{k}}-i\Phi_{0}+\frac{i}{\left(-g^{00}\right)}\gamma^{0}\gamma^{k}\Phi_{k}$$

$\hat{H}\neq\hat{H}^{\dagger}$ **and depend on the choice of tetrad vectors**

The Kerr solution

(solution of Einstein equations for an axially symmetrical stationary gravitational field generated by a rotating point body with mass M and moment $\mathbf{J}$)

The metric of the Kerr solution in the first-order approximation in "mass" with length dimensionality ($M \to GM/c^2$, $\mathbf{J} \to \mathbf{J}/Mc$)

$$\left.\begin{aligned}
&g_{00} = -1 + 2\frac{M}{R}; \quad g_{0k} = 2\frac{M\left(J_{kl}R_l\right)}{R^3}; \quad g_{mn} = \delta_{mn} + 2\frac{M}{R}\delta_{mn};\\
&\sqrt{-g} = 1 + 2\frac{M}{R};\\
&g^{00} = -1 - 2\frac{M}{R}; \quad g^{0k} = 2\frac{M\left(J_{kl}R_l\right)}{R^3}; \quad g^{mn} = \delta_{mn} - 2\frac{M}{R}\delta_{mn}.
\end{aligned}\right\}$$

The systems of tetrad vectors

The Killing system of tetrad vectors:

$$\tilde{H}_{\underline{0}}^{0}=\left[1+\frac{M}{R}\right];\quad \tilde{H}_{\underline{0}}^{k}=0;\quad \tilde{H}_{\underline{k}}^{0}=2\frac{M\left(J_{kl}R_{l}\right)}{R^{3}};\quad \tilde{H}_{\underline{k}}^{m}=\delta_{mk}\left[1-\frac{M}{R}\right]$$

System of tetrad vectors used in papers of Hehl, Ni, Obukhov, Silenko, Teryaev:

$$H_{\underline{0}}^{\prime 0}=\left[1+\frac{M}{R}\right];\quad H_{\underline{0}}^{\prime k}=-2\frac{M\left(J_{kl}R_{l}\right)}{R^{3}};\quad H_{\underline{k}}^{\prime 0}=0;\quad H_{\underline{k}}^{\prime m}=\delta_{mk}\left[1-\frac{M}{R}\right]$$

System of tetrad vectors in symmetric gauge:

$$H_{\underline{0}}^{0}=\left[1+\frac{M}{R}\right];\quad H_{\underline{0}}^{k}=-\frac{M\left(J_{kl}R_{l}\right)}{R^{3}};\quad H_{\underline{k}}^{0}=\frac{M\left(J_{kl}R_{l}\right)}{R^{3}};\quad H_{\underline{k}}^{m}=\delta_{mk}\left[1-\frac{M}{R}\right]$$

HAMILTONIANS FOR THE KERR SOLUTION

The Killing system of tetrad vectors:

$$\hat{\tilde{\tilde{H}}} =$$

$$= im\gamma_{\underline{0}} - im\frac{M}{R}\gamma_{\underline{0}} - i\gamma_{\underline{0}}\gamma^{\underline{k}}\frac{\partial}{\partial x^k} + 2i\frac{M}{R}\gamma_{\underline{0}}\gamma^{\underline{k}}\frac{\partial}{\partial x^k} + \frac{i}{2}\frac{MR_k}{R^3}\gamma_{\underline{0}}\gamma^{\underline{k}} +$$

$$+2i\frac{M\left(J_{kl}R_l\right)}{R^3}\frac{\partial}{\partial x^k} -$$

$$-2im\frac{M\left(J_{kl}R_l\right)}{R^3}\gamma^{\underline{k}} + 2i\frac{M\left(J_{ml}R_l\right)}{R^3}S_{\underline{mk}}\frac{\partial}{\partial x^k} -$$

$$-\frac{i}{2}\left\{\frac{M}{R^3}J_k - 3\frac{M\left(J_lR_l\right)R_k}{R^5}\right\}\gamma_{\underline{5}}\gamma_{\underline{0}}\gamma_{\underline{k}}.$$

HAMILTONIANS FOR THE KERR SOLUTION

System of tetrad vectors used in papers of Hehl, Ni, Obukhov, Silenko, Teryaev:

$$\hat{H}' = im\gamma_{\underline{0}} - im\frac{M}{R}\gamma_{\underline{0}} - i\gamma_{\underline{0}}\gamma_{\underline{k}}\frac{\partial}{\partial x^k} + 2i\frac{M}{R}\gamma_{\underline{0}}\gamma^{\underline{k}}\frac{\partial}{\partial x^k} + \frac{i}{2}\frac{MR_k}{R^3}\gamma_{\underline{0}}\gamma^{\underline{k}} +$$

$$+2i\frac{M\left(J_{kl}R_l\right)}{R^3}\frac{\partial}{\partial x^k} +$$

$$+\frac{i}{2}\left\{\frac{M}{R^3}J_k - 3\frac{M\left(J_l R_l\right)R_k}{R^5}\right\}\gamma_{\underline{5}}\gamma_{\underline{0}}\gamma_{\underline{k}}.$$

HAMILTONIANS FOR THE KERR SOLUTION

System of tetrad vectors in symmetric gauge:

$$\hat{H} = im\gamma_{\underline{0}} - i\gamma_{\underline{0}}\gamma_{\underline{k}}\frac{\partial}{\partial x^k} - im\frac{M}{R}\gamma_{\underline{0}} + 2i\frac{M}{R}\gamma_{\underline{0}}\gamma^{\underline{k}}\frac{\partial}{\partial x^k} + \frac{i}{2}\frac{MR_k}{R^3}\gamma_{\underline{0}}\gamma^{\underline{k}} +$$

$$+2i\frac{M\left(J_{kl}R_l\right)}{R^3}\frac{\partial}{\partial x^k} -$$

$$-im\frac{M\left(J_{kl}R_l\right)}{R^3}\cdot\gamma_{\underline{k}} + i\frac{M\left(J_{ml}R_l\right)}{R^3}S_{\underline{mk}}\frac{\partial}{\partial x^k}.$$

Pseudo-Hermitian quantum mechanics

$\rho \hat{H} \rho^{-1} = \hat{H}^{\dagger}$ **- the condition of pseudo-Hermiticity of the Hamiltonian**

If $\rho = \eta^{\dagger}\eta$ **, then** $\hat{\mathrm{H}} = \eta \hat{H} \eta^{-1}$ **and** $\hat{\mathrm{H}} = \hat{\mathrm{H}}^{\dagger}$

$\hat{\mathrm{H}}$ **- the Hamiltonian in** η **-representation**

$$i\frac{\partial \Psi}{\partial t} = \hat{\mathrm{H}}\Psi$$

$$(\Phi, \Psi) = \int d^3x \left(\Phi^{\dagger}\Psi\right)$$ **-standard scalar product for Hermitian Hamiltonians**

$$\Psi = \hat{\eta}\psi$$

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi$$

$$\langle \varphi, \psi \rangle_{\rho} = \int d^3x \left(\varphi^{\dagger}\rho\psi\right)$$ **-scalar product for pseudo-Hermitian Hamiltonians**

Operators ρ and η for three systems of system vectors

Table 1

Tetrad vectors	Operator ρ	Operator η
Killing system	$1+3\frac{M}{R}+2\frac{M\left(J_{km}R_m\right)}{R^3}\gamma_{\underline{0}}\gamma_{\underline{k}}$	$1+\frac{3}{2}\frac{M}{R}+\frac{M\left(J_{km}R_m\right)}{R^3}\gamma_{\underline{0}}\gamma_{\underline{k}}$
System used by Hehl, Ni, Obukhov, Silenko, Teryaev	$1+3\frac{M}{R}$	$1+\frac{3}{2}\frac{M}{R}$
System of tetrad vectors in symmetric gauge	$1+3\frac{M}{R}+\frac{M\left(J_{km}R_m\right)}{R^3}\gamma_{\underline{0}}\gamma_{\underline{k}}$	$1+\frac{3}{2}\frac{M}{R}+\frac{1}{2}\frac{M\left(J_{km}R_m\right)}{R^3}\gamma_{\underline{0}}\gamma_{\underline{k}}$

$$\rho\,\hat{H}\,\rho^{-1}=\hat{H}^{\dagger}$$

$$\hat{\mathrm{H}}=\eta\,\hat{H}\,\eta^{-1}=\hat{\mathrm{H}}^{\dagger}$$

- the same for any choice of tetrad vectors

The Hermitian Hamiltonian operator $\hat{\mathrm{H}}$

$\hat{\mathrm{H}} = \eta \hat{H} \eta^{-1} = \hat{\mathrm{H}}^{\dagger}$ **is the same for any system of tetrad vectors** $=$

$$
\begin{aligned}
&= im\gamma_{\underline{0}} - im\frac{M}{R}\gamma_{\underline{0}} - i\gamma_{\underline{0}}\gamma_{\underline{k}}\frac{\partial}{\partial x^{k}} + 2i\frac{M}{R}\gamma_{\underline{0}}\gamma^{\underline{k}}\frac{\partial}{\partial x^{k}} - i\frac{MR_{k}}{R^{3}}\gamma_{\underline{0}}\gamma^{\underline{k}} + \\
&+ 2i\frac{M\left(J_{kl}R_{l}\right)}{R^{3}}\frac{\partial}{\partial x^{k}} + \\
&+ \frac{i}{2}\left\{\frac{M}{R^{3}}J_{k} - 3\frac{M\left(J_{l}R_{l}\right)R_{k}}{R^{5}}\right\}\gamma_{\underline{5}}\gamma_{\underline{0}}\gamma_{\underline{k}}.
\end{aligned}
$$

THE PARKER SCALAR PRODUCT

The matrix: $\gamma_{\underline{0}}$ $\qquad \gamma_{\underline{\alpha}}^{\dagger} = \gamma_{\underline{0}}\gamma_{\underline{\alpha}}\gamma_{\underline{0}}, \quad \gamma_{\alpha}^{\dagger} = \gamma_{\underline{0}}\gamma_{\alpha}\gamma_{\underline{0}}$

$$\langle \varphi, \psi \rangle = \int dx^3 \sqrt{-g}\, \left(\varphi^{\dagger} \gamma_{\underline{0}} \gamma^{0} \psi \right)$$

For three system of tetrad vectors considered by us

$$\sqrt{-g}\,\gamma_{\underline{0}}\gamma^{0} = \rho$$

that is

$$\langle \varphi, \psi \rangle = \langle \varphi, \psi \rangle_{\rho}$$

Hermiticity of Hamiltonian concerning Parker scalar product

$$\Delta \equiv \left\langle \varphi, (\hat{H}\psi) \right\rangle - \left\langle (\hat{H}\varphi), \psi \right\rangle \qquad \Delta \equiv \Delta_1 + \Delta_2 + \Delta_3$$

$$\Delta_1 = \int dx^3 \sqrt{-g} \left(\varphi^\dagger \gamma_{\underline{0}} \gamma^0 \left\{ -\frac{im}{\left(-g^{00}\right)} \gamma^0 \right\} \psi \right) -$$

$$-\int dx^3 \sqrt{-g} \left(\left(\left\{ -\frac{im}{\left(-g^{00}\right)} \gamma^0 \right\} \varphi \right)^\dagger \gamma_{\underline{0}} \gamma^0 \psi \right)$$

$$\Delta_2 = \int dx^3 \sqrt{-g} \left(\varphi^\dagger \gamma_{\underline{0}} \gamma^0 \left\{ -i\Phi_0 \right\} \psi \right) - \int dx^3 \sqrt{-g} \left(\left(\left\{ -i\Phi_0 \right\} \varphi \right)^\dagger \gamma_{\underline{0}} \gamma^0 \psi \right)$$

$$\Delta_3 = \int dx^3 \sqrt{-g} \left(\varphi^\dagger \gamma_{\underline{0}} \gamma^0 \left\{ \frac{i}{\left(-g^{00}\right)} \gamma^0 \gamma^k \nabla_k \right\} \psi \right) -$$

$$\Delta = 0$$

$$-\int dx^3 \sqrt{-g} \left(\left(\left\{ \frac{i}{\left(-g^{00}\right)} \gamma^0 \gamma^k \nabla_k \right\} \varphi \right)^\dagger \gamma_{\underline{0}} \gamma^0 \psi \right).$$

Comparison of quantum mechanics treatment of dynamics of Dirac particle in stationary gravitational field (I)

Table 2

	Methods of quantum mechanics treatment		
	A	B	C
	Hermitian quantum mechanics in η - representation	Pseudo-Hermitian quantum mechanics in initial representation	Approach based on Parker scalar product
Hamiltonian	$\hat{\mathrm{H}} = \hat{\mathrm{H}}^{\dagger}$	$\hat{H} = \rho^{-1}\hat{H}^{\dagger}\rho$	$\hat{H} = \left(\sqrt{-g}\gamma_{\underline{0}}\gamma^{0}\right)^{-1}\hat{H}^{\dagger}\left(\sqrt{-g}\gamma_{\underline{0}}\gamma^{0}\right)$
Dependence of Hamiltonian type on choice of system of tetrad vectors	No	Yes	Yes

Comparison of quantum mechanics treatment of dynamics of Dirac particle in stationary gravitational field (II)

Table 3

	Methods of quantum mechanics treatment		
	A	**B**	**C**
	Hermitian quantum mechanics in η - representation	**Pseudo-Hermitian quantum mechanics in initial representation**	**Approach based on Parker scalar product**
Scalar product	**Standard scalar product for Hilbert space:** $(\Phi,\Psi)=\int d^3x\left(\Phi^\dagger\Psi\right)$	**With an weight operator** $\rho=\eta^\dagger\eta$ **:** $\langle\varphi,\psi\rangle_\rho=\int d^3x\left(\varphi^\dagger\rho\psi\right)$	$\rho=\eta^\dagger\eta=\left(\sqrt{-g}\,\gamma_{\underline{0}}\gamma^0\right)$ $\langle\varphi,\psi\rangle=$ $=\int d^3x\,\sqrt{-g}\left(\varphi^\dagger\gamma_{\underline{0}}\gamma^0\psi\right)$
Connection between Hamiltonians	$\mathrm{\hat{H}}=\eta\hat{H}\eta^{-1}$		
Connection between scalar products	$(\Phi,\Psi)=\langle\varphi,\psi\rangle_\rho=\langle\varphi,\psi\rangle$		
Connection between wave function	$\Psi=\eta\psi$		

Conclusions

The results obtained in this study make it possible to look at the description of quantum mechanics of spin 1/2 particles in stationary gravitational fields from a new point of view.

For example, before this study, the problem of non-uniqueness of Hamiltonians and their sensitivity to the choice of tetrad vectors in our opinion was unresolved. The apparatus of pseudo-Hermitian quantum mechanics used in this study allowed us to resolve this issue at least as applied to the Schwarzschild and the Kerr solution. It was proven that the resulting Hamiltonian $\hat{\mathrm{H}} = \eta \hat{H} \eta^{-1}$ **does not depend on the choice of tetrad vectors and is Hermitian. The scalar products also do not dependent on the choice of tetrad vectors, if they were calculated using operators** ρ **shown in Table 1.**

In our opinion, the uniqueness of the Hamiltonian $\hat{\mathrm{H}}$ is nontrivial. Indeed, all of the three initial expressions for Hamiltonians, first, differ from each other and, second, are non-Hermitian concerning standard scalar product in Hilbert space. After applying procedures for the transformion of initial Hamiltonians $\hat{H}$ into their Hermitian expressions $\hat{\mathrm{H}}$, the latter could in principle differ in some Hermitian summands. It is not the case, however, and the expressions for $\hat{\mathrm{H}}$ are the same in all the three cases. Such a coincidence means that whatever the choice of tetrad vectors in a gravitational field there will always exist a single Hermitian Hamiltonian $\hat{\mathrm{H}}$, which has the same spectrum of energy levels as any of the starting operators $\hat{H}$.

Upon transition to the Hamiltonian $\hat{\mathrm{H}}$, one can use the quantum mechanics apparatus in its standard form. In particular, the left member of the Schrödinger equation will contain the operator $i\left(\partial/\partial t\right)$, in which the time coordinate t is understood to be the time of an infinitely distant observer.

Interestingly, the expression derived in this study for the Hamiltonian $\hat{\mathrm{H}}$ is the same as the expression proposed Obukhov, Silenko, Teryaev for the weak Kerr field. The formalism of pseudo-Hermitian Hamiltonians in fact validates the expressions for Hermitian Hamiltonians used Obukhov, Silenko, Teryaev .

The results of this study allow us to claim that pseudo-Hermitian Hamiltonians provides for the application of the relativistic quantum mechanics formalism practically in its standard form. The expression for the operator $\hat{\mathrm{H}}$ in η - representation allows to get rid of "ambiguity" connected with different type of the initial Hamiltonians at the use of different system of tetrad vectors. We think that this feature of the pseudo-Hermitian method makes it preferable as applied to the problems, in which gravitational effects are resolved and their quantitative characteristics are analyzed.

Thank you for your attention