Relationship between Conformal Geometrodynamics Equations and Dirac Equation

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Subject of Discussion:

I	**What is Conformal Geometrodynamics (CGD) understood to mean?** Minimum conformally invariant extension of GR equations. CGD equations in flat space in terms of vector and anti-symmetric second-rank tensor.
II	**Algebraic mapping of a set of tensors** {scalar, pseudoscalar, vector, pseudovector, anti-symmetric second-rank tensor} **to a set of four complex bispinors** {bispinor matrix}.
III	Main Statement: **Dynamics of a set of tensors {vector, anti-symmetric second-rank tensor} governed by CGD equations can be mapped to the dynamics of a set of four real bispinors {bispinor matrix} governed by the Dirac equation.**

Properties of Conformal Transformations

Conformal transformations are the best candidates to play the role of the base group of transformations for the extension of GR equations. The best because conformal transformations are the transformations that preserve cause and effect relations between events.

The conformal transformations include inversion transformations, i.e. transformations that can lead to relations between solutions that describe macro- and micro-scale processes.

What is "Conformal Geometrodynamics"?

CGD equations have the form of equations of General Relativity

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}$$

in which the energy-momentum tensor has a specific structure

$$T_{\alpha\beta} = -2A_\alpha A_\beta - g_{\alpha\beta} A^2 - 2 g_{\alpha\beta} A^\nu{}_{;\nu} + A_{\alpha;\beta} + A_{\beta;\alpha} + \lambda g_{\alpha\beta}$$

CGD equations are invariant with respect to conformal transformations

$$g_{\alpha\beta} \to g_{\alpha\beta} \cdot e^{2\sigma}, \quad A_\alpha \to A_\alpha - \sigma_{;\alpha}, \quad \lambda \to \lambda \cdot e^{-2\sigma}$$

Properties of CGD Equations

1) For them, the Cauchy problem is defined without links to Cauchy data on the initial hypersurface.

2) They lead to the equations $T_{\alpha}{}^{\mu}{}_{;\mu} = 0$, which in turn lead to the existence of the vector $J_{\alpha} \equiv \lambda_{,\alpha} - 2\lambda A_{\alpha}$, satisfying the relationship $J^{\alpha}{}_{;\alpha} = 0$.

3) The presence of the vector J^{α} allows us to introduce, using the Eckart method, thermodynamic functions of state of geometrodynamic continuum: energy density, pressure, specific volume, viscosity coefficient, heat conductivity coefficient, entropy density, temperature.

CGD Equations in Flat Space

Equations that follow strictly from CGD equations

$$T_{\alpha}{}^{\mu}{}_{;\mu} = 0$$

in flat space:

$$\left(J_{\beta,\alpha} - J_{\alpha,\beta}\right) = 4m \cdot H_{\alpha\beta} \qquad (1)$$

$$H_{\alpha}{}^{\nu}{}_{;\nu} = -m \cdot J_{\alpha} \qquad (2)$$

$$J^{\nu}{}_{;\nu} = 0 \qquad m = \mathrm{Const} \qquad (3)$$

Algorithm for Bispinor Mapping to Tensors

Direct problem:

We have a bispinor ψ . We seek tensor quantities generated by the bispinor. Solution:

1)Scalar $$a = \frac{1}{4}\mathrm{Sp}\left\{\psi^{+}D\psi\right\}$$

2)Pseudoscalar $$b = -\frac{i}{4}\mathrm{Sp}\left\{\psi^{+}D\gamma_5\psi\right\}$$

3)Vector $$j^{\alpha} = \frac{1}{4}\mathrm{Sp}\left\{\psi^{+}D\gamma^{\alpha}\psi\right\}$$

4)Pseudovector $$s^{\alpha} = -\frac{i}{4}\mathrm{Sp}\left\{\psi^{+}D\gamma_5\gamma^{\alpha}\psi\right\}$$

5)Antisymmetric tensor $$h_{\alpha\beta} = -\frac{1}{8}\mathrm{Sp}\left\{\psi^{+}DS_{\alpha\beta}\psi\right\}$$

Algorithm for Tensor Mapping to Bispinors

Inverse problem:

At the point of Riemann space, we have arbitrarily defined tensor quantities:

1)Scalar	a
2)Pseudoscalar	b
3)Vector	j^{α}
4)Pseudovector	s^{α}
5)Antisymmetric tensor	$h_{\alpha\beta}$

We seek a set of bispinors reproducing this set of tensor quantities.

Formalism of Dirac Matrices

Definition of Dirac matrix: $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta}$

Anti-hermitizing matrix: $D\gamma_\mu D^{-1} = -\gamma_\mu^+$

Bispinor matrix: Z

Operation of covariant differentiation:
$$\left.\begin{aligned} \nabla_\alpha Z &= Z_{;\alpha} - Z\Gamma_\alpha \\ \nabla_\alpha Z^+ &= Z^+{}_{;\alpha} + \Gamma_\alpha Z^+ \end{aligned}\right\}$$

Dirac equation: $\gamma^\nu \left(\nabla_\nu Z\right) = m \cdot Z$

Solving the Inverse Problem with Defined Vector and Antisymmetric Tensor

1) Based on the vector and antisymmetric tensor, construct the matrix *M:*

$$M = j^{\alpha} \cdot \left(\gamma_{\alpha} D^{-1}\right) + h^{\mu\nu} \cdot \left(S_{\mu\nu} D^{-1}\right)$$

2) Calculate eigenvalues of *M*.

3) If the matrix *M* is positively defined, extract "square root" from it.

$$M = ZZ^{+}$$

The matrix Z is a direct sum of four real bispinors reproducing the given quantities j_{α} and $h_{\alpha\beta}$.

$$j_{\alpha} \equiv \frac{1}{4}\mathrm{Sp}\left\{Z^{+} D \gamma_{\alpha} Z\right\} \qquad h_{\alpha\beta} \equiv -\frac{1}{8}\mathrm{Sp}\left\{Z^{+} D S_{\alpha\beta} Z\right\}$$

Summary on the Algorithm for Tensor Mapping to Bispinors

The algorithm provides for the following operations:

(1) Construct the matrix M. The resulting matrix automatically turns out to be Hermitian and, consequently, having real eigenvalues or pairs of complex-conjugate ones.

(2) Calculate eigenvalues of the matrix M. Find regions in space, where all eigenvalues of the matrix M are positive.

(3) Extract the positive “square root” of the matrix M in these regions of space. Subject to the additional positivity condition, the resulting matrix Z is found uniquely, and it is the proper matrix, which reproduces the field of the vector j^{α} and anti-symmetric tensor $h^{\alpha\beta}$ using the formulas shown at the bottom of the slide.

This mapping procedure is algebraic and valid at any point for the fields j^{α}, $h^{\alpha\beta}$ irrespective of whether or not they are governed by some equations or are arbitrary.

What happens with the quantities $j^{\alpha}, h^{\alpha\beta}$, if the bispinor matrix satisfies the Dirac equation

$j^{\alpha}{}_{;\alpha}=0$
$\left(j_{\beta,\alpha}-j_{\alpha,\beta}\right)=4m\cdot h_{\alpha\beta}+E_{\alpha\beta\mu}{}^{\nu}\frac{1}{4}\mathrm{Sp}\left\{\left(\nabla_{\nu}Z^{+}\right)D\gamma_{5}\gamma^{\mu}Z-Z^{+}D\gamma_{5}\gamma^{\mu}\left(\nabla_{\nu}Z\right)\right\}$
$h_{\alpha}{}^{\nu}{}_{;\nu}=-m\cdot j_{\alpha}-\frac{1}{8}\mathrm{Sp}\left(\left(\nabla_{\alpha}Z^{+}\right)DZ-Z^{+}D\left(\nabla_{\alpha}Z\right)\right)$
$E^{\lambda\alpha\beta\varepsilon}h_{\alpha\beta;\varepsilon}=\frac{1}{4}\cdot\mathrm{Sp}\left\{\left(\nabla^{\lambda}Z^{+}\right)D\gamma_{5}Z-Z^{+}D\gamma_{5}\left(\nabla^{\lambda}Z\right)\right\}$

Comparison of equations

Consequences of Dirac equation for $j^{\alpha}, h^{\alpha\beta}$	CGD equations for $J_{\alpha}, H_{\alpha\beta}$
$j^{\alpha}{}_{;\alpha} = 0$	$J^{\alpha}{}_{;\alpha} = 0$
$\left(j_{\beta,\alpha} - j_{\alpha,\beta}\right) = 4m \cdot h_{\alpha\beta} +$ $+E_{\alpha\beta\mu}{}^{\nu} \frac{1}{4} \mathrm{Sp}\left\{\left(\nabla_{\nu} Z^{+}\right) D\gamma_5 \gamma^{\mu} Z - Z^{+} D\gamma_5 \gamma^{\mu} \left(\nabla_{\nu} Z\right)\right\}$	$\left(J_{\beta;\alpha} - J_{\alpha;\beta}\right) = 4mH_{\alpha\beta}$
$h_{\alpha}{}^{\nu}{}_{;\nu} = -m \cdot j_{\alpha}$ $-\frac{1}{8} \mathrm{Sp}\left(\left(\nabla_{\alpha} Z^{+}\right) DZ - Z^{+} D \left(\nabla_{\alpha} Z\right)\right)$	$H_{\alpha\cdot}{}^{\beta}{}_{;\beta} = -m \cdot J_{\alpha}$
$E^{\lambda\alpha\beta\varepsilon} h_{\alpha\beta;\varepsilon} = \frac{1}{4} \cdot \mathrm{Sp}\left\{\left(\nabla^{\lambda} Z^{+}\right) D\gamma_5 Z - Z^{+} D\gamma_5 \left(\nabla^{\lambda} Z\right)\right\}$	$E^{\alpha\beta\mu\nu} H_{\beta\mu,\nu} \equiv 0$

$$j_{\alpha} \equiv \frac{1}{4} \mathrm{Sp}\left\{Z^{+} D\gamma_{\alpha} Z\right\}$$

$$h_{\alpha\beta} \equiv -\frac{1}{8} \mathrm{Sp}\left\{Z^{+} D S_{\alpha\beta} Z\right\}$$

$J_{\alpha}, H_{\alpha\beta}$ - Quantities, included in CGD equations

Main Statement

Assume we have defined quantities J_α and $H_{\alpha\beta}$, satisfying the equations (1), (2) and (3) (see slide 10), and the Z matrix constructed at each point and satisfying the relationship

$$ZZ^{+} = J^{\alpha} \cdot \left(\gamma_{\alpha} D^{-1}\right) + H^{\alpha\beta} \cdot \left(S_{\alpha\beta} D^{-1}\right)$$

Then, the matrix Z satisfies the Dirac equation

$$\gamma^{\nu}\left(\nabla_{\nu} Z\right) = m \cdot Z \quad ,$$

if the connectivity Γ_α has the form $\Gamma_{\alpha} = \frac{1}{2}\left(\left(Z^{-1} Z_{,\alpha}\right) - \left(Z^{+}{}_{,\alpha} Z^{-1+}\right)\right)$.

If the connectivity has the above form, all spur terms, which disturb the consistency of the two dynamics, are removed, and during gauge transformation, the connectivity itself transforms according to the law standard for the gauge field.
The connectivity corresponds to the gauge group O(4).

Other Formulation of the Main Result

If the vector J^{α} and the tensor $H_{\alpha\beta}$ are governed by CGD equations, and if they initially coincide with the vector j^{α} and tensor $h_{\alpha\beta}$, constructed based on the bispinor matrix Z , and if the bispinor connectivity in the chosen gauge is found from the rule

$$\Gamma_{\alpha} = \frac{1}{2}\left(\left(Z^{-1}Z_{,\alpha}\right) - \left(Z^{+}{}_{,\alpha}Z^{-1+}\right)\right),$$

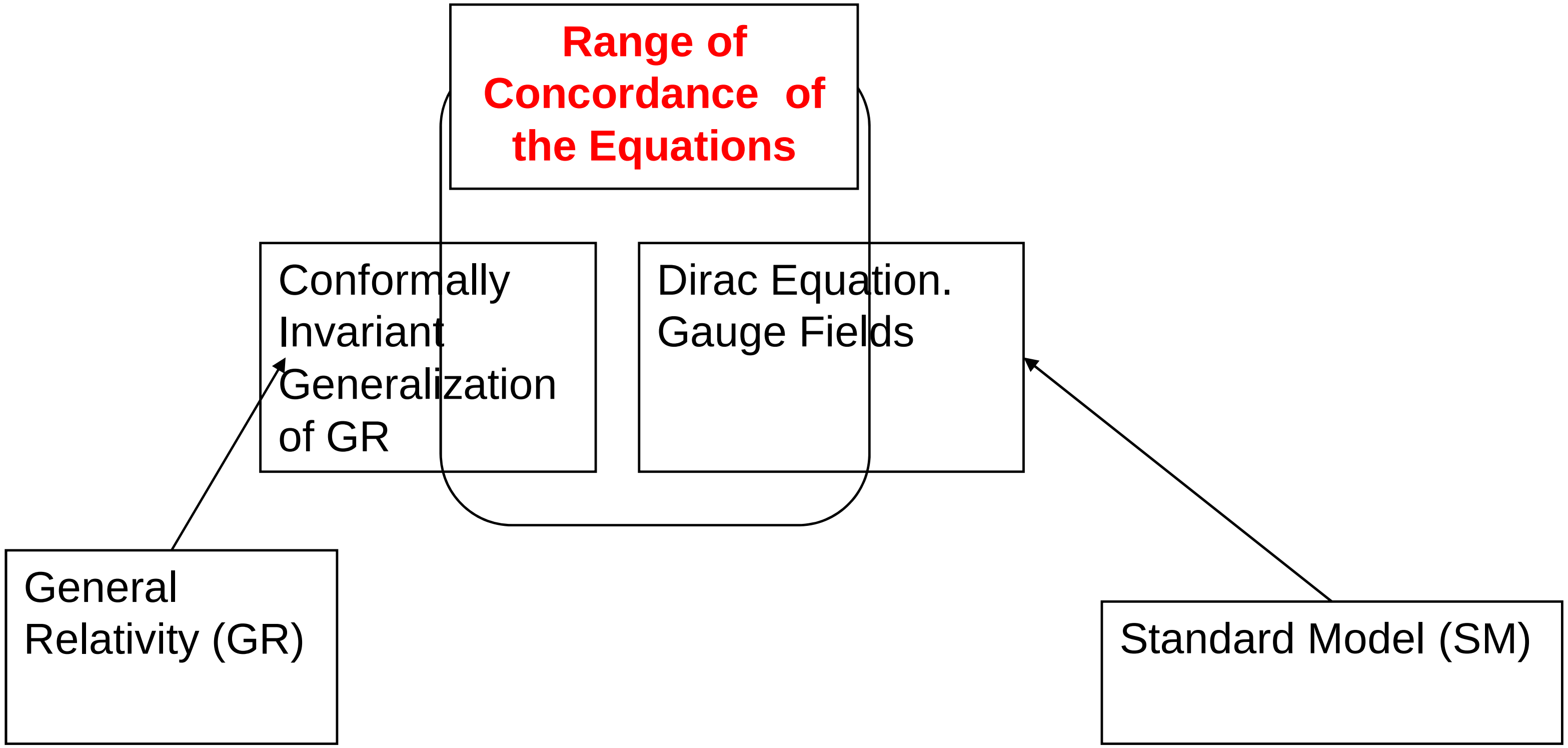
then the equalities

$$j^{\alpha} = J^{\alpha}, \qquad h_{\alpha\beta} = H_{\alpha\beta},$$

will also be satisfied at any time, and the matrix Z will satisfy the Dirac equation

$$\gamma^{\nu}\left(\nabla_{\nu}Z\right) = m \cdot Z \, .$$

What do the Obtained Results Mean?



Reasonable Hypothesis I

- **The geometrodynamics energy-momentum tensor $T_{\alpha\beta}^{CGD}$ can model any tensor $T_{\alpha\beta}^{GR}$, which is used in GR to describe any type of matter. This is associated with the possibility of defining the Cauchy problem for CGD equations without links to the Cauchy data on the initial surface.**
- **However, the modeling is approximate, as it is possible in a finite time span in a spatial domain of finite dimensions.**

The hypothesis was validated by expansions of centrally symmetric solutions and solutions for Friedman models.

Reasonable Hypothesis II

Existing hypotheses for the Weyl vector:

- It is associated with scale conversion for the measurement of space-time intervals.
- It is associated with dark matter and energy parameters of the universe.
- It is associated with cosmological red shift.
- It is an attribute of the integrable Weyl space (i.e. the space, in which the Weyl vector is a gradient of a scalar function), leading to the appearance of the Schroedinger equation.

Our hypothesis II:

In the regions, where the quantum-field interpretation of CGD equations is possible, the Weyl vector has a meaning of a current vector of a complete set of bispinor fields, to which the Weyl degrees of freedom are mapped. No other variants!