VACUUM ENERGY IN QUANTUM FIELD THEORY

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Advances of Quantum Field Theory

Dubna, 4 – 7 October 2011

INTRODUCTION

Is it possible to calculate the vacuum energy in the standard Quantum Field Theory?

"Let us settle that, when dealing with the expressions quadratic in field operators with the same arguments, such as the Lagrangian, energy–momentum tensor, current and so on, we shall write them in the form of the normal product." ... "As a result, the pseudo–physical quantities like zero–point energy, zero–point charge and so on are eliminated in our consideration from the very beginning."

■ ***Bogoliubov, Shirkov, "Introduction to the Theory of Quantized Fields".***

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{\sqrt{2\omega(\mathbf{k})}} \left[e^{-ikx} a(\mathbf{k}) + e^{ikx} a^+(\mathbf{k}) \right], \qquad (1)$$

where

$$\omega(\mathbf{k}) = k_0 = \sqrt{\mathbf{k}^2 + m^2}\,, \qquad [a(\mathbf{k}),\, a^+(\mathbf{k}')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')\,.$$

$$H_{\rm cl} = \frac{1}{2} \int d^3\mathbf{k}\, \omega(\mathbf{k})\, [a^+(\mathbf{k}) a(\mathbf{k}) + a(\mathbf{k}) a^+(\mathbf{k})] \qquad (2)$$

$$\Downarrow$$

$$H_{\rm qu} = \frac{1}{2} \int d^3\mathbf{k}\, \omega(\mathbf{k}) :[a^+(\mathbf{k}) a(\mathbf{k}) + a(\mathbf{k}) a^+(\mathbf{k})]: = \int d^3\mathbf{k}\, \omega(\mathbf{k})\; a^+(\mathbf{k}) a(\mathbf{k}), \qquad (3)$$

$$\langle\, 0 \,|\, H_{\rm qu} \,|\, 0 \,\rangle = 0$$

IS THIS CORRECT ?

Yes, this is correct for scattering processes in the unbounded Minkowski space–time (Bogoliubov's R–operation). In the general case there is no rigorous answer.

Another choice of $H_{\rm qu}$:

$$H'_{\rm qu} = \int d^3\mathbf{k}\,\omega(\mathbf{k})\left[a^+(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\right]. \tag{4}$$

In this case there is an explicit relation to the quantum–mechanical oscillator with the frequency $\omega(\mathbf{k})$.

$$H'_{\rm qu} - H_{\rm qu} = \frac{1}{2}\int d^3\mathbf{k}\,\omega(\mathbf{k}) \equiv E_0, \tag{5}$$

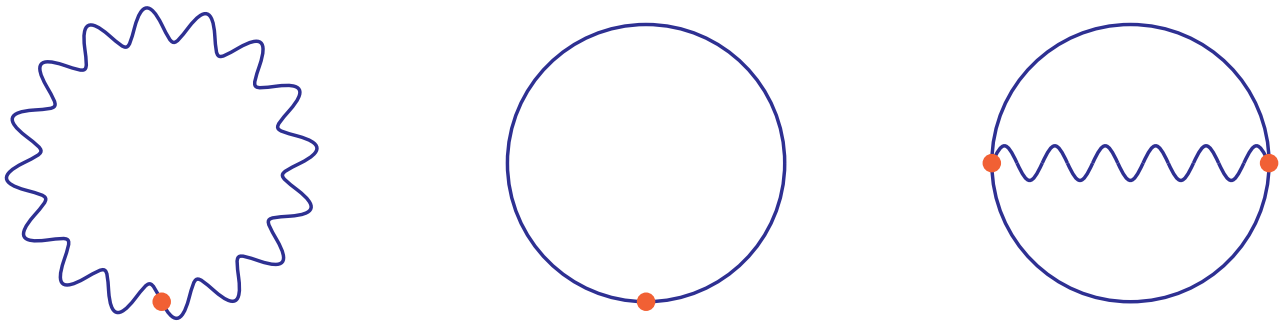
$$\langle\, 0 \,|\, H'_{\rm qu} \,|\, 0 \,\rangle = E_0. \tag{6}$$

Obviously, $E_0 \to \infty$ for any reasonable dispersion law $\omega(\mathbf{k})$.

ZERO–POINT ENERGY IN QUANTUM MECHANICS AND IN QUANTUM FIELD THEORY

The notion of zero–point energy (ZPE) was introduced by M. Planck in 1912.

In Quantum Mechanics one deals with a finite number of quantum oscillators and their ZPE is an observed physical quantity (specific heat of rigid body, vibration spectra of diatomic molecules, and so on). Quantum field is equivalent to the infinite number of oscillators.



BASIC RESULTS FOR VACUUM ENERGY

A "naive" subtraction procedure

$$E_0 = \frac{1}{2}\sum_n (\omega_n - \bar{\omega}_n)\,, \tag{7}$$

where $\bar{\omega}_n$ are calculated for a some "limiting" configuration (without any boundaries). The progress in calculating the Casimir energy in this approach was very slow, especially for boundaries with curvature.

- **Perfectly conducting parallel plates in vacuum**

$$E_0 = -c\hbar\,\frac{\pi^2}{720}\,\frac{L_x L_y}{a^3} \tag{8}$$

- ***Casimir, 1948.***

The Casimir force is very weak, however it increases rapidly as the separation a decreases and it becomes measurable when $a \sim 1\ \mu\mathrm{m}$ or less. For plates $1\ \mathrm{cm}^2$ in area with $a = 0.5\ \mu\mathrm{m}$, the Casimir force is about $0.2\ \mathrm{dyn}$. Now experimental precision in this field is about 1%.

Perfectly conducting spherical shell

$$E_0 = c\hbar \frac{1}{a} 0.046361\ldots \tag{9}$$

■ ***Boyer, 1968.***

(three years of numerical computations!) This energy is positive, hence the respective forces are repulsive.

For nanometer size, that is for $a = 10^{-7}\ \mathrm{cm}$, the energy (9) is about $10\ \mathrm{eV}$ which is of a considerable magnitude.

- **Perfectly conducting infinite cylindrical shell**

$$E_0 = -c\hbar \frac{1}{a^2} 0.01356\ldots \tag{10}$$

- ***DeRaad and Milton, 1981; Milton, A.V.Nesterenko, NVV, 1999.***

Accounting for the material properties of boundaries:

- **Two material (dielectric) semi–spaces separated by a plane gap (Lifshitz formula is important for practical use)**
- ***Lifshitz, 1955.***

- **Dielectric compact ball**
- ***Brevik, Marachevsky, Milton; Barton, 1999; Lambiase, Scarpetta, NVV, 2001.***

- **Dilute dielectric compact cylinder**

$$E_0 = 0 \tag{11}$$

- ***Cavero–Pelaez, Milton, 2004.***

MATHEMATICAL TOOLS FOR CALCULATING THE VACUUM ENERGY

Here there is a considerable progress.

Spectral zeta functions

In the Casimir studies we are dealing with the following spectral problem

$$L\varphi_n(x) = \lambda_n \varphi_n(x) \qquad \textbf{or} \qquad L\,|\,n\,\rangle = \lambda_n\,|\,n\,\rangle, \qquad \lambda_n = \omega_n^2/c^2, \tag{12}$$

where $L = -\Delta + \ldots$ with corresponding boundary (or matching) conditions. By making use of the completeness relation

$$I = \sum_n |\,n\,\rangle\langle\,n\,|\,, \tag{13}$$

we can define the inverse operator L^{-s}

$$L^{-s} = \sum_n \frac{|n\rangle\langle n|}{\lambda_n^s}. \tag{14}$$

The global spectral zeta function (Hawking) is defined by

$$\zeta_L(s) = \mathrm{Tr}\, L^{-s} = \sum_n \lambda_n^{-s}. \tag{15}$$

Definition of the spectral zeta function (15) is a direct extension of the Riemann zeta function

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}\, s > 1 \tag{16}$$

to the spectrum of the operator L.

Within the approach on hand the vacuum energy E_0 is defined by

$$E_0 = \frac{\hbar}{2}\zeta_L\left(s = -\frac{1}{2}\right). \tag{17}$$

For calculating the density of the vacuum energy, the local spectral zeta function $\zeta_L(s;x)$ can be introduced as a diagonal element of the operator L^{-s}

$$\zeta_L(s;x) = \sum_n \frac{\langle x\,|n\rangle\langle n|\,x\rangle}{\lambda_n^s} = \sum_n \lambda^{-s}\varphi_n^*(x)\,\varphi_n(x). \tag{18}$$

Obviously

$$\zeta_L(s) = \int \zeta_L(s;x)\,dx. \tag{19}$$

Some examples of the spectral zeta–functions

- **The zeta function in the string models:**

$$\zeta_{\text{string}}(s) \equiv \zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{20}$$

Therefore

$$\sum_{n=1}^{\infty} n \Rightarrow -\frac{1}{12}. \tag{21}$$

This result leads, in a straightforward way, to the anomalous dimensions of the space-time in the string theory: $D = 26$ **(bosonic Nambu–Goto string),** $D = 10$ **(superstring).**

- **Perfectly conducting plates:**

$$\zeta(s) = \frac{L_x\, L_y}{c^{2s}} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left\{ 2 \sum_{n=1}^{\infty} \left[\mathbf{k}^2 + \left(\frac{n\pi}{a} \right)^2 \right]^{-s} + \left(\mathbf{k}^2 + \mu^2 \right)^{-s} \right\}, \tag{22}$$

where L_x **and** L_y **are the dimensions of the plates and the photon mass** μ **is an infrared regularization. Integrating Eq. (22) and substituting the sum over** n **by the Riemann zeta function we get**

$$\zeta(s) = \frac{L_x\, L_y}{2\pi\, c^{2s}} \left[\left(\frac{\pi}{a} \right)^{2-2s} \frac{\zeta_{\mathrm{R}}(2s-2)}{s-1} + \frac{1}{2} \frac{\mu^{2-2s}}{s-1} \right]. \tag{23}$$

The zeta function (23) immediately leads to the Casimir result

$$E_0 = \frac{\hbar}{2}\,\zeta\!\left(-\frac{1}{2}\right) = -c\,\hbar\,\frac{\pi^2}{720}\,\frac{L_x L_y}{a^3} \tag{24}$$

or for the vacuum energy density

$$\frac{E_0}{V} = -\frac{c\hbar\pi^2}{720a^4}, \quad \text{where} \quad V = a\,L_x\,L_y. \tag{25}$$

The differentiation of vacuum energy (24) with respect to the distance a gives the Casimir force

$$F = -\frac{\pi^2 c\hbar}{240a^4}. \tag{26}$$

- **Perfectly conducting sphere:**

$$\zeta_{\text{sphere}}(s) \simeq \frac{a^{2s}}{4}\,s(1+s)(2+s)\{(2^{1+2s}-1)\zeta_{\text{R}}(1+2s) - 2^{1+2s} + \\ +q(s)[(2^{3+2s}-1)\zeta_{\text{R}}(3+2s) - 2^{3+2s}] + \ldots\}\,, \tag{27}$$

$$q(s) = \frac{1}{480}(60 + 217s + 252s^2 + 71s^3). \tag{28}$$

The nearest singularity in this formula is a simple pole at $s = -3$. The Casimir energy is

$$E_{\text{sphere}} = \frac{1}{2}\zeta_{\text{shell}}\left(-\frac{1}{2}\right) = \frac{3}{64a}\left[1 - \frac{3}{256}\left(\frac{\pi^2}{2} - 4\right) + \ldots\right]$$
$$= \frac{1}{a}\,0.046361\ldots. \tag{29}$$

■ ***NVV, I.G.Pirozhenko, 1998.***

In Casimir calculations another spectral function is also useful, namely, the heat kernel

$$K(\tau) = \mathrm{Tr}\,(e^{-\tau L}) = \sum\nolimits_n e^{-\lambda_n \tau}, \tag{30}$$

where τ is an auxiliary variable ranging from 0 to $+\infty$. Such a name of this function is due to the following. By making use of the unity operator (13) one can write

$$e^{-\tau L} = \sum\nolimits_n e^{-\tau\lambda_n}\,|n\rangle\langle n|\,. \tag{31}$$

The matrix element of this operator

$$K(x,y;\tau) \equiv \langle\, x\,|e^{-\tau L}|\, y\,\rangle = \sum_n e^{-\tau\lambda_n}\,\langle\, x\,|\,n\,\rangle\langle\, n\,|\,y\,\rangle$$
$$= \sum_n e^{-\tau\lambda_n}\varphi_n^*(x)\varphi_n(y) \tag{32}$$

is the Green function of the heat conduction equation with the operator L

$$\left(L_x + \frac{\partial}{\partial\tau}\right) K(x,y;\tau) = 0, \tag{33}$$

$$K(x,y;\tau) = \delta(x,y), \quad \tau \to +0. \tag{34}$$

For the functions $K(\tau)$ **and** $K(x,y;\tau)$ **the relation analogous to (19) holds**

$$K(\tau) = \int dx\, K(x,x;\tau). \tag{35}$$

In physical applications the coefficients in the asymptotic expansion of the heat kernel, when $\tau \to +0$, are important

$$K(\tau) = \sum_n e^{-\lambda_n \tau} = (4\pi\tau)^{-d/2} \sum_{n=0,1,2,\ldots}^{\infty} \tau^{n/2} B_{n/2} + \mathrm{ES}. \qquad (36)$$

ES denotes exponentially small corrections as $\tau \to +0$.

$$B_0 = V, \quad B_{1/2} = \mp \frac{\sqrt{\pi}}{2} S, \quad \ldots . \qquad (37)$$

The upper sign is for the Dirichlet boundary conditions and the lower sign is for the Neumann conditions.

The first few coefficients $B_{n/2}$ yield the ultraviolet divergencies of the vacuum energy.

If $B_2 = 0$, then the zeta regularization gives a finite value for E_0.

CONCLUSION

- **One–loop calculation of the vacuum energy is incorporated into the extended formalism of Quantum Field Theory.**

- **A new mathematical technique, effective and beautiful, is developed for this purpose (spectral geometry methods).**

- **Predicted results are in a good agreement with the experimental data.**

This talk is based on the studies of Casimir effect conducted in collaboration with:

I.G. Pirozhenko (BLTP)

M. Bordag (Leipzig)

K.A. Milton (Norman, USA)

I. Brevik (Trondheim, Norway)

G. Lambiase, G. Scarpetta (Salerno, Italy)

H. Kleinert (FU, Berlin)

N.R. Shvetz (USA)

A.V. Nesterenko (BLTP)

J. Dittrich (Czech Rep.)

L. Hadasz (Poland)

THANK YOU FOR YOUR ATTENTION