CP^n supersymmetric mechanics in the $U(n)$ background gauge fields

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Our goal is to construct a $\mathcal{N}=4$ supersymmetric mechanics describing the motion of the particle over CP^n manifold in background $U(n)$ gauge fields.

- Quantum Hall effect in higher (greater then two) dimensions: Zhang and Hu considered the Landau problem for charged fermions on S^4 with a background magnetic field which is $SU(2)$ instanton.
- Karabali and Nair have extended the original idea of quantum Hall effect to the complex projective spaces CP^n
- The corresponding gauge potentials are proportional to the connections on $SO(4)/SO(3)$ and $SU(n+1)/U(n)$
- The case of CP^n allows both Abelian ($U(1)$) and non-Abelian ($SU(n)$) background fields
- The system describing the motion of the particles over CP^n manifold in the absence of gauge fields can be easily extended to possess $\mathcal{N}=4$ supersymmetry

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1. Preliminaries: Motivation and free CP^n mechanics
   - Motivation
   - CP^n mechanics with $\mathcal{N}=4$ supersymmetry
   - Geometry of CP^n model
2. Gauge fields: construction
   - Interaction in N=2 and N=4 Supersymmetric systems
   - Isospin currents
   - Supercharges
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3. Conclusion and Discussion

The construction of $\mathcal{N}=4$ supersymmetric mechanics on CP^n manifold is almost trivial. Let us take n complex $\mathcal{N}=4$ chiral superfields $Z^\alpha, \overline{Z}_\alpha$

$$D^i Z^\alpha = 0, \quad \overline{D}_i \overline{Z}_\alpha = 0, \qquad \alpha = 1\ldots n, \quad i,j = 1,2, \qquad \left\{D^i, \overline{D}_j\right\} = 2\mathrm{i}\delta^i_j \partial_t,$$

then the superfields action S

$$S = \int dt\, d^4\theta \, \log\left[1 + Z^\alpha \overline{Z}_\alpha\right]$$

did all job, completely defining the model. The explicit form of Lagrangian density immediately follows from invariance of the action with respect to $SU(n+1)$ group, which is realized on the superfields $Z, \overline{Z}$ as

$$\delta Z^\alpha = a^\alpha + Z^\alpha \left(Z^\beta \bar{a}_\beta\right), \; \delta \overline{Z}_\alpha = \bar{a}_\alpha + \overline{Z}_\alpha \left(a^\beta \overline{Z}_\beta\right),$$

where $a^\alpha, \bar{a}_\alpha$ are the parameters of the coset $SU(n+1)/U(n)$ transformations.

The Hamiltonian description of the $\mathcal{N}=4$ supersymmetric CP^n mechanics directly follows from the action after passing to the components and removing the auxiliary fields.
So, our basic ingredients are bosonic variables $\{z^\alpha, \bar{z}_\alpha\}$ which parameterized the coset $SU(n+1)/U(n)$ and fermionic variables $\{\psi_i^\alpha, \bar{\psi}^i_\alpha\}$:

$$z^\alpha = Z^\alpha|, \quad \bar{z}_\alpha = \overline{Z}_\alpha|, \qquad \psi_i^\alpha = \overline{D}_i Z^\alpha|, \quad \bar{\psi}^i_\alpha = D^i \overline{Z}_\alpha|.$$

In what follows we will pay a great attention to $U(n)$ properties of our model. That is why we decided to keep the corresponding indices α, β of our fields in a proper position. For $SU(n+1)$ group we will fix the commutation relations to be

$$i\left[R_\alpha, \overline{R}^\beta\right] = J_\alpha{}^\beta, \quad i\left[J_\alpha{}^\beta, J_\gamma{}^\sigma\right] = \delta_\gamma^\beta J_\alpha{}^\sigma - \delta_\alpha^\sigma J_\gamma{}^\beta,$$
$$i\left[J_\alpha{}^\beta, R_\gamma\right] = \delta_\gamma^\beta R_\alpha + \delta_\alpha^\beta R_\gamma, \quad i\left[J_\alpha{}^\beta, \overline{R}^\gamma\right] = -\delta_\alpha^\gamma \overline{R}^\beta - \delta_\alpha^\beta \overline{R}^\gamma.$$

Thus, the generators $R_\alpha, \overline{R}^\alpha$ belong to the coset $SU(n+1)/U(n)$ while $J_\alpha{}^\beta$ form $U(n)$. In addition we choose these generators to be anti-hermitian ones

$$(R_\alpha)^\dagger = -\overline{R}^\alpha, \quad \left(J_\alpha{}^\beta\right)^\dagger = -J_\beta{}^\alpha.$$

After introducing the momenta for all our variables and passing to Dirac brackets we will obtain the following set of relations (As usually, the bosonic momenta are shifted by $\psi \cdot \bar\psi$ terms in this basis)

$$\left\{\psi_i^\alpha, \bar\psi^j_\beta\right\} = i\delta_i^j \left(g^{-1}\right)_\beta{}^\alpha, \qquad \left\{p_\alpha, \bar p^\beta\right\} = -i\left(g_\alpha{}^\beta g_\mu{}^\nu + g_\alpha{}^\nu g_\mu{}^\beta\right)\bar\psi^i_\nu \psi_i^\mu,$$
$$\left\{p_\alpha, \psi_i^\beta\right\} = -\frac{1}{(1+z\cdot\bar z)}\left[\bar z_\alpha \psi_i^\beta + \delta_\alpha^\beta\, \psi_i^\gamma \bar z_\gamma\right],$$
$$\left\{\bar p^\alpha, \bar\psi^i_\beta\right\} = -\frac{1}{(1+z\cdot\bar z)}\left[z^\alpha \bar\psi^i_\beta + \delta_\beta^\alpha\, z^\gamma \bar\psi^i_\gamma\right].$$

Here, CP^n metric $g_\alpha{}^\beta$ has standard Fubini-Studi form

$$g_\alpha{}^\beta = \frac{1}{(1+z\cdot\bar z)}\left[\delta_\alpha^\beta - \frac{\bar z_\alpha z^\beta}{(1+z\cdot\bar z)}\right], \quad z\cdot\bar z \equiv z^\alpha \bar z_\alpha.$$

Now, it is not too hard to check that the supercharges $Q^i, \overline{Q}_i$ have extremely simple form

$$Q^i = \bar{p}^\alpha \bar{\psi}^i_\alpha, \qquad \overline{Q}_i = \psi_i^\alpha p_\alpha .$$

They are perfectly anticommute as

$$\left\{Q^i, \overline{Q}_j\right\} = i\delta^i_j H, \qquad \left\{Q^i, Q^j\right\} = \left\{\overline{Q}_i, \overline{Q}_j\right\} = 0,$$

where Hamiltonian H reads

$$H = \bar{p}^\alpha \left(g^{-1}\right)_\alpha{}^\beta p_\beta + \frac{1}{4}\left(g_\mu{}^\alpha g_\rho{}^\sigma + g_\mu{}^\sigma g_\rho{}^\alpha\right) \bar{\psi}_{\alpha\, i} \bar{\psi}^i_\sigma \, \psi^{\rho\, j} \psi^\mu_j .$$

In principle, one may modify the supercharges and Hamiltonian by including the potential terms, but here we will be interesting in including the interaction with non-Abelian gauge fields which looks itself a rather complicated. Therefore we will ignore such possible modifications in what follows.

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Finally, we will need the explicit expressions for the vielbeins $e_\alpha{}^\beta$ and $U(n)$-connection $\omega_\alpha{}^\beta$ on CP^n manifold which we choose as

$$e_\alpha{}^\beta = \frac{1}{\sqrt{1+z\cdot\bar z}}\left[\delta_\alpha^\beta - \frac{\bar z_\alpha z^\beta}{\sqrt{1+z\cdot\bar z}\,\left(1+\sqrt{1+z\cdot\bar z}\right)}\right],$$

$$\omega_\alpha{}^\beta = \frac{1}{\sqrt{1+z\cdot\bar z}\,\left(1+\sqrt{1+z\cdot\bar z}\right)}\left[\delta_\alpha^\beta - \frac{\bar z_\alpha z^\beta}{2\sqrt{1+z\cdot\bar z}\,\left(1+\sqrt{1+z\cdot\bar z}\right)}\right].$$

With our definition of $SU(n+1)$ algebra these quantities enter the standard Cartan forms as

$$g^{-1}dg = dz^\alpha\, e_\alpha{}^\beta R_\beta + \overline{R}^\alpha e_\alpha{}^\beta d\bar z_\beta + iJ_\alpha{}^\beta\left(z^\alpha\,\omega_\beta{}^\gamma d\bar z_\gamma - dz^\gamma\,\omega_\gamma{}^\alpha \bar z_\beta\right),$$

where

$$g = e^{x^\alpha R_\alpha + \bar x_\alpha \overline{R}^\alpha}, \quad \text{and} \quad z^\alpha \equiv \frac{\tan\sqrt{x\cdot\bar x}}{\sqrt{x\cdot\bar x}}x^\alpha .$$

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How to introduce the interactions?

- N=2: There are no problems!

$$S = \int dtd^2\theta \left[F(z,\bar{z}) D\bar{z}\overline{D}z + G(z,\bar{z}) \right]$$

  $G(z,\bar{z})$ is an arbitrary function – pre-potential

- N=4: The straightforward generalization gives a σ-model action

$$S = \int dtd^4\theta \, F(z,\bar{z})$$

  In the case of chiral superfileds one may add the interactions

$$S_{int} = \int dtd^2\theta G(\bar{z}) + \int dtd^2\bar{\theta}\bar{G}(z)$$

- The interaction breaks the symmetries of the free action. Thus we need some additional ingredients.

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It is curious, but the simplest form of the supercharges does not help in the coupling with gauge fields. One may easily check that the standard coupling by shifting bosonic momenta in supercharges does not work. Our idea is to introduce the coupling simultaneously with all currents spanning $SU(n+1)$ and/or $SU(1,n)$ group. Thus, let us introduce the isospin currents spanning $SU(n+1)$ and/or $SU(1,n)$, respectively

$$\left\{R_\alpha, \overline{R}^\beta\right\} = \pm J_\alpha{}^\beta, \quad \left\{J_\alpha{}^\beta, J_\gamma{}^\sigma\right\} = \delta^\beta_\gamma J_\alpha{}^\sigma - \delta^\sigma_\alpha J_\gamma{}^\beta,$$
$$\left\{J_\alpha{}^\beta, R_\gamma\right\} = \delta^\beta_\gamma R_\alpha + \delta^\beta_\alpha R_\gamma, \quad \left\{J_\alpha{}^\beta, \overline{R}^\gamma\right\} = -\delta^\gamma_\alpha \overline{R}_\beta - \delta^\beta_\alpha \overline{R}^\gamma.$$

The $\pm$ sign in the first line corresponds to the choice of $SU(n+1)$ or $SU(1,n)$. It will be clear soon, why we are going to consider both these cases.

The currents $\left\{R_\alpha, \overline{R}^\beta, J_\alpha{}^\beta\right\}$ commute with all dynamic variables $\left\{z^\alpha, \bar{z}_\alpha, p_\alpha, \bar{p}^\alpha, \psi_i^\alpha, \bar{\psi}^i_\alpha\right\}$! They should be realized (on the Lagrangian level) in terms of new semi-dynamical variables $\left\{u^A, \bar{u}_A\right\}$ as
$S \sim \int dt \left(\dot{u}\cdot\bar{u} - u\cdot\dot{\bar{u}}\right) + \ldots$

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Now, we are ready to write the anzatz for the supercharges. This anzatz is a direct generalization of those supercharges for $SU(2)$ case which were explicitly constructed within superspace approach.

$$Q^i = \left(\bar{p}^\alpha - z^\gamma J_\gamma{}^\beta h_\beta{}^\alpha\right)\bar{\psi}^i_\alpha + \psi^{i\,\alpha} f_\alpha{}^\beta R_\beta, \quad \overline{Q}_i = \psi_i^\alpha\left(p_\alpha + h_\alpha{}^\beta J_\beta{}^\gamma \bar{z}_\gamma\right) + \overline{R}^\beta f_\beta{}^\alpha \bar{\psi}_{i\,\alpha}.$$

Here, $h_\alpha{}^\beta$ and $f_\alpha{}^\beta$ are arbitrary, for the time being, functions depending on the bosonic fields $z^\alpha, \bar{z}_\alpha$ only. Moreover, due to explicit $U(n)$ symmetry of our construction, we are going to keep unbroken, one may further restrict these functions as

$$h_\alpha{}^\beta = h_1\,\delta_\alpha^\beta + h_2\,\bar{z}_\alpha z^\beta, \quad f_\alpha{}^\beta = f_1\,\delta_\alpha^\beta + f_2\,\bar{z}_\alpha z^\beta,$$

where scalar functions h_1, h_2, f_1, f_2 depend now on $z \cdot \bar{z}$ only.

The supercharges have to obey the standard $\mathcal{N}=4$ Poincaré superalgebra relations. Therefore, the closure of superalgebra is achieved if the following equations on functions in our supercharges are satisfied

$$\{Q,Q\}=0 \quad \Rightarrow \quad \begin{cases} f_1^{'} = -(f_1 h_1 + x f_1 h_2), \quad f_2^{'} = -(2f_2 h_1 + f_1 h_2 + 2x f_2 h_2), \\ h_1^{'} = -(h_1^2 - h_2 + x h_1 h_2), \quad f_2 = -f_1 h_1 \end{cases}$$

$$\left\{Q^i, \overline{Q}_j\right\} = \mathrm{i}\delta^i_j H \quad \Rightarrow \quad h_2^{'} = \frac{1}{2}\left(A f_1^2 h_1^2 + h_1^3\right), \quad h_2 = -\frac{1}{2} h_1^2, \quad A f_1^2 = (2h_1 - x h_1^2),$$

where the derivatives are taken with respect to x.
The simplest, almost trivial solution of these equations reads

$$f_1 = f_2 = 0, \qquad h_1 = \frac{2}{z\cdot\bar{z}}, \quad h_2 = -\frac{2}{(z\cdot\bar{z})^2}.$$

The functions h_1, h_2 have the singularities at $(z,\bar{z}) \to 0$. In addition, in the case of CP^1

$$h = h_1 + h_2\, z\cdot\bar{z} = 0 \ \Rightarrow \ \text{No interaction !}$$

Moreover, this solution has no any geometric meaning within CP^n geometry. Thus, without $R, \bar{R}$ terms in the Ansatz the reasonable interaction can not be constructed.

In contrast, with non-zero f_1, f_2 functions the solution of the equations is fixed to be

$$
\begin{aligned}
f_1 &= \frac{1}{\sqrt{1+A\, z\cdot\bar z}}, \quad f_2 = -\frac{A}{(1+A\, z\cdot\bar z)\left(1+\sqrt{1+A\, z\cdot\bar z}\right)},\\
h_1 &= \frac{A}{\sqrt{1+A\, z\cdot\bar z}\left(1+\sqrt{1+A\, z\cdot\bar z}\right)},\\
h_2 &= -\frac{1}{2\,(1+A\, z\cdot\bar z)\left(1+\sqrt{1+A\, z\cdot\bar z}\right)^2}.
\end{aligned}
$$

Here, $A=+1$ for the $SU(1,n)$ currents and $A=-1$ for $SU(n+1)$ case. Thus, we see that the matrix valued function ${f_\alpha}^\beta$ is perfectly coincides with the vielbeins for CP^n manifold if we choose $A=1$. The function ${h_\alpha}^\beta$, defining the gauge fields, is the part of the spin connection for CP^n.

This gauge field is identical to those one constructed previously in H. Kihara, M. Nitta, “Generalized Instantons on Complex Projective Spaces”, J.Math.Phys. **50** (2009) 012301, as the solution of the Bogomol'nyi equation for the Tchrakian's type of self-duality relations in $U(n)$ gauge theory.

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Last step is to write the Hamiltonian

$$
\begin{aligned}
H \;=\;& \left(\bar{p}\, g^{-1} p\right) + \left(\bar{p}\, g^{-1} h\, J\bar{z}\right) - \left(z\, J\, h\, g^{-1} p\right) - \left(\overline{R}\, f\, g^{-1} f\, R\right) - \left(z\, J\, h\, g^{-1} h\, J\, \bar{z}\right) \\
&+ i \left(\frac{1-A}{(1+z\cdot\bar{z})(1+Az\cdot\bar{z})}\right) \left[(z\,\bar{\psi})^i \left(\overline{R}\, f\, \bar{\psi}\right)_i - (\psi\, f\, R)^i\, (\psi\,\bar{z})_i \right] - iA\, \left(\psi_i f\, J\, f \bar{\psi}^i\right) \\
&+ \frac{1}{4}\, \left(g_\mu{}^\alpha g_\rho{}^\sigma + g_\mu{}^\sigma g_\rho{}^\alpha\right) \bar{\psi}_{\alpha\, i} \bar{\psi}^i_\sigma\, \psi^{\rho\, j} \psi^\mu_j .
\end{aligned}
$$

Here, we used concise notations - all indices in parenthesize are in the proper positions and they are converted from top-left to down-right, e.g. $(\psi\,\bar{z})_i = \psi_i^\alpha\,\bar{z}_\alpha$, etc.

Our Hamiltonian commutes with all our supercharges, as it should be. Its bosonic part (the first two lines) contains the terms describing the interaction with $U(n)$ gauge fields and a specific potential term.

The parameter A takes two values $A = \pm 1$, according with the algebra. If we take $A = 1$, so the algebra of currents of the internal group is $SU(1,n)$, then the Hamiltonian drastically simplified to be

$$\begin{aligned} H_{A=1} \;&=\; \left(\bar{p}\, g^{-1} p\right) + \left(\bar{p}\, g^{-1} h J \bar{z}\right) - \left(z J h g^{-1} p\right) - \left(\overline{R} R\right) - \left(z J h g^{-1} h J \bar{z}\right) \\ &-\; \mathrm{i}\left(\psi_i f J f \bar{\psi}^i\right) + \frac{1}{4}\left(g_\mu{}^\alpha g_\rho{}^\sigma + g_\mu{}^\sigma g_\rho{}^\alpha\right) \bar{\psi}_{\alpha\, i} \bar{\psi}^i_\sigma\, \psi^{\rho\, j} \psi^\mu_j . \end{aligned}$$

Clearly, the $R, \overline{R}$ dependent term in the Hamiltonian can be rewritten through the Casimir operator $\mathcal{K}$ of $SU(1,n)$ algebra

$$\mathcal{K} = \overline{R}^\alpha R_\alpha - \frac{1}{2} J_\alpha{}^\beta J_\beta{}^\alpha + \frac{1}{2(n+1)} J_\alpha{}^\alpha J_\beta{}^\beta .$$

Thus, the Hamiltonian depends only on $U(n)$ currents $J_\alpha{}^\beta$ and $SU(1,n)$ Casimir operator.

The $U(1)$ gauge potential presented in the Hamiltonian has standard form (for A=1 case)

$$\mathcal{A}_{U(1)} = i\,\frac{\dot{z}\bar{z} - z\dot{\bar{z}}}{2\,(1 + z\cdot\bar{z})}.$$

In the simplest case of CP^1 we have only this gauge potential in the theory, while the scalar potential term acquires form (This is just example of super-oscillator potential on CP^n manifolds constructed in S. Bellucci, A. Nersessian, "(Super)Oscillator on CP(N) and Constant Magnetic Field", Phys.Rev. **D67** (2003) 065013; Erratum-ibid. **D71** (2005) 089901)

$$\mathcal{V}_{CP^1} = -\overline{R}^{\alpha}R_{\alpha} - \frac{z\cdot\bar{z}}{4}J^2.$$

Let us remind that we choose the matrix-valued operators $\overline{R}, R, J$ to be anti-hermitian. Thus, the potential is positively defined.

Thus, we have constructed $\mathcal{N}=4$ supersymmetric extension of mechanics describing the motion of particle over CP^n manifold in the presence of background $U(n)$ gauge fields.

- The gauge potential is proportional to the $U(n)$-connection on $SU(n+1)/U(n)$. Such type of background gauge fields is well known for a long time in a bosonic case.
- This gauge potential appears in our system automatically as a result of imposing $\mathcal{N}=4$ supersymmetry.
- In addition to gauge fields $\mathcal{N}=4$ supersymmetry demand additional potential terms to be present in the Hamiltonian. In the simplest case of CP^1 system this potential is just a harmonic oscillator one.

One of the most unexpected features of the present model is a strange interplay between isospin group to which our gauge fields are coupled to and the form of these fields. It turns out that the standard $SU(n+1)/U(n)$ connection appears as a gauge fields potential only in case if isospin group is chosen to be $SU(1,n)$. Alternatively, the choice of $SU(n+1)$ group for isospin variables gives rize to a $U(n)$ connection on $SU(1,n)/U(n)$ group. Any case, the both cases are compatible with $\mathcal{N}=4$ supersymmetry.

Another interesting peculiarity of our model is the presence of the isospin variables on the whole $SU(n+1)$ (or $SU(1,n)$) group, despite the fact that only $U(n)$ gauge fields appear in the Hamiltonian. Again, this situation is not new. The same effect has been noted in the recently constructed $\mathcal{N}=4$ supersymmetric mechanics coupled to non-Abelian gauge fields.

One of the possible immediate application of the constructed model is the analysis of the role the additional fermionic variables play in the quantum Hall effect on CP^n. In this respect it could be important that $\mathcal{N}=4$ supersymmetry insists on the simultaneous appearance of the gauge fields on $U(1)$ and $SU(n)$ with a proper fixed relative coefficient. The role of the special type of the scalar potential which appears due to $\mathcal{N}=4$ supersymmetry also has to be clarify.

Another interesting possibility to describe $\mathcal{N}=4$ supersymmetric CP^n mechanics is to replace from beginning the linear chiral supermultiplets by the nonlinear ones. This case is under investigation at present.