# Green Functions in Stochastic Field Theory

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# Outline

- Stochastic differential equation
- Fokker-Planck equation
- Field theory for Fokker-Planck equation
- Master equation
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- Generating functions of Green functions
- Functional representation of Schwinger-Keldysh formalism
- Functional integral representation

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yields Stratonovich interpretation. Mathematicians prefer Ito. Sometimes second-order SDE is discussed

$$m\frac{\partial^2 \varphi}{\partial t^2} + \gamma\frac{\partial \varphi}{\partial t} = -K\varphi + U(\varphi) + fb(\varphi)\,.$$

# Iterative solution for correlation functions

Tree-graph solution $\varphi[\chi, f] = (\partial_t + K)^{-1}\chi +$ *tree-graphs* yields correlation functions with the aid of Wick's theorem

$$G(J) = \left\langle e^{\varphi[\chi,f]J} \right\rangle = \int \mathcal{D}f \, e^{-\frac{1}{2} f \overline{D}^{-1} f} e^{\varphi[\chi,f]J} \, .$$

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$$\times \, e^{-\frac{1}{2} f \overline{D}^{-1} f + \tilde{\varphi}(-\partial_t \varphi - K\varphi + U(\varphi) + f) + \varphi J} \, .$$

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Loop expansion of $|\det(-\partial_t - K + U')|$ to remove $\Delta$ loops.

# Fokker-Planck equation

The SDE in the Stratonovich sense yields the FPE:

$$\frac{\partial}{\partial t} p\left(\varphi, t|\varphi_0, t_0\right) = -\frac{\partial}{\partial \varphi}\left\{\left[-K\varphi + U(\varphi)\right] p\left(\varphi, t|\varphi_0, t_0\right)\right\} + \frac{1}{2}\frac{\partial}{\partial \varphi}\left\{b(\varphi)\frac{\partial}{\partial \varphi}\left[Db(\varphi) p\left(\varphi, t|\varphi_0, t_0\right)\right]\right\} .$$

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Equations coincide, when $b(\varphi)$ is a constant (additive noise).

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The Fokker-Planck equation (Ito) for the PDF $p(\varphi, t) = \langle\, \varphi \,|\, p_t \,\rangle$

$$\frac{\partial}{\partial t} |\, p_t \,\rangle = \hat{L} |\, p_t \,\rangle\,, \quad \hat{L} = \hat{\pi}\,[-K\hat{\varphi} + U(\hat{\varphi})] + \frac{1}{2}\hat{\pi}^2 b(\hat{\varphi}) D b(\hat{\varphi})\,.$$

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Operators in the Heisenberg picture and Dirac picture (Euclidean, imaginary time)

$$\hat{\varphi}_H(t) = e^{-\hat{L}t} \hat{\varphi} e^{\hat{L}t}\,, \quad \hat{\varphi}(t) = e^{-\hat{L}_0 t} \hat{\varphi} e^{\hat{L}_0 t}\,, \quad \hat{L}_0 = -\hat{\pi} K \hat{\varphi}\,.$$

# Green functions for expectation values

Consider the $n$-point Green function of Heisenberg operators

$$G_n(t_1, t_2, \ldots t_n) = \mathrm{Tr}\ \{\hat{p}_0 T\left[\hat{\varphi}_H(t_1)\hat{\varphi}_H(t_2)\cdots\hat{\varphi}_H(t_n)\right]\}$$

with the density operator $\hat{p}_0 = \int d\varphi |\, p_0 \,\rangle\langle\, \varphi \,|$.

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$$p\,(\varphi, t|\varphi_0, t_0) = \left\langle\, \varphi \,\middle|\, \exp[\hat{L}(t - t_0)] \,\middle|\, \varphi_0 \,\right\rangle$$

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to conclude that for $t_1 > t_2 > t_3 > \ldots > t_{n-1} > t_n > t_0$

$$\int d\varphi_1 \ldots \int d\varphi_n\, \varphi_1 \cdots \varphi_n p\,(\varphi_1, t_1; \ldots; \varphi_n, t_n) = G_n(t_1, \ldots t_n)\,.$$

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Use the QFT to evaluate expectation values for the FPE!

# Master equation

Discontinuous sample paths, use the master equation

$$\frac{\partial}{\partial t} p\left(\varphi, t\right) = \int d\chi \left[W(\varphi|\chi, t) p\left(\chi, t\right) - W(\chi|\varphi, t) p\left(\varphi, t\right)\right] .$$

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Description of reactions, population dynamics etc; use the (integer valued) occupation number $n$ and the probability density $P\left(t, n\right)$.

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Classic example: stochastic Verhulst model

$$\frac{\mathrm{d}P(t,n)}{\mathrm{d}t} = [\beta(n+1) + \gamma(n+1)^2]P(t,n+1) + \lambda(n-1)P(t,n-1) - \left(\beta n + \lambda n + \gamma n^2\right) P(t,n)$$

with death rate $\beta$, birth rate $\lambda$ and damping coefficient $\gamma$.

# Kinetic equation in Fock space

Construct (Doi 1976) a single kinetic equation in the a Fock space spanned by operators $\hat{a}$, $\hat{a}^+$ and basis vectors $|\, n \,\rangle$:

$$\hat{a}|\, 0 \,\rangle = 0\,, \quad \hat{a}^+|\, n \,\rangle = |\, n+1 \,\rangle\,, \quad [\,\hat{a}, \hat{a}^+] = 1\,, \quad \langle\, n \,|\, m \,\rangle = n!\delta_{nm}\,.$$

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Master equations yield kinetic equation for state vector $|\, P_t \,\rangle$:

$$\frac{\mathrm{d}|\, P_t \,\rangle}{\mathrm{d}t} = \hat{L}(\hat{a}^+,\, \hat{a})|\, P_t \,\rangle\,, \quad |\, P_t \,\rangle = \sum_{n=0}^{\infty} P(t,n)|\, n \,\rangle\,.$$

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The operator $\hat{L}$ is determined by the rules:
$nP(t,n)|\, n \,\rangle = \hat{a}^+\hat{a}P(t,n)|\, n \,\rangle$, $nP(t,n)|\, n-1 \,\rangle = \hat{a}P(t,n)|\, n \,\rangle \ldots$

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Liouville operator for the stochastic Verhulst model:

$$\hat{L}(\hat{a}^+, \hat{a}) = \beta(I - \hat{a}^+)\hat{a} + \gamma(I - \hat{a}^+)\hat{a}\hat{a}^+\hat{a} + \lambda(\hat{a}^+ - I)\hat{a}^+\hat{a}\,.$$

# Green functions of number density operators

Consider the Green function of operators $\hat{n}_H(t) = \hat{a}_H^+(t)\hat{a}_H(t)$:

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to conclude that for $t_1 > t_2 > t_3 > \ldots > t_{m-1} > t_m > t_0$

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# Generating function

Generic form of the *generating function* of the moments

$$G(J) = \mathrm{Tr}\,\hat{\rho}_0 T\left[\exp\left(\hat{S}_J\right)\right] \ , \quad \hat{\rho}_0 = \int d\varphi |\, p_0 \,\rangle\langle\, \varphi \,| \text{ or } \hat{\rho}_0 = |\, P_0 \,\rangle\langle\, P \,| \,,$$

where $\hat{S}_J = \int_{t_i}^{t_f} dt\, \hat{\varphi}_H(t) J(t)$ or $\hat{S}_J = \int_{t_i}^{t_f} dt\, \hat{a}_H^+(t) \hat{a}_H(t) J(t)$.

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In the Dirac picture ($\hat{L} = \hat{L}_0 + \hat{L}_I\,, \quad t_f > t_i > t_0$)

$$T\, e^{\hat{S}_J} = e^{\hat{L}_0 t_0}\hat{U}(t_0, t_f) T\left[e^{\hat{S}_J + \hat{S}_I}\right]\hat{U}(t_i, t_0) e^{-\hat{L}_0 t_0}$$

$$= e^{\hat{L}_0 t_0}\tilde{T}\, e^{-\int_{t_0}^{t_f}\hat{L}(t)\,dt} T\left[e^{\hat{S}_J + \hat{S}_I}\right] T\, e^{\int_{t_0}^{t_i}\hat{L}(t)\,dt} e^{-\hat{L}_0 t_0}\,,$$

where $\hat{U}(t, t') = e^{-t\hat{L}_0} e^{(t-t')\hat{L}} e^{t'\hat{L}_0}$, $\hat{S}_I = \int_{t_i}^{t_f} \hat{L}_I(t)\, dt$ and $\tilde{T}$ is the *anti-chronological* product.

# Generic functional representation

$T$ products fuse due to Wick's theorems in a *normal product*:

$$G(J) = \mathrm{Tr}\left( N \left\{ \exp\left[ \frac{1}{2}\frac{\delta}{\delta\phi_1}\tilde{\Delta}\frac{\delta}{\delta\phi_1} + \frac{1}{2}\frac{\delta}{\delta\phi_2}\Delta\frac{\delta}{\delta\phi_2} + \frac{\delta}{\delta\phi_1} n \frac{\delta}{\delta\phi_2} \right] \right.\right.$$

$$\left.\left. \times \exp\left[ S_J(\phi_2) - \int_{t_0}^{t_f} L_I(\phi_1)\, du + \int_{t_0}^{t_f} L_I(\phi_2)\, du \right] \right|_{\phi_i = \hat{\phi}} \right\} e^{-\hat{L}_0 t_0} \hat{\rho}_0 e^{\hat{L}_0 t_0} \Bigg) .$$

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In QFT $\hat{L} \to -i(\hat{H} - \mu\hat{N})/\hbar$, $\hat{\rho}_0 \to e^{-(\hat{H}-\mu\hat{N})/T}/Z_G$ yield finite-temperature Green functions and Keldysh graphs.

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$$\left.\left.\times\exp\left[S_J(\phi_2) - \int_{t_0}^{t_f} L_I(\phi_1)\,du + \int_{t_0}^{t_f} L_I(\phi_2)\,du\right]\Bigg|_{\phi_i=\hat{\phi}}\right\} e^{-\hat{L}_0 t_0}\hat{\rho}_0 e^{\hat{L}_0 t_0}\right)$$

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Separate evaluation of $\mathrm{Tr}\, e^{-\hat{L}_0 t_0}\hat{\rho}_0 e^{\hat{L}_0 t_0} N[\ldots]$ for FPE and ME.

# Functional representation for FPE

For any operator functional $F[\hat{\pi}, \hat{\varphi}]$ calculation yields

$$\mathrm{Tr}\, e^{-\hat{L}_0 t_0} \hat{\rho}_0 e^{\hat{L}_0 t_0} N \left\{ F[\hat{\pi}, \hat{\varphi}] \right\} = \int \mathcal{D}\varphi \, p_0(\varphi) F\left[0, n\varphi\right] \ ,$$

where $p_0(\varphi) = p(\varphi, t_0)$.

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$$\mathrm{Tr}\, e^{-\hat{L}_0 t_0}\hat{\rho}_0 e^{\hat{L}_0 t_0} N\left\{F[\hat{\pi},\hat{\varphi}]\right\} = \int \mathcal{D}\varphi\, p_0(\varphi) F\left[0, n\varphi\right] ,$$

where $p_0(\varphi) = p(\varphi, t_0)$. Therefore

$$G(J) = \int \mathcal{D}\varphi\, p_0(\varphi) \exp\left[\frac{\delta}{\delta\varphi_1}\tilde{\Delta}\frac{\delta}{\delta\pi_1} + \frac{\delta}{\delta\varphi_2}\Delta\frac{\delta}{\delta\pi_2} + \frac{\delta}{\delta\varphi_1} n \frac{\delta}{\delta\pi_2}\right]$$

$$\times \exp\left[\int_{t_i}^{t_f} dt\, \varphi_2(t) J(t) - \int_{t_0}^{t_f} L_I(\pi_1,\varphi_1)\, dt + \int_{t_0}^{t_f} L_I(\pi_2,\varphi_2)\, dt\right]\Bigg|_{\substack{\pi_i=0\\ \varphi_i=n\varphi}} .$$

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In the limit $t_f \to \infty$, $t_i \to -\infty$ we arrive at Keldysh rules. Cancelation of closed propagator loops is produced by the auxiliary set of fields $\pi_1$, $\varphi_1$.

# Functional integral for FPE

In first-order models closed loops of $\Delta$, $\tilde{\Delta}$ vanish. The contribution of fields $\pi_1$, $\varphi_1$ is reduced to a constant:

$$G(J) = \int \mathcal{D}\varphi \, p_0(\varphi) \left[ e^{\frac{\delta}{\delta\varphi_2}\Delta\frac{\delta}{\delta\pi_2}} e^{S_I(\pi_2,\varphi_2)+\varphi_2 J} \right] \Bigg|_{\substack{\pi=0 \\ \varphi_2=n\varphi}} .$$

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Introduce functional integral through the Gaussian trick

$$e^{\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\pi}} = \iint \mathcal{D}\phi \, \mathcal{D}\tilde{\phi} \, e^{-\tilde{\phi}\Delta^{-1}\phi+\tilde{\phi}\frac{\delta}{\delta\pi}+\phi\frac{\delta}{\delta\varphi}} .$$

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Obtain generating function of Martin-Siggia-Rose theory:

$$G(J) = \iiint \mathcal{D}\varphi\mathcal{D}\phi\mathcal{D}\tilde{\phi}\, p_0(\varphi)\, e^{-\tilde{\phi}(\partial_t+K)\phi+S_I(\tilde{\phi},\phi+n\varphi)+(\phi+n\varphi)J} .$$

# Functional integral for Schwinger-Keldysh

The functional $L_I$ is quadratic in $\pi$. Therefore (constant $b$)

$$G(J) = \iiint \mathcal{D}\varphi\, \mathcal{D}\eta_1 \mathcal{D}\eta_2\, p_0(\varphi) \exp\Big\{ J(\Delta\eta_2 + n\varphi)$$
$$+ \frac{1}{2}\eta_2 b D b \eta_2 - \frac{1}{2}\eta_1 b D b \eta_1 + \eta_1 (bDb)^{-1} U_1 + \eta_2 (bDb)^{-1} U_2$$
$$+ \frac{1}{2} U_1 (bDb)^{-1} U_1 - \frac{1}{2} U_2 (bDb)^{-1} U_2 \Big\},$$

where $U_1 = U(\tilde{\Delta}\eta_1 + n\eta_2 + n\varphi)\,,\ U_2 = U(\Delta\eta_2 + n\varphi)\,.$

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Cancelations are now explicit in the functional integral.

Should be more convenient numerically than use of ghosts.

# Functional representation for ME

Generating function of Green functions of number density operators has a more complicated expression

$$G(J) = \int \frac{ds}{2\pi i}\, e^s \tilde{G}(s)\, \exp\left[ \frac{\delta}{\delta a_1} \tilde{\Delta} \frac{\delta}{\delta a_1^+} + \frac{\delta}{\delta a_2} \Delta \frac{\delta}{\delta a_2^+} + \frac{\delta}{\delta a_1} n \frac{\delta}{\delta a_2^+} \right]$$
$$\times \exp\left\{ \int \left[ -L_I(a_1^+ + 1, a_1) + L_I(a_2^+ + 1, a_2) \right] dt \right\}$$
$$\times \exp\left\{ \int \left[ (a_2^+(t) + 1) a_2(t) \right] J(t)\, dt \right\} \Bigg|_{\substack{a_i^+ = 0 \\ a_i = ns}} ,$$

where

$$\tilde{G}(s) = \sum_{n=1}^{\infty} \frac{\Gamma(n)}{s^n} P(0, n-1)\,.$$

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- Solution of the determinant problem with the aid of Schwinger-Keldysh approach.
- Schwinger-Keldysh approach advantageous for numerical calculation.