Superstrong B: photon “mass” and LLL spectrum in hydrogen atom

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Recently solved QM + QED (almost) textbook problem.

A.E.Shabad, V.V.Usov (2007,2008) - numerically;
M.I.Vysotsky, JETP Lett. **92** (2010)15;
B.Machet, M.I.Vysotsky, PR D **83** (2011)025022 - analytically;

For this talk:

strong magnetic field: $B > m_e^2 e^3$
(Gauss units; $e^2 = \alpha = 1/137$);
$B_{cr} = m_e^2/e$;
superstrong magnetic field: $B > m_e^2/e^3$;
$a_H = 1/\sqrt{eB}$ - Landau radius, magnetic length.

plan

- $D=2$ QED with massive fermions, radiative "corrections" to Coulomb potential in $d=1$; analytical formula for $\Phi(z)$, $g>m$ - photon "mass" $m_\gamma \sim g$, screening at ALL z when $g>m$
- $D=4$ QED in external B, photon "mass" $m_\gamma^2 = e^3 B$ at superstrong magnetic fields $B >> m_e^2/e^3$, analytical formula for $\Phi(z)$
- Electron in magnetic field: LLL - nonrelativistic at all B
- The Karnakov-Popov equation for atomic energies
- Conclusions

$D = 2$ QED: screening of Φ

$$\Phi(\bar{k}) \equiv A_0(\bar{k}) = \frac{4\pi g}{\bar{k}^2}\ ; \quad \mathbf{\Phi} \equiv \mathbf{A}_0 = D_{00} + D_{00}\Pi_{00}D_{00} + \ldots$$

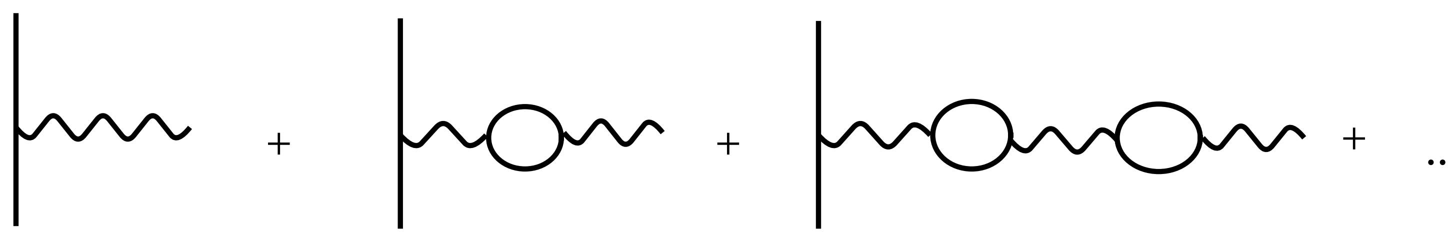


Fig 1. *Modification of the Coulomb potential due to the dressing of the photon propagator.*

Summing the series we get:

$$\mathbf{\Phi}(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)}\ , \quad \Pi_{\mu\nu} \equiv \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi(k^2)$$

$$\Pi(k^2) = 4g^2 \left[\frac{1}{\sqrt{t(1+t)}} \ln(\sqrt{1+t} + \sqrt{t}) - 1 \right] \equiv -4g^2 P(t) \ ,$$

$t \equiv -k^2/4m^2, \ \ [g] =$mass.

Taking $k = (0, k_{\|})$, $k^2 = -k_{\|}^2$ for the Coulomb potential in the coordinate representation we get:

$$\mathbf{\Phi}(z) = 4\pi g \int\limits_{-\infty}^{\infty} \frac{e^{ik_{\|}z} dk_{\|}/2\pi}{k_{\|}^2 + 4g^2 P(k_{\|}^2/4m^2)} \ ,$$

and the potential energy for the charges $+g$ and $-g$ is finally: $V(z) = -g\mathbf{\Phi}(z)$.

Asymptotics of $P(t)$ are:

$$P(t) = \begin{cases} \frac{2}{3}t & , \quad t \ll 1 \\ 1 & , \quad t \gg 1 \end{cases} .$$

Let us take as an interpolating formula for $P(t)$ the following expression:

$$\overline{P}(t) = \frac{2t}{3+2t} \quad .$$

The accuracy of this approximation is not worse than 10% for the whole interval of t variation, $0 < t < \infty$.

$$
\begin{aligned}
\mathbf{\Phi} &= 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_\parallel z} dk_\parallel / 2\pi}{k_\parallel^2 + 4g^2(k_\parallel^2/2m^2)/(3 + k_\parallel^2/2m^2)} = \\
&= \frac{4\pi g}{1 + 2g^2/3m^2} \int_{-\infty}^{\infty} \left[\frac{1}{k_\parallel^2} + \frac{2g^2/3m^2}{k_\parallel^2 + 6m^2 + 4g^2} \right] e^{ik_\parallel z} \frac{dk_\parallel}{2\pi} = \\
&= \frac{4\pi g}{1 + 2g^2/3m^2} \left[-\frac{1}{2}|z| + \frac{g^2/3m^2}{\sqrt{6m^2 + 4g^2}} \exp(-\sqrt{6m^2 + 4g^2}|z|) \right]
\end{aligned}
$$

In the case of heavy fermions ($m \gg g$) the potential is given by the tree level expression; the corrections are suppressed as g^2/m^2.

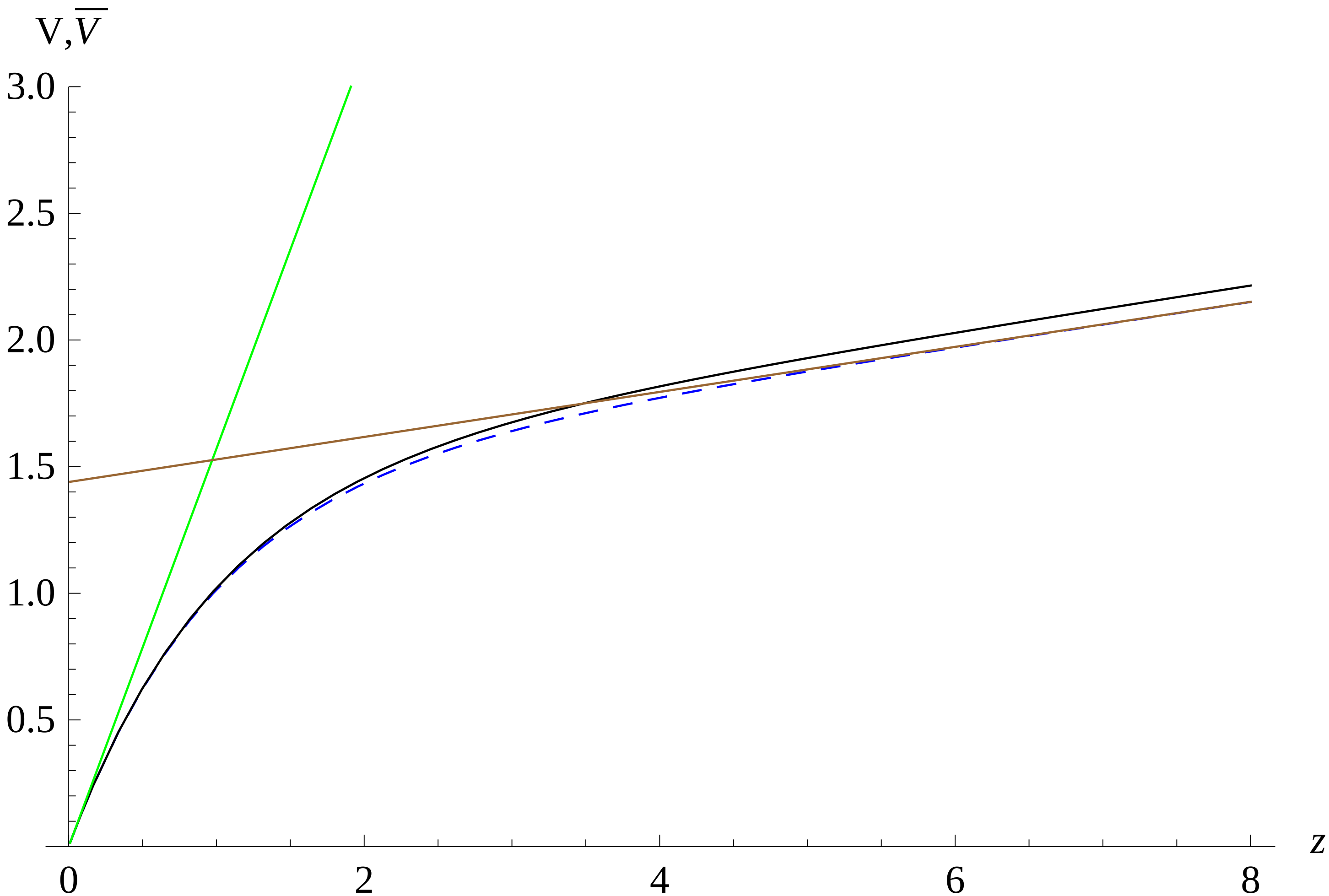
In case of light fermions ($m \ll g$):

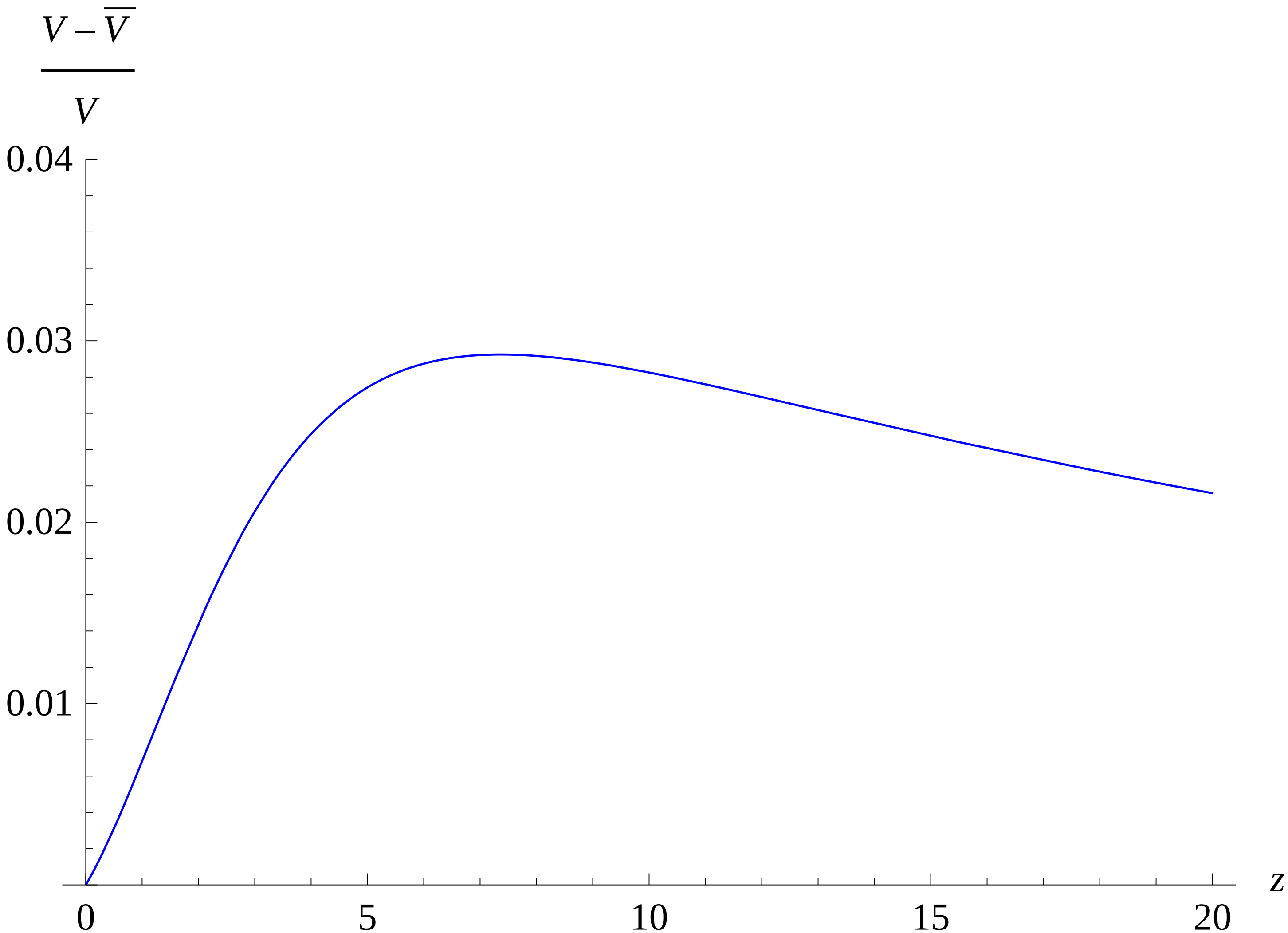
$$\mathbf{\Phi}(z)\Big|_{\; m \ll g} = \begin{cases} \pi e^{-2g|z|} & , \quad z \ll \frac{1}{g}\ln\left(\frac{g}{m}\right) \\ -2\pi g\left(\frac{3m^2}{2g^2}\right)|z| & , \quad z \gg \frac{1}{g}\ln\left(\frac{g}{m}\right) \end{cases} \quad .$$

$m = 0$ - Schwinger model; photon get mass.

Light fermions - continuous transition from $m > g$ to $m = 0$.

Next two figures correspond to $g = 0.5, \;\; m = 0.1$:


V,$\overline{V}$
3.0
2.5
2.0
1.5
1.0
0.5
0
2
4
6
8
z


$\frac{V - \overline{V}}{V}$
0.04
0.03
0.02
0.01
0
5
10
15
20
z

$D = 4$ QED in superstrong B

For $B >> B_{cr} \equiv m^2/e$ and moderate photon momentum $k^2 \lesssim eB$ the following expression for polarization operator was obtained long ago (Skobelev, 1975; Loskutov, Skobelev, 1976):

$$\Pi_{\mu\nu} \simeq e^3 B * exp(-\frac{k_\perp^2}{2eB}) * \Pi_{\mu\nu}^{(2)}(k_\| \equiv k_z);$$

$$\mathbf{\Phi} = \frac{4\pi e}{(k_\|^2 + k_\perp^2)\left(1 - \frac{\alpha}{3\pi}\ln\left(\frac{eB}{m^2}\right)\right) + \frac{2e^3 B}{\pi}\exp\left(-\frac{k_\perp^2}{2eB}\right) P\left(\frac{k_\|^2}{4m^2}\right)} \, .$$

$$\mathbf{\Phi}(z) =$$

$$= 4\pi e \int \frac{e^{ik_\parallel z} dk_\parallel d^2 k_\perp/(2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi} \exp(-k_\perp^2/(2eB))(k_\parallel^2/2m_e^2)/(3 + k_\parallel^2/2m_e^2)} \ ,$$

$$\mathbf{\Phi}(z) = \frac{e}{|z|}\left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}|z|}\right] \ .$$

For magnetic fields $B \ll 3\pi m^2/e^3$ the potential is Coulomb up to small power suppressed terms:

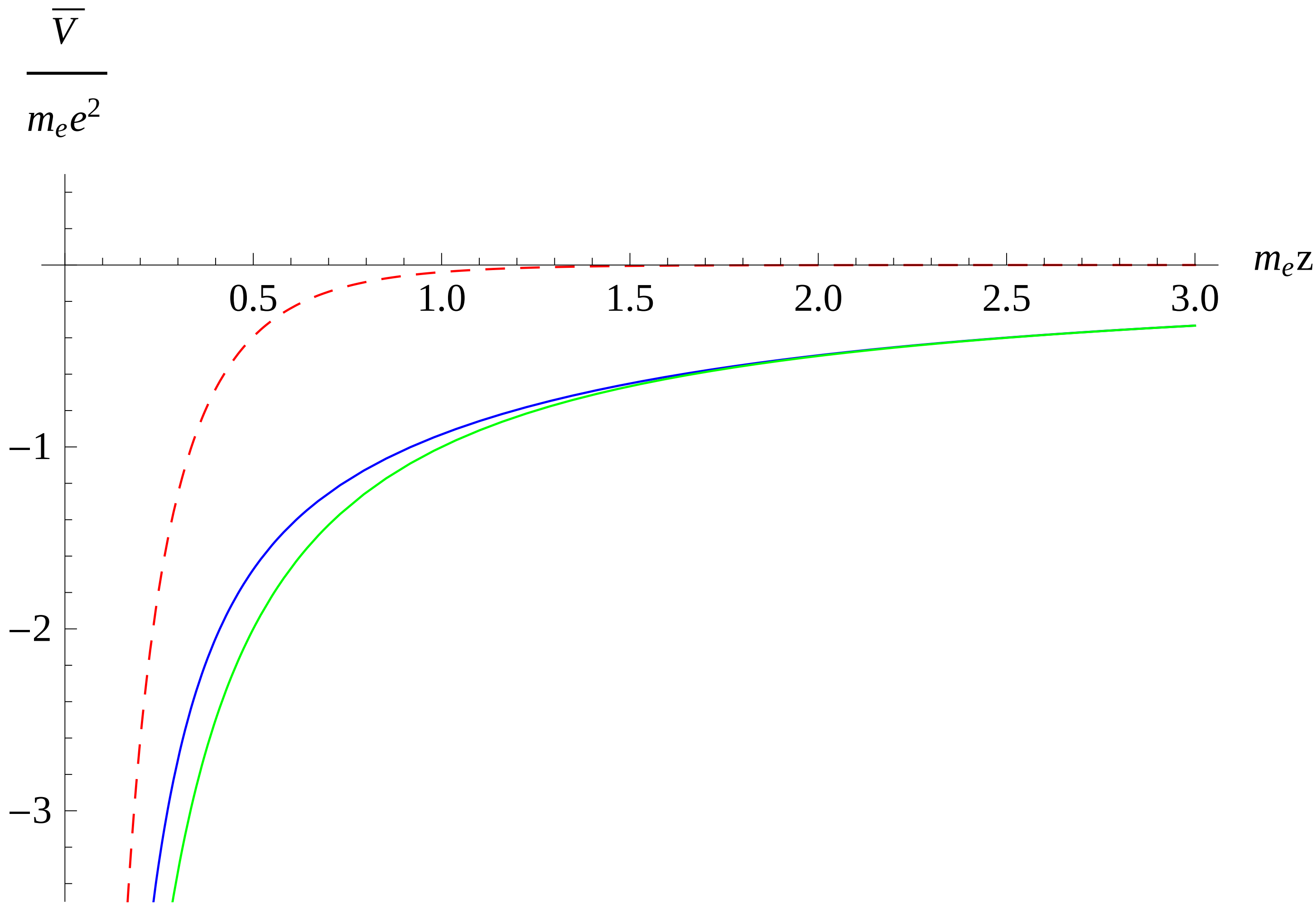
$$\mathbf{\Phi}(z)\Bigg|_{e^3 B \ll m_e^2} = \frac{e}{|z|}\left[1 + O\left(\frac{e^3 B}{m_e^2}\right)\right]$$

in full accordance with the $D = 2$ case, where g^2 plays the role of $e^3 B$.

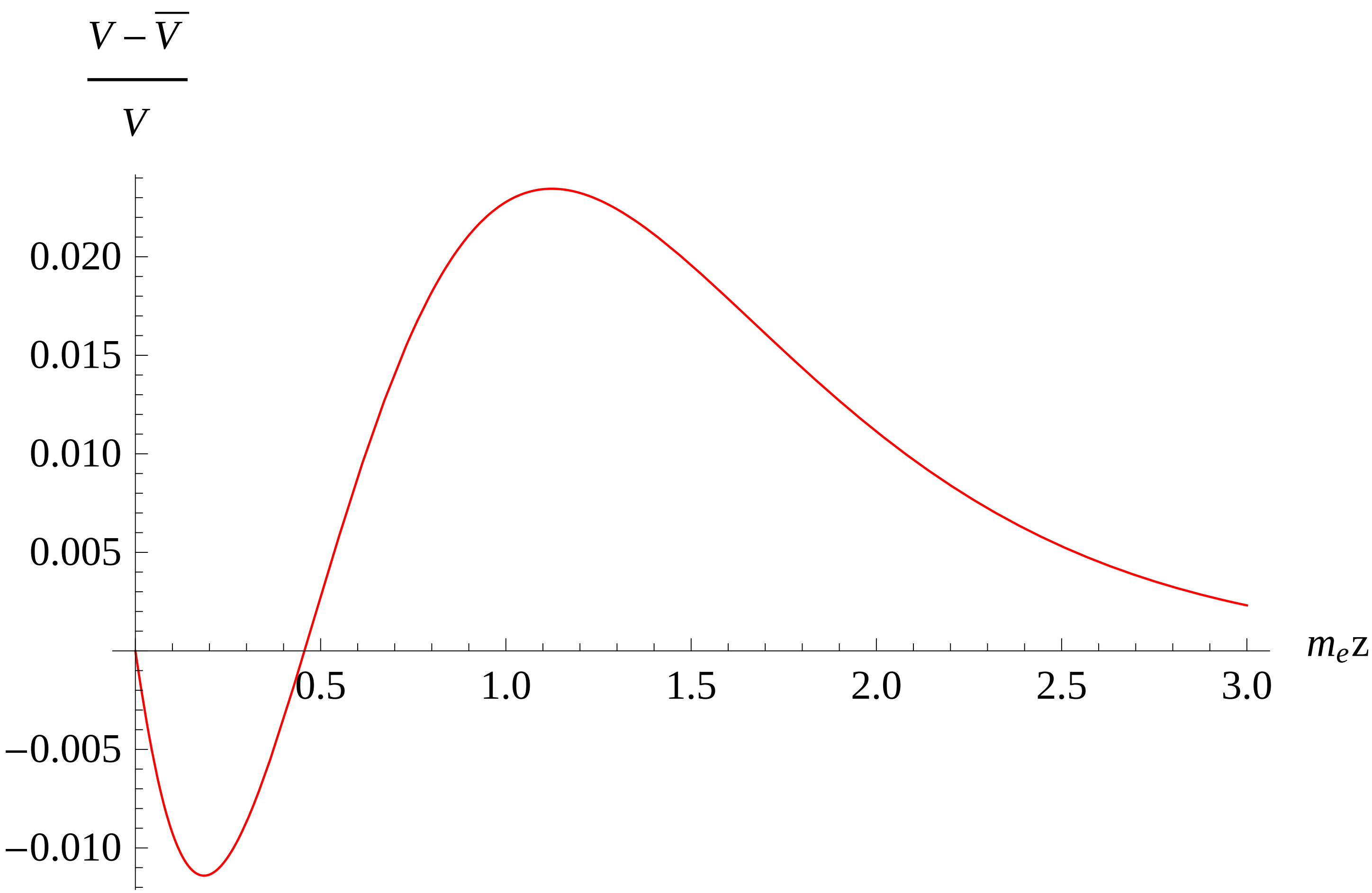
In the opposite case of superstrong magnetic fields $B \gg 3\pi m_e^2/e^3$ we get:

$$\mathbf{\Phi}(z) = \begin{cases} \frac{e}{|z|} e^{(-\sqrt{(2/\pi)e^3 B}|z|)}, & \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln\left(\sqrt{\frac{e^3 B}{3\pi m_e^2}}\right) > |z| > \frac{1}{\sqrt{eB}} \\ \frac{e}{|z|}(1 - e^{(-\sqrt{6m_e^2}|z|)}), & \frac{1}{m} > |z| > \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln\left(\sqrt{\frac{e^3 B}{3\pi m_e^2}}\right) \\ \frac{e}{|z|} \quad , & |z| > \frac{1}{m} \end{cases}$$

$$V(z) = -e\mathbf{\Phi}(z)$$



Modified Coulomb potential at $B = 10^{17}$G (blue) and its long distance (green) and short distance (red) asympotics.



Relative accuracy of analytical formula for modified Coulomb potential at $B = 10^{17}$G.

electron in magnetic field

spectrum of Dirac eq in cylindrical coordinates $(\bar{\rho}, z)$ in the gauge, where $\bar{A} = \frac{1}{2}[\bar{B}\bar{r}]$:

$$\varepsilon_n^2 = m_e^2 + p_z^2 + (2n_\rho + |m| + m + \sigma_z + 1)eB \ \ ,$$

$n_\rho = 0, 1, 2, 3, ...; \ \ m = 0, \pm 1, \pm 2, ...; \ \ \sigma_z = \pm 1$

for $B > B_{cr} = m_e^2/e$ the electrons are relativistic

with only one exception: electrons from lowest Landau level (LLL: $n_\rho = 0; \ \ m = 0, -1, -2, ...; \ \ \sigma_z = -1$) can be nonrelativistic.

In what follows we will study the spectrum of electrons from LLL in the Coulomb field of the proton modified by the superstrong B.

Spectrum of Schrödinger eq.:

$$E_{p_z n_\rho m \sigma_z} = \left(n_\rho + \frac{|m| + m + 1 + \sigma_z}{2} \right) \frac{eB}{m_e} + \frac{p_z^2}{2m_e} \quad ,$$

LLL: $n_\rho = 0, \sigma_z = -1, m = 0, -1, -2, ...$

$$R_{0m}(\bar{\rho}) = \left[\pi (2a_H^2)^{1+|m|} (|m|!) \right]^{-1/2} \rho^{|m|} e^{(im\varphi - \rho^2/(4a_H^2))} \quad ,$$

Now we should take into account electric potential of atomic nuclei situated at $\bar{\rho} = z = 0$. For $a_H \ll a_B$ adiabatic approximation is applicable and the wave function in the following form should be looked for:

$$\Psi_{n0m-1} = R_{0m}(\bar{\rho}) \chi_n(z) \quad ,$$

where $\chi_n(z)$ is the solution of the Schrödinger equation for electron motion along a magnetic field:

$$\left[-\frac{1}{2m}\frac{d^2}{dz^2}+U_{eff}(z)\right]\chi_n(z)=E_n\chi_n(z) \quad .$$

Without screening the effective potential is given by the following formula:

$$U_{eff}(z)=-e^2\int\frac{|R_{0m}(\rho)|^2}{\sqrt{\rho^2+z^2}}d^2\rho \quad ,$$

For $|z|\gg a_H$ the effective potential equals Coulomb:

$$U_{eff}(z)\Bigg|_{z\gg a_H}=-\frac{e^2}{|z|}$$

and is regular at $z = 0$:

$$U_{eff}(0) \sim -\frac{e^2}{|a_H|} \quad .$$

Since $U_{eff}(z) = U_{eff}(-z)$, the wave functions are odd or even under reflection $z \to -z$; the ground states (for $m = 0, -1, -2, ...$) are described by even wave functions.

Karnakov - Popov equation

It provides a several percent accuracy for the energies of even states for $B > 10^3(m_e^2 e^3)$.
Main idea: to integrate Sh eq with effective potential from $x = 0$ till $x = z$, where $a_H << z << a_B$ and to equate obtained expression for $\chi'(z)$ to the logarithmic derivative of Whittaker function - the solution of Sh eq with Coulomb potential, which exponentially decreases at $z >> a_B$:

$$2\ln\left(\frac{z}{a_H}\right) + \ln 2 - \psi(1+|m|) + O(a_H/z) =$$

$$2\ln\left(\frac{z}{a_B}\right) + \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2\ln 2 + O(z/a_B)$$

$$E = -(m_e e^4/2)\lambda^2$$

The energies of the odd states are:

$$E_{\rm odd} = -\frac{m_e e^4}{2n^2} + O\left(\frac{m_e^2 e^3}{B}\right) \ , \quad n = 1, 2, ... \quad .$$

So, for superstrong magnetic fields $B \sim m_e^2/e^3$ the deviations of odd states from the Balmer series are negligible.

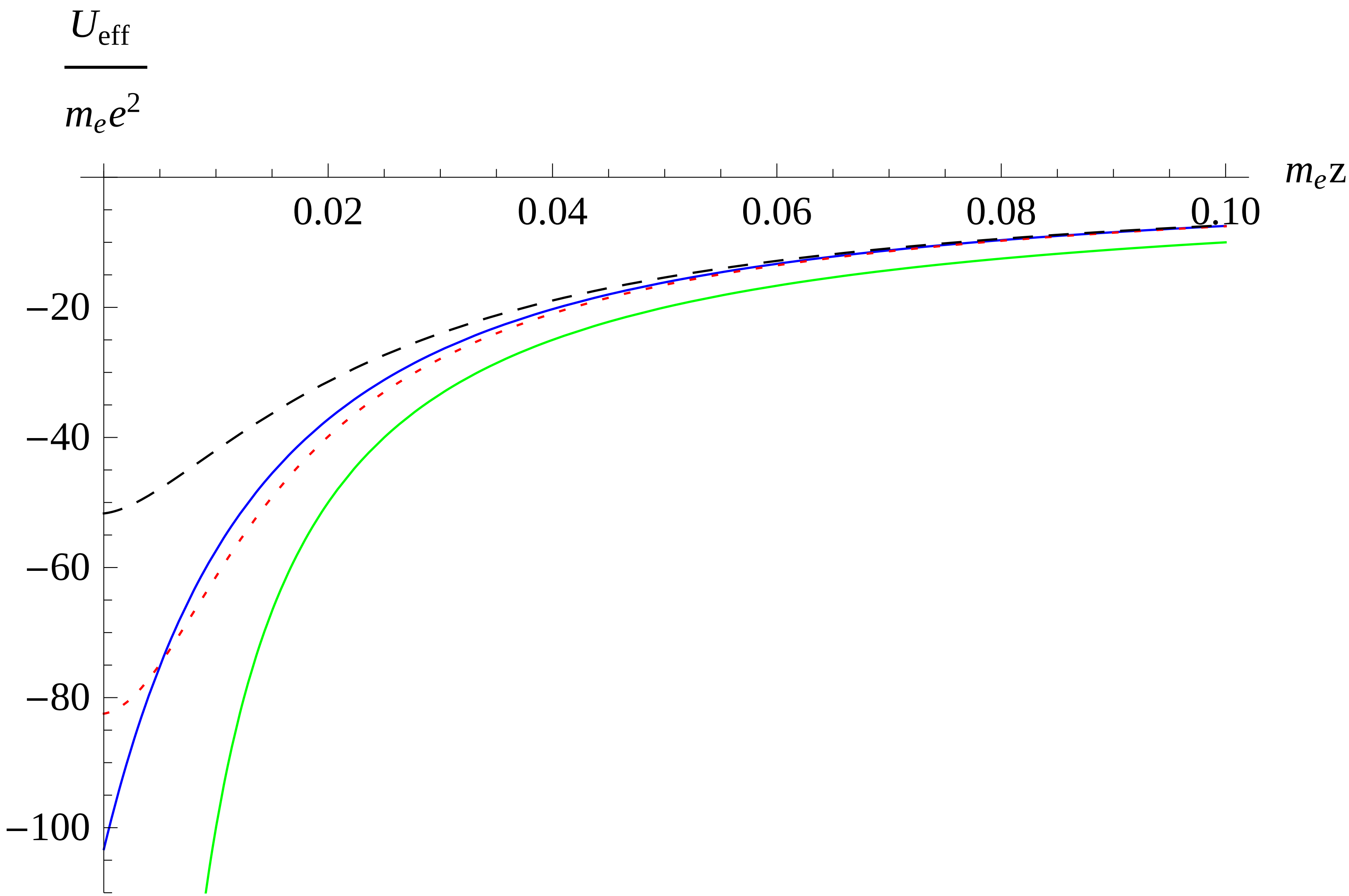
Energies of even states; screening

When screening is taken into account an expression for effective potential transforms into

$$\tilde{U}_{eff}(z) = -e^2 \int \frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2 + z^2}} d^2\rho \left[1 - e^{-\sqrt{6m_e^2}\,z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\,z}\right]$$

$$U_{simpl}(z) = -e^2 \frac{1}{\sqrt{a_H^2 + z^2}} \left[1 - e^{-\sqrt{6m_e^2}\,z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\,z}\right]$$

Eff potential - figures



Effective potentials at $B = 10^{17}$*G*

KP equation with screening

The original KP equation for LLL splitting by Coulomb potential:

$$\ln(B/(m_e^2 e^3)) = \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1+|m|) \quad .$$

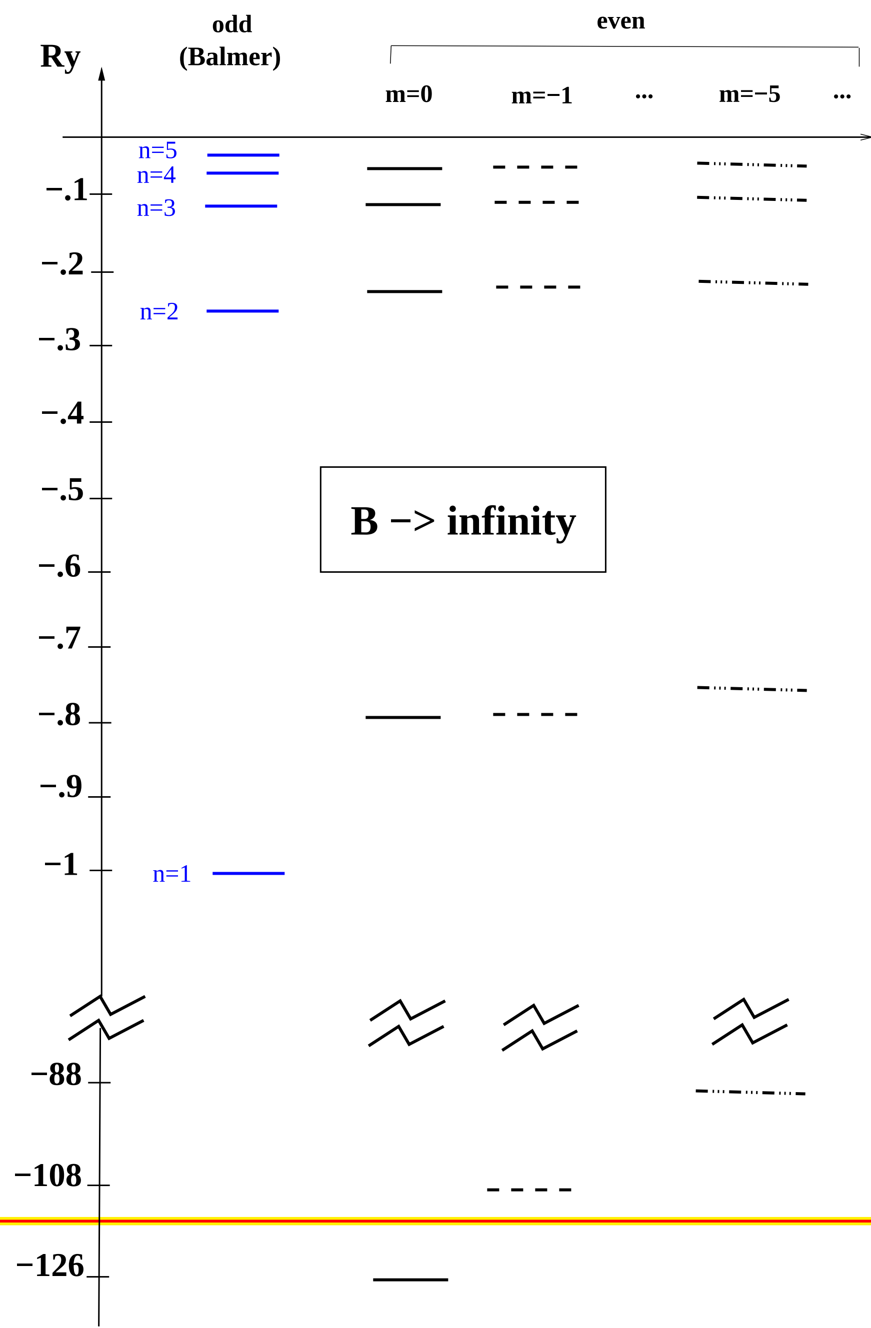
The modified KP equation, which takes screening into account:

$$\ln\left(\frac{B/(m_e^2 e^3)}{1 + B/(3\pi m_e^2/e^3)}\right) = \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) +$$

$$+ \ln 2 + 4\gamma + \psi(1 + |m|) \quad ,$$

$E = -(m_e e^4/2)\lambda^2$, for $B \to \infty : \quad \lambda \to 11.2, \ E_0 \to -1.7KeV$.

spectrum

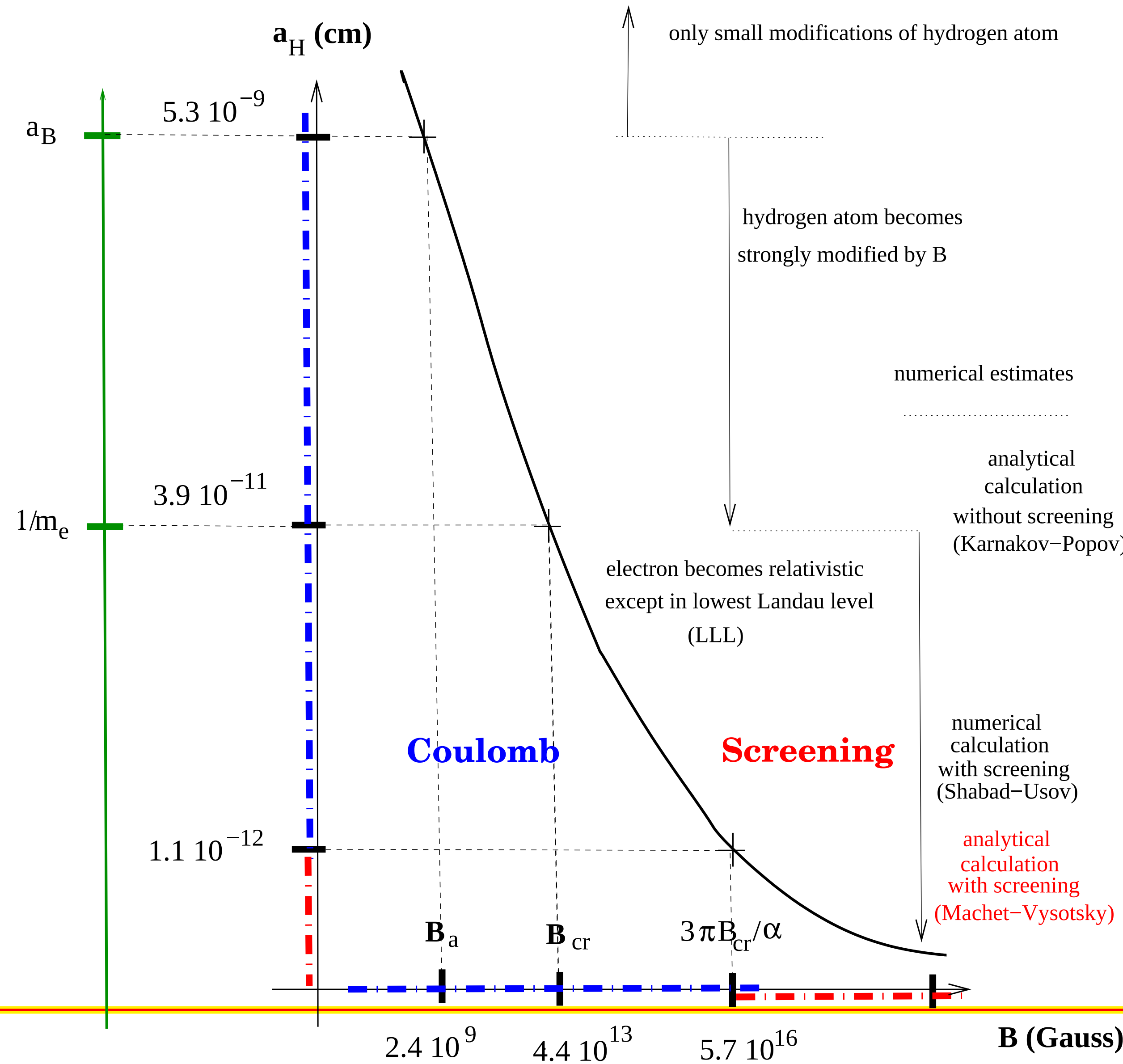

Ry
odd
(Balmer)
even
m=0
m=−1
...
m=−5
...
n=5
n=4
n=3
n=2
n=1
−.1
−.2
−.3
−.4
−.5
−.6
−.7
−.8
−.9
−1
−88
−108
−126
B −> infinity

Conclusion

An algebraic formula for the energy levels of a hydrogen atom originating from the lowest Landau level in superstrong B has been obtained.

backup slides

Landau radius a_H versus B



The shallow-well approximation

$$E^{sw} = -2m_e \left[\int_{a_H}^{a_B} U(z) dz \right]^2 = -(m_e e^4/2) ln^2(B/(m_e^2 e^3))$$

Used to calculate the ground state energy of hydrogen in strong B in LL QM (after 1974 editions); GKK; Shabad, Usov.

Analogous formula for $m \neq 0$ published in 1971 by Barbieri.

$$-\frac{1}{2\mu}\frac{d^2}{dz^2}\chi(z) + U(z)\chi(z) = E_0\chi(z)$$

Neglecting E_0 in comparison with U and integrating we get:

$$\chi'(a) = 2\mu \int\limits_0^a U(x)\chi(x)dx \quad ,$$

where we assume $U(x) = U(-x)$, that is why χ is even. The next assumptions are: 1. the finite range of the potential energy: $U(x) \neq 0$ for $a > x > -a$; 2. χ undergoes very small variations inside the well. Since outside the well $\chi(x) \sim e^{-\sqrt{2\mu|E_0|}\,x}$, we readily obtain:

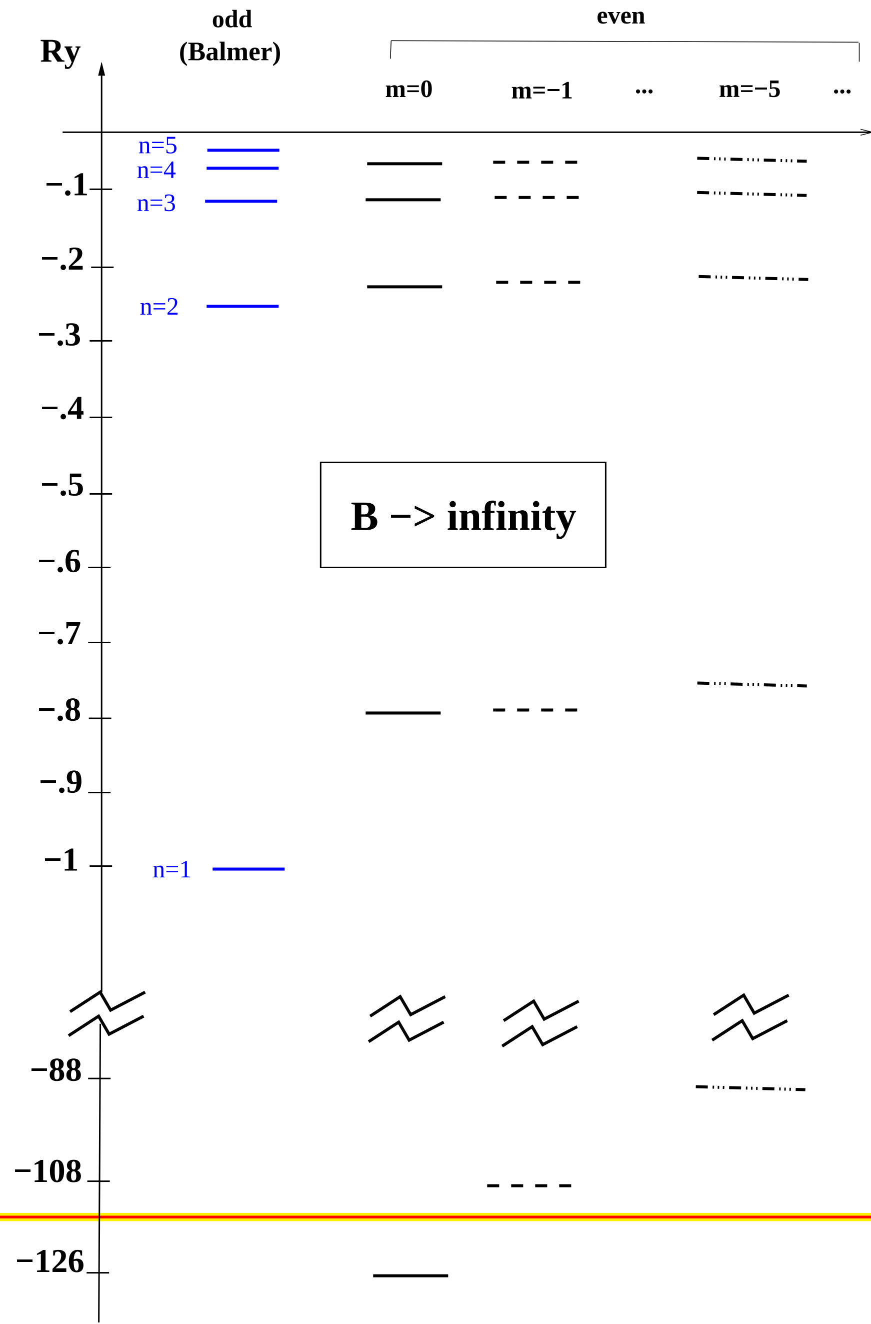
$$|E_0| = 2\mu \left[\int\limits_0^a U(x)dx\right]^2 \quad .$$

For

$$\mu|U|a^2 \ll 1$$

(condition for the potential to form a shallow well) we get that, indeed, $|E_0| \ll |U|$ and that the variation of χ inside the well is small, $\Delta\chi/\chi \sim \mu|U|a^2 \ll 1$.
Concerning the one-dimensional Coulomb potential, it satisfies this condition only for $a \ll 1/(m_e e^2) \equiv a_B$.
This explains why the accuracy of log^2 formula is very poor.

spectrum


Ry
odd
(Balmer)
even
m=0
m=−1
...
m=−5
...
n=5
n=4
n=3
n=2
n=1
−.1
−.2
−.3
−.4
−.5
−.6
−.7
−.8
−.9
−1
−88
−108
−126
B −> infinity

B values

$B > m_e^2 e^3 = 2.4 * 10^9$Gauss - strong B,
$B > m_e^2/e^3 = 6 * 10^{15}$Gauss - superstrong B.

$B_{cr} = m_e^2/e = 4.4 * 10^{13}$Gauss - critical B

B in laboratories:
$10^6 - 10^7$Gauss - magnetic cumulation, A.D.Saharov, 1952,
$H * r^2 =$const

Pulsars: $B \sim 10^{13}$Gauss; Magnetars: $B \sim 10^{15}$Gauss

Elliott, Loudon: excitons in semiconductors,
$m^* \ll m_e, \;\; e^* << e \;\; B > 2000$ Gauss - strong B

superstrong B - graphene: $m << m_e, \;\; \alpha \sim 1$???

References

Shabad, Usov (2007,2008): $D=4$ screening of Coulomb potential, freezing of the energy of ground state for $B>>m^2/e^3$ - numerical calculations;
Batalin, Shabad (1971): Π at $B>B_{cr}$ calculation;
Skobelev(1975), Loskutov, Skobelev(1976): linear in B term and $D=4 \Longrightarrow D=2$ correspondence in photon polarization operator for $B>m^2/e$;
Loskutov, Skobelev(1983); Kuznetsov, Mikheev, Osipov (2002): in $B>>m^2/e^3$ photon “mass” emerge;
Loudon(1959), Elliott, Loudon(1960) - atomic energies in strong $B>m^2e^3$ - numerical calculations;
Karnakov, Popov(2003) - analytical formulas for atomic energies in strong $B>m^2e^3$;
Vysotsky; Machet, Vysotsky(2010) - analytical formulas for Coulomb potential screening and LLL spectrum.