A Few Lessons from QCD perturbative Analysis at Low Energies

[Divergent Series, Summation, Practical Alternatives]

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Power Series with Factorial Coefficients

Formal Divergent series

$F(\alpha_s) \sim \sum_n n!\,(\alpha_s)^n$

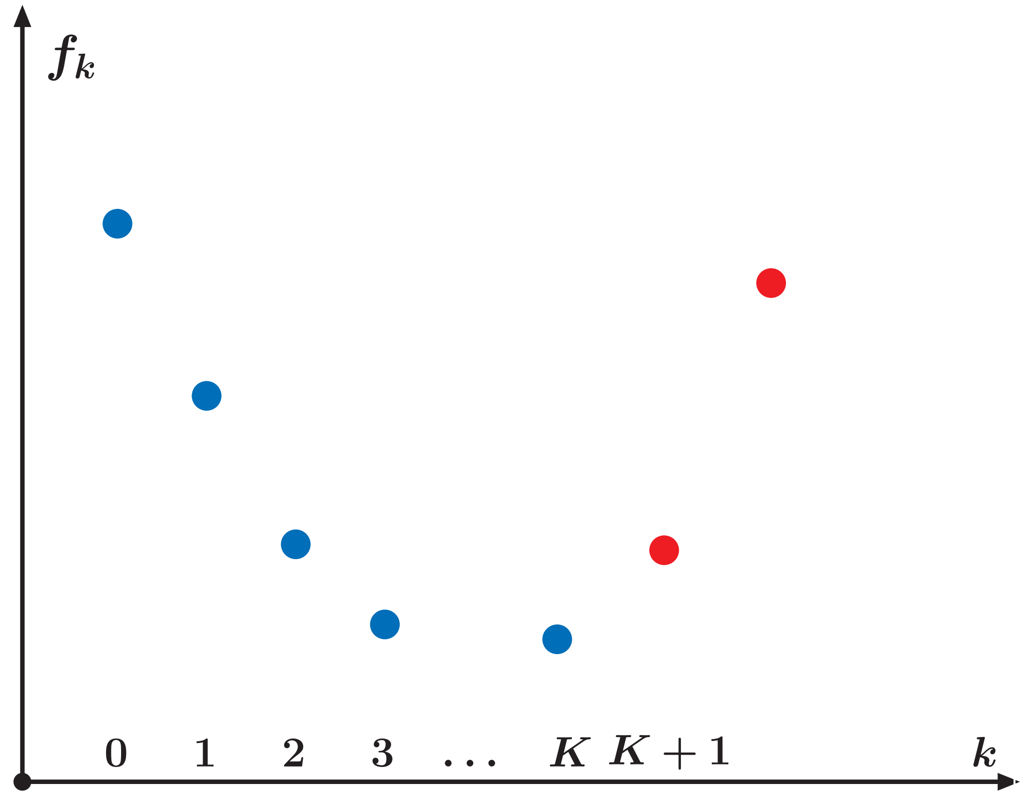
Finite Sum

$F_K(\alpha_s) = \sum_n^K f_n;$

$f_n = n!\,(\alpha_s)^n$

Poincaré estimate

$\Delta F(\alpha_s) \sim f_K$

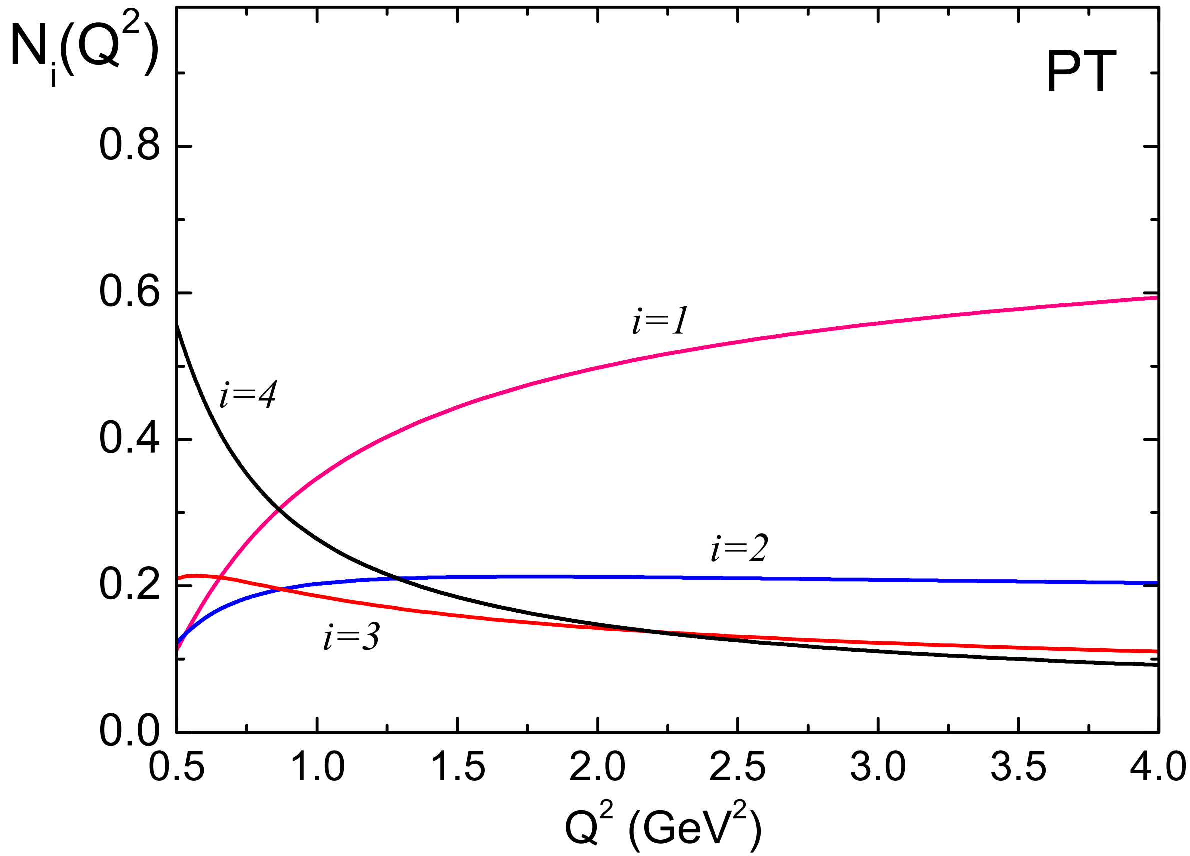


Optimal number of terms $K_* \sim 1/\alpha_s$ for numerical estimation with lower limit of possible accuracy, $\mathbf{f}_{\mathbf{K}_*}$

4-loop Evidence from the Bjorken Sum Rule

of the PT series "blow up" at $Q^2 \lesssim 2 - 3\mathrm{GeV}^2$ from

[O.Teryaev, Khandramaj, Pasechnik, Solovtsova, D.Sh (2011)]



Relative weight of 1-, 2-, 3-, 4-loop terms.

Asympt.Series (AS) born by Essential Singularity $e^{-1/g}$

The singularity is usual in Theory of Big Systems (representable via Functional or Path Integral) :

- Turbulence
- Classic and Quantum Statistics
- Quantum Fields

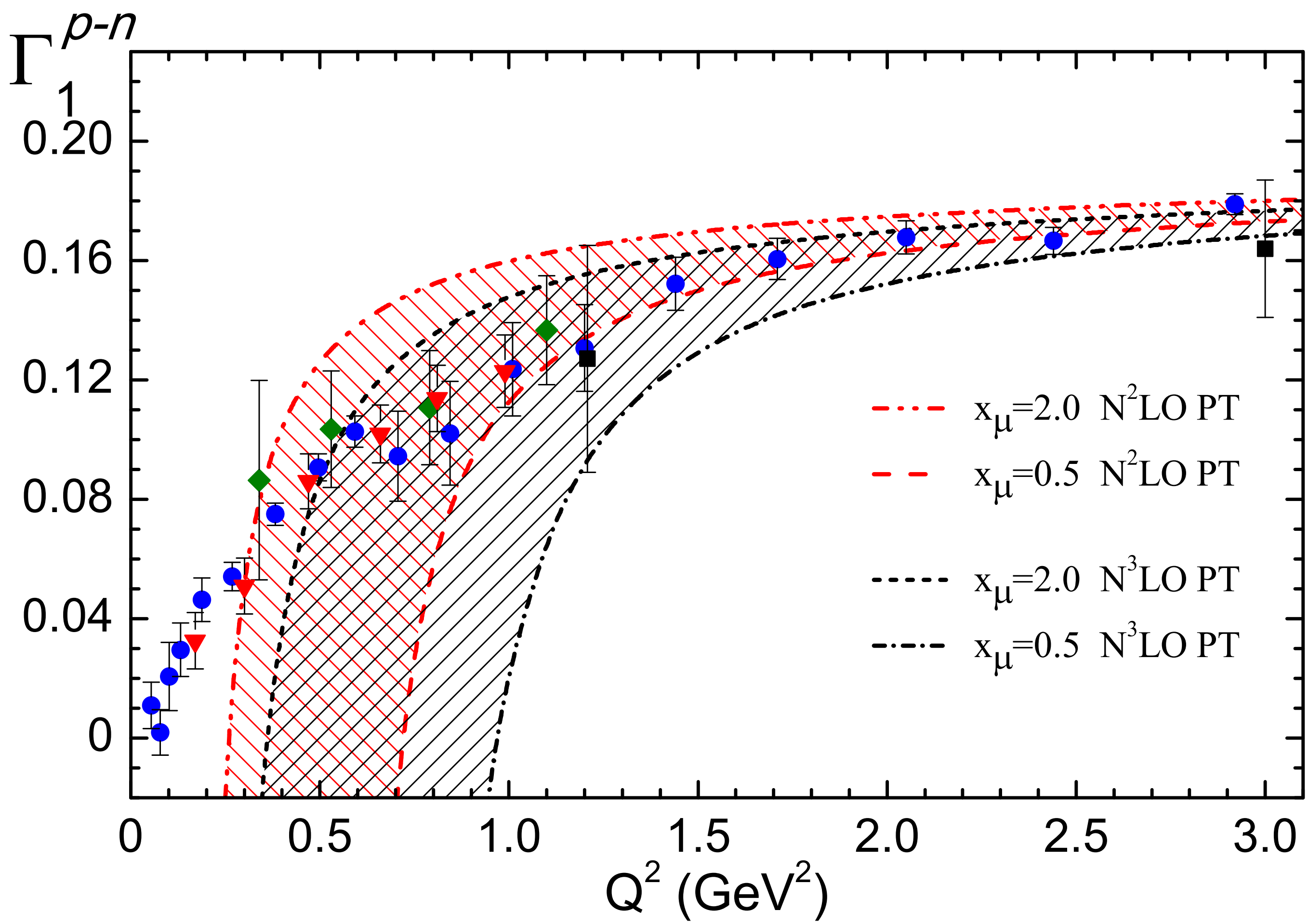
Reason : small parameter $g << 1$ at nonlinear structure

- Energy Gap in SuperFluidity and SuperConductivity
- Tunneling in QM
- Quantum Fields (Dyson singularity), ...

Generally, a certain AsymptSeries can correspond to set of various functions.

. Their "summation" is an Art.

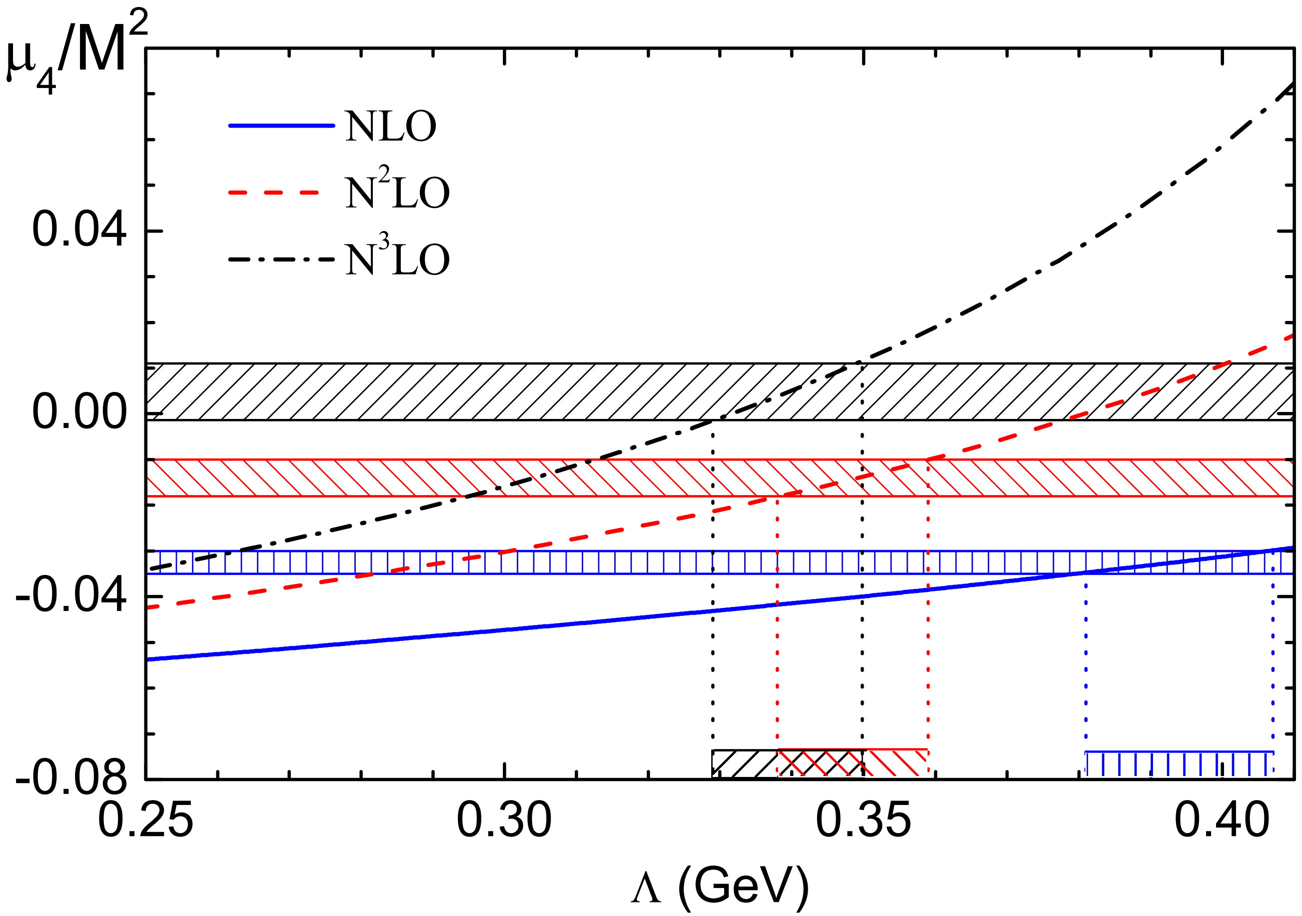
3- and 4-loop pQCD for Bjorken SumRule



Description of JLab data for the 1st moment Γ_1^{p-n}

4-loop fit is slightly worse than the 3-loop

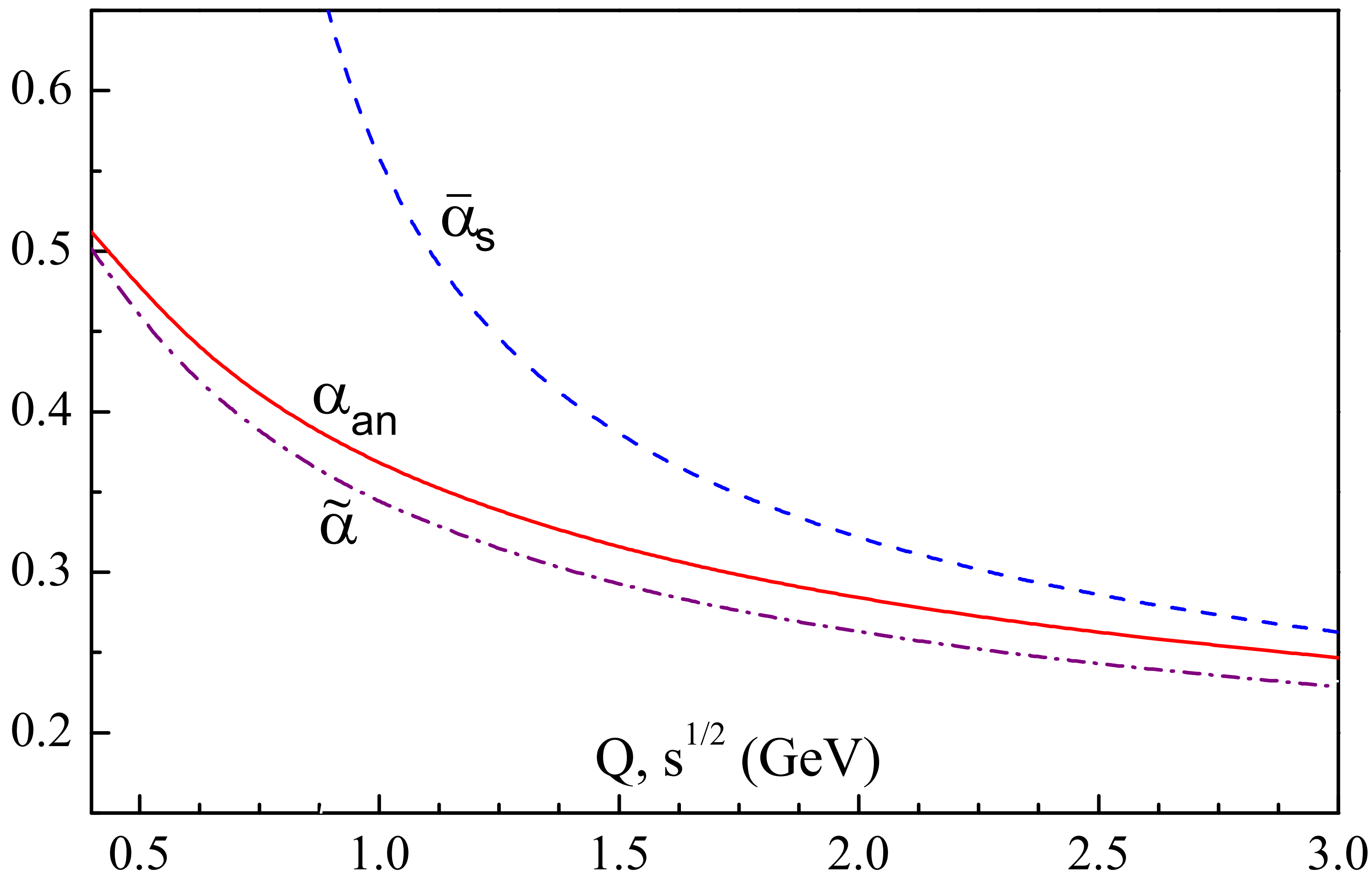
Extracting Λ_{QCD} from Bjorken SR



Extracting Λ_{QCD} from 3- and 4-loop fits for JLab data

Again no profit from the 4-loop fit !

Comparing APT couplings with $\alpha_s(Q^2)$



Red line $\alpha_{APT}(Q)$, black dash-dotted - $\tilde{\alpha}_{APT}(\sqrt{s})$

Few Words about APT

"Analytic Perturbation Theory"(APT) in QCD, the closed theor. scheme [Solovtsov+Sh-90s] without Landau-poles and additional parameters. It stems from imperative of Q^2-analyticity and compatibility with linear integral (like, Fourrier) transformations. Incorporates e^{-1/α_s} (algebraic in Q^2) structures.

Instead of power PT set $\bar{\alpha}_s(Q^2), \bar{\alpha}_s(Q^2)^2$, $\bar{\alpha}_s(Q^2)^3, \ldots$ one has

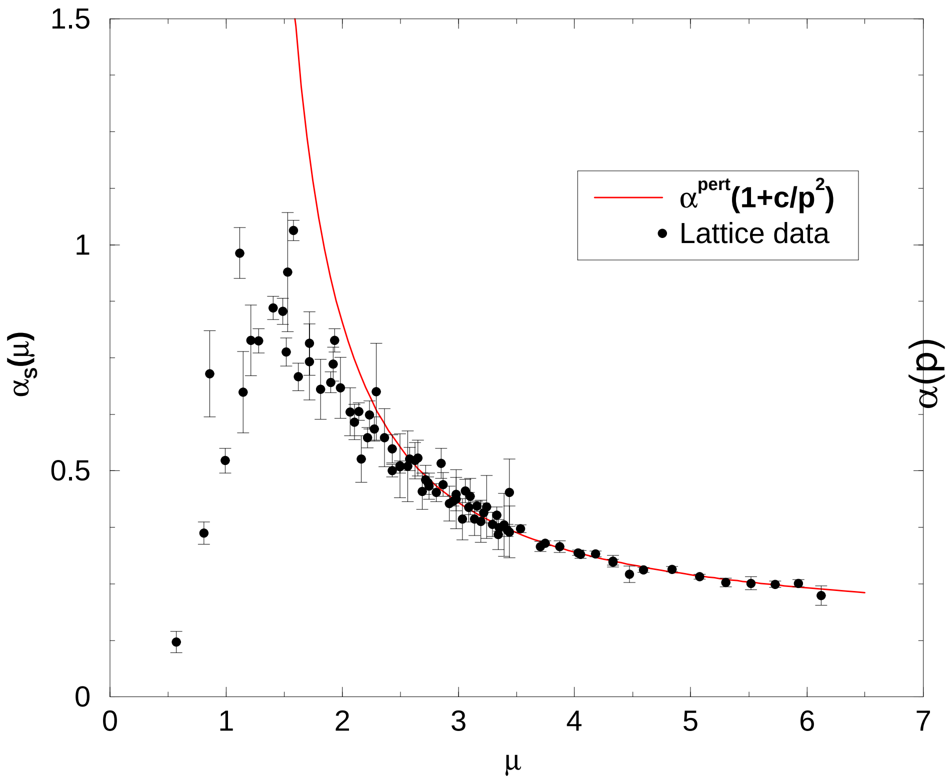
non-power APT expansion set $\{\mathcal{A}_k(Q^2)\}k = 1, 2, \ldots$

with all $\mathcal{A}_k(Q^2)$ regular in the IR

$\mathcal{A}_1(Q)$ **quantitatively corresponds to Lattice Simulation results down to $Q \sim 500\mathrm{MeV}$ [+Slide with lattice results]**

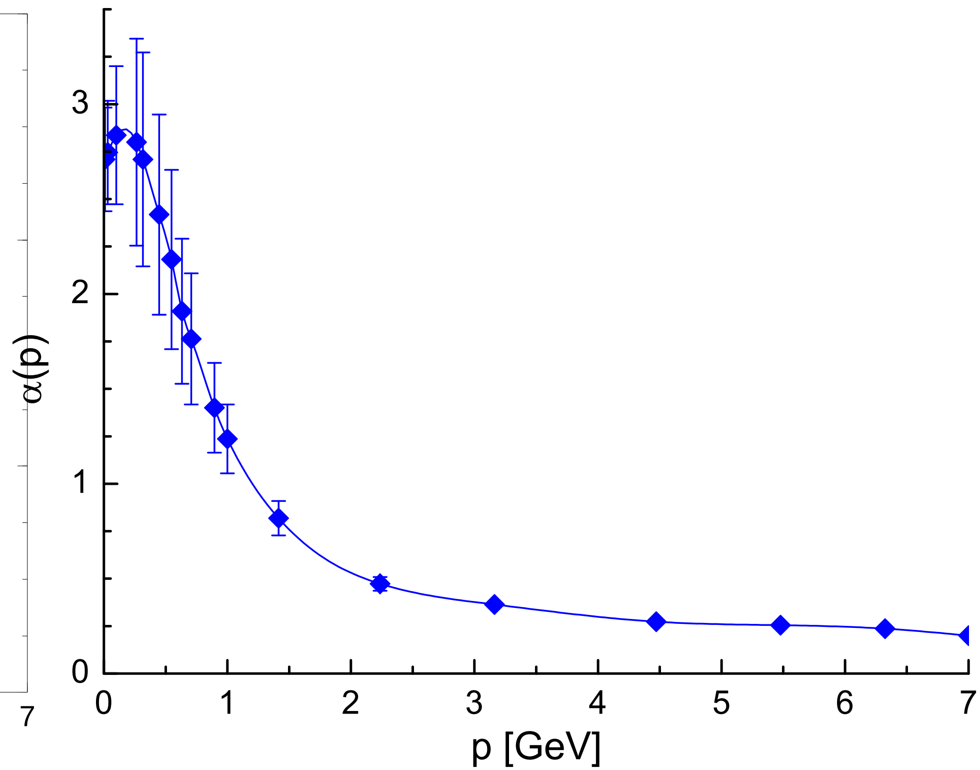
Lattice evidences for smooth α_s^{latt} behavior below 1 GeV

Asymptotic fit of α_s



α_s **based on Three-gluon vertex**

[P. Boucaud *et al.*,JHEP **0201**, 046 (2002)]

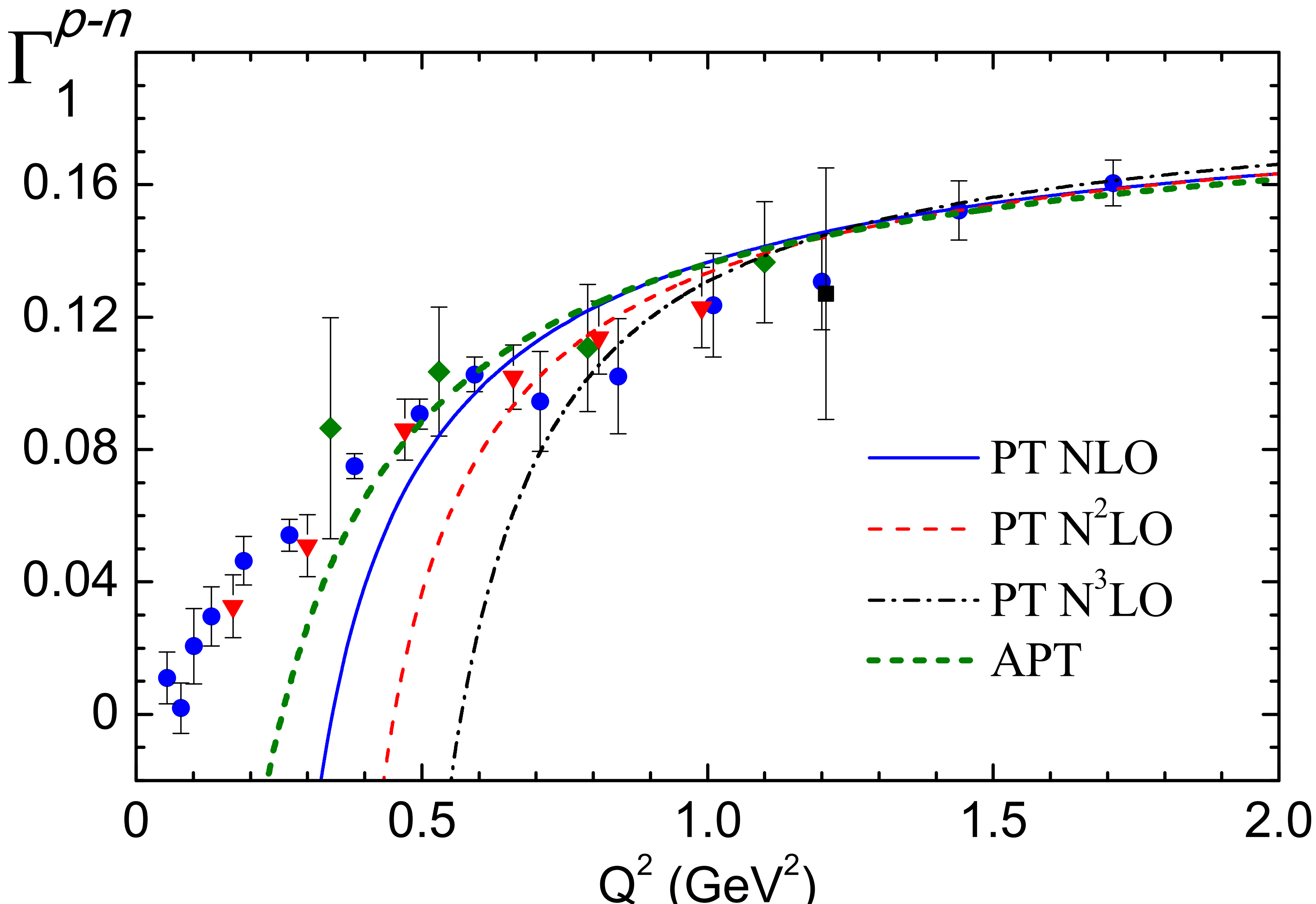


α_s **based on ghost-gluon vertex**

[from C.S. Fischer and R. Alkofer, Phys. Lett. B **536**, 177 (2002)]

Note the various IR behavior !!

The JLab-data Description by PT and by APT+HT



Anti-progress as 2– $\to 3- \to$ **4-loop below** $Q < 1\mathrm{GeV}$

Higher Twists for PT and APT+HT

Table 1: HT extraction from JLab data on BSR in PT – **uncertain ?**

PT	Q^2_{min},	μ_4/M^2	μ_6/M^4	μ_8/M^6
NLO	0.5	$-0.028(5)$	$-$	$-$
N^2LO	0.66	$-0.014(7)$	$-$	$-$
N^3LO	0.66	$0.005(9)$	$-$	$-$

Table 2: HT extraction from JLab data in APT – stable !.

APT	Q^2_{min}, GeV^2	μ_4/M^2	μ_6/M^4	μ_8/M^6
NLO	0.078	$-0.061(4)$	$0.009(1)$	$-0.0004(1)$
N^2LO	0.078	$-0.061(4)$	$0.009(1)$	$-0.0004(1)$
N^3LO	0.078	$-0.061(4)$	$0.009(1)$	$-0.0004(1)$

On the $[Q^2\, exp 1/\alpha_s]$ structure

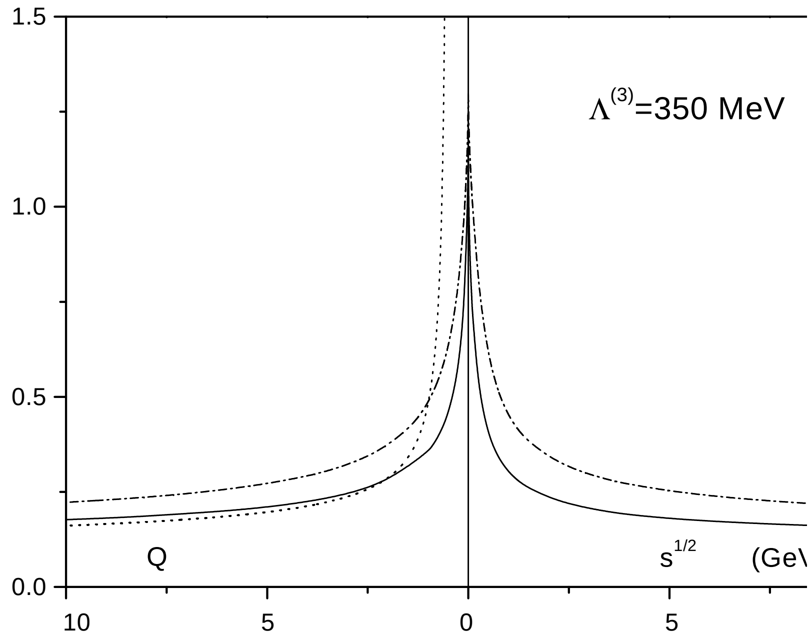
RG-invariance reduces the number of independent arguments

$$f(Q^2, \alpha_s) \to F_{RGinv}(\frac{1}{\alpha_s} + \beta_0 \ln\left(\frac{Q^2}{\mu^2}\right)) = \tilde{F}_{RG}\left(\frac{Q^2}{\mu^2}\, e^{1/\alpha_s}\right);$$

together with Q^2 analyticity yields one more statement on

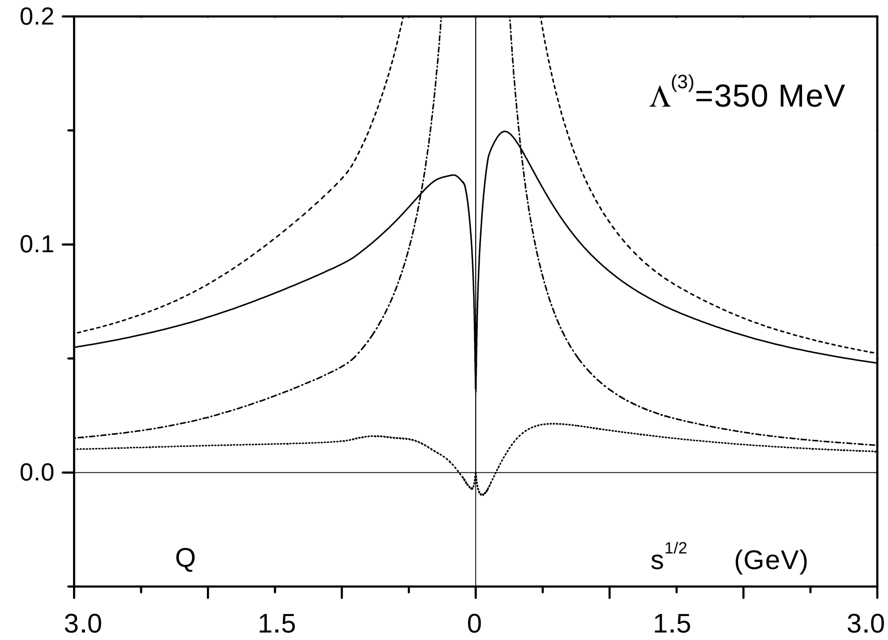
inevitable not-perturb nature $\sim e^{-\dfrac{1}{\alpha_s}}$ of all algebraic -in Q^2- structures, like HT terms (and singularity-subtraction terms in APT)

Q- and s-dependence of APT functions



Loop dependence of $\alpha_{APT}(Q)$ and $\tilde{\alpha}_{APT}(s$

[2- and 3-loops are very close each other]



Higher APT expansion functions

[All higher APT functions vanishes at IR limit]

Relative contributions (in %) of 1– , 2–, 3– and 4–loop terms

Process		Scale/Gev	*PT(in %)*				*APT+HT* *		
Bjorken SR	t	1	35	20	**19**	**26**	80	19	1
Bjorken SR	t	1.78	56	21	13	**11**	80	19	1
GLS SumRule	t	1.78	65	24	**11**		75	21	4
Incl. τ-decay	s	1.78	51	27	14	**7**	88	11	1

* The 4-loop APT contributions are negligible everywhere

Higher PT terms for $e^+e^- \to$ hadrons

Relative contributions of 1- ... 4–loop terms in $e^+e^- \to$ hadrons

Function	Scale/Gev	*PT terms (in %)*				Comment
r(s)	1	65	19	55	-39	**?!?**
r(s)	1.78	73	13	24	-10	**? ! ?**
d(Q)	1	56	17	11	16	
d(Q)	1.78	75	14	6	5	

In the $r(s)$ higher PT coefficients –

— terrible effect of the π^2 terms

Outlook and Appeal

I. Invitation for Work

- Methods of summation, including integral and conformal tricks,
- Devising Generating Function for HT terms
- Either generalizing minimal APT

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II. Appeal for Speculating

- Toy models for the 4-loop term predicting for a process P_i
- Set of couplings $\alpha_s^i(Q^2)$ each adequate to the given process P_i ?
- Generating HT function for the each P_i ?

Theoretical prediction of higher coefficients

Nice old example: The $g\,\phi^4$ beta-function was known up to the N^3LO term $\beta_{\overline{\rm MS}} = \frac{3}{2}\,g^2 - \frac{17}{6}g^3 + 16.27\,g^4 - 135.8\,g^5$

The Kazakov-Sh.-80 "summed" expression by Conform-Borel method

$$\beta^{CB}_{\overline{\rm MS}}(g) = \int_0^\infty \frac{dx}{g} e^{-x/g} \left(\frac{d}{dx}\right)^5 B(x) \quad \text{with} \tag{1}$$

$$B(x) = a\,x^2(1 - b_2 w + \cdots - b_4 w^3); \quad w(x) - \text{conform variable}$$

contains N^4LO term $\beta_6^{CB} = 1409.6\,.$

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(with Dmitrii Kazakov participation) $\beta_6 = 1420.6$ via Feynman diagrams .

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Congratulations just on the Dmitrii 60-birthday