Supersymmetric QCD and lower dimensions

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Supersymmetry

Supersymmetric gauge theories in four dimensions:
$\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$?

- $\mathcal{N} = 1$ is "realistic" ... roughly - almost nothing is known, physically - counting of vacua (determined by F-terms;

- $\mathcal{N} = 4$ - "everything" is known, called even "exactly soluble", but ... lack of physical questions in *conformal* theories, frozen coupling.
  Certainly - nice as a theory in UV;

- $\mathcal{N} = 2$ - "happy medium": still a little is known - but these are important answers to certain questions about quantum dynamics at strong coupling ...

- Couplings - holomorphic functions of vacuum condensates, not frozen, but fixed by complex geometry.

A partial reason for this - "multiple" relations with low-dimensional (holomorphic) theories.

$\mathcal{N} = 2$ **supersymmetric QCD**:

- Adjoint vector multiplet: $A = A_\mu dx^\mu$, Φ; $(\lambda_\alpha, \psi_\alpha)$, all matrices $N \times N$, $N = \#$ of colours. Gauge field necessarily requires adjoint (complex!) scalar;

- Fundamental matter (scalar quarks): Q^f, $\tilde{Q}_f$, $(q^f_\alpha, \tilde{q}^\alpha_f)$ with masses m_f; $f = 1, \ldots, N_f = \#$ of flavors

Scalars can condense:
If $\langle \Phi \rangle \neq 0$: the Coulomb or Abelian gauge theory in IR;
If $\langle \tilde{Q} Q \rangle \neq 0$: gauge group is (totally) "Higgsed", as in superconductor.

Coulomb phase:
$\langle\Phi\rangle = \text{diag}(a_1, \ldots, a_N)$, $[A, \Phi]_{ij} = A_{ij}(\phi_i - \phi_j)$, $U(N) \to U(1)^N$.

Scalars cause monopole (and string) solutions: there are the (BPS) monopoles in the spectra:

$$M_{BPS} \sim |Z_{\boldsymbol{\gamma}}(a)| \tag{1}$$

complex BPS masses, given by the central charges of $\mathcal{N} = 2$ SUSY algebra, $\boldsymbol{\gamma}$ - an element of charge's lattice.

Dirac quantization - symplectic pairing on BPS charges

$$\boldsymbol{\gamma} = (\mathbf{n}, \mathbf{m}), \quad \boldsymbol{\gamma} \circ \boldsymbol{\gamma}' = \mathbf{n} \cdot \mathbf{m}' - \mathbf{m} \cdot \mathbf{n}' = -\boldsymbol{\gamma}' \circ \boldsymbol{\gamma} \tag{2}$$

measures non-locality. $(\mathbf{n}, \mathbf{m})$ - electric and magnetic charges (w.r.t. many $U(1)$ factors)

Nontrivial IR dynamics: $U(N)$ SQCD in UV

$$\mathcal{L}_{\mathrm{UV}} = \mathrm{Tr}\left(\frac{1}{4g_0^2}\mathrm{F}_{\mu\nu}^2 - i\frac{\theta_0}{8\pi^2}F\wedge F\ldots\right) \tag{3}$$

flows to the IR Abelian effective theory ($i,j = 1,\ldots,N$):

$$\mathcal{L}_{\mathrm{IR}} \sim \mathrm{Im}\ T_{ij}(a)\ F_{\mu\nu}^i F_{\mu\nu}^j + \ldots \tag{4}$$

with holomorphic

$$T_{ij}(a) \underset{\text{weak coupling}}{\longrightarrow} \frac{i\beta}{4\pi}\log\frac{a_i - a_j}{\Lambda} + O\left(\left(\frac{\Lambda}{a}\right)^{\beta}\right)$$

$\beta = 2N_c - N_f$ is 1-loop (perturbatively exact) beta function, corrected by (!?) instantons.

$\mathcal{N} = 2$ SUSY: the holomorphic prepotential $T_{jk} = \frac{\partial^2 \mathcal{F}}{\partial a_j \partial a_k}$

$$F_{UV} = \frac{1}{2}\tau_0 \sum_i a_i^2 \;\rightarrow\; F_{IR} = \mathcal{F}(a) \stackrel{?}{=} \mathcal{F}_{\text{UV}} + \mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{inst}} \quad (5)$$

where $\tau_0 = \frac{\theta_0}{2\pi} + \frac{4\pi i}{g_0^2}$ and $T_{jk}(a) = \frac{\theta_{jk}(a)}{2\pi} + \frac{4\pi i}{g_{jk}(a)^2}$, QFT gives

$$\frac{1}{g_{jk}(a)^2} \sim \beta \log \frac{|a_j - a_k|}{\Lambda} + \ldots \quad (6)$$

and the perturbative formula *must* be corrected, when $M_W = |a_i - a_j| \lesssim \Lambda$.

Advanced QFT:
Already the relation $T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$ can be called an *integrable system*, an equation in *special* variables - special geometry.

Special variables $\mathbf{a} = (a_1, \ldots, a_{N-1})$ from the BPS mass formula:

$$
\begin{gathered}
Z_{\boldsymbol{\gamma}} = \oint_{\boldsymbol{\gamma}} z \frac{dw}{w} = \mathbf{na} + \mathbf{ma}_D (+B_f m_f) \\
\boldsymbol{\gamma} = \mathbf{nA} + \mathbf{mB}, \quad A_i \circ B_j = -B_j \circ A_i = \delta_{ij} \\
\Sigma : \ \Lambda^N \left(w + \frac{1}{w} \right) = \langle \mathrm{det}(z - \Phi) \rangle
\end{gathered}
\tag{7}
$$

Dirac pairing of charges $\gamma_i \circ \gamma_j$: intersection form of the cycles on Riemann surface Σ, charges are measured by cycles

$$
\boldsymbol{\mu}_i \cdot \boldsymbol{\alpha}_j = \delta_{ij} = A_i \circ B_j \tag{8}
$$

though in physical convention $|\boldsymbol{\mu}| \neq |\boldsymbol{\alpha}|$.

In geometric normalization - *no fractional* charges!

An *integrable system*: period matrix

$$a_i = \oint_{A_i} z \frac{dw}{w}, \quad a_i^D = \oint_{B_i} z \frac{dw}{w}$$
$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i}, \quad T_{ij} = \frac{\partial a_i^D}{\partial a_j} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} \tag{9}$$

integrability condition, and no problems with the positivity

$$\text{Im } T_{ij}(a) \geq 0,$$
$$T_{ij}(a) \underset{|a| \gg \Lambda}{=} \frac{i\beta}{4\pi} \log \frac{a_i - a_j}{\Lambda} + O\left(\left(\frac{\Lambda}{a}\right)^{\beta}\right) \tag{10}$$

with

$$\beta = 2N - N_f \underset{\text{pure gauge}}{=} 2N \tag{11}$$

for $U(N) \to U(1)^N$ effective gauge theory.

How to compute the non-perturbative (instanton) corrections to the perturbative logarithm?

- *No* honest way in four-dimensional theory (divergent integrals, how to fix?);

- Nekrasov functions: integrals over moduli spaces of instantons after two-parametric ϵ-deformation (IR regularization)

$$\mathcal{F}_{\text{inst}}(a) = \epsilon_1 \epsilon_2 \log Z_{\text{inst}}(a; \epsilon_{1,2})\Big|_{\epsilon_{1,2}=0} \tag{12}$$

- A way from two-dimensions: representation theory of (extended) Virasoro algebra.

4d $\to$ 2d:

- From two-dimensional conformal theory: conformal blocks and/or "coherent" or Whittaker states;

- Conformal blocks (two-dimensions) for conformal (four-dimensional) theory, coherent states - for the pure gauge.

Example $N = 1$: $[\alpha, \alpha^\dagger] = 1$, $t = \log \Lambda^2$, $\epsilon_1 = -\epsilon_2 = 1$, introduce

$$\begin{gathered} \alpha|0\rangle = 0, \quad \mathcal{H} = \bigoplus_{n\geq 0} \frac{(\alpha^\dagger)^n}{\sqrt{n!}}|0\rangle \\ |\Psi\rangle \in \mathcal{H}: \quad \alpha|\Psi\rangle = |\Psi\rangle \end{gathered} \tag{13}$$

Then

$$\begin{aligned} Z_{\text{inst}} &= \langle \Psi | e^{t\alpha^\dagger \alpha} | \Psi \rangle = e^{e^t} = e^{\mathcal{F}_{\text{inst}}} \\ \mathcal{F}(a,t) &= F_{UV} + \mathcal{F}_{\text{inst}} = \tfrac{1}{2} a^2 t + e^t \end{aligned} \tag{14}$$

no perturbative corrections - no nontrivial flows in $U(1)$ gauge theory.

- Summation over instantons in 4d from matrix elements of coherent states (towards 2d bosons/fermions, 2d CFT);

- An integrable system: $\frac{\partial^2 \mathcal{F}}{\partial t^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2}$ - the famous Toda equation (exponential potential).

Free two-dimensional scalar field (massless $\equiv$ conformal) with the holomorphic spin-1 current

$$J(w) = i\partial\phi(w) = \sum_{n\in\mathbb{Z}} \frac{J_n}{w^{n+1}}, \quad [J_n, J_m] = n\delta_{n+m,0} \tag{15}$$

and holomorphic spin-2 stress tensor (Virasoro algebra with $c = 1$)

$$\begin{gathered} T(w) = -\tfrac{1}{2}(\partial\phi)^2 = \sum_{n\in\mathbb{Z}} \frac{L_n}{w^{n+2}} \\ [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \end{gathered} \tag{16}$$

Then ($|0_a\rangle : \; J_0|0_a\rangle = a|0_a\rangle$)

$$\begin{gathered} |\Psi\rangle = e^{\alpha^\dagger}|0_a\rangle = e^{J_{-1}}|0_a\rangle, \quad J_1|\Psi\rangle = |\Psi\rangle \\ Z = \langle\Psi|\Lambda^{2L_0}|\Psi\rangle = e^{ta^2/2}\langle\Psi|e^{t\alpha^\dagger\alpha}|\Psi\rangle = e^{\mathcal{F}} \end{gathered} \tag{17}$$

Pure $U(N)$ gauge theory: coherent state for (spin $K = 1, 2, \ldots, N$ extended Virasoro $\mathcal{W}^{(K)}(z) = \sum_{n\in\mathbb{Z}} \frac{\mathcal{W}_n^{(K)}}{w^{n+K}}$) W_N-algebra:

$$\begin{gathered} \mathcal{W}_1^{(N)}|\Psi\rangle = |\Psi\rangle \\ \mathcal{W}_n^{(N)}|\Psi\rangle = 0,\ n > 1, \qquad \mathcal{W}_n^{(K)}|\Psi\rangle = 0,\ n > 0,\ K < N \end{gathered} \tag{18}$$

in the representation with "vacuum"

$$\mathcal{W}_0^{(K)}|0_a\rangle \sim \sum_{j=1}^{N} a_j^K |0_a\rangle, \quad K = 1, \ldots, N \tag{19}$$

Integrable system: the curve Σ is ($D|\Psi\rangle = \frac{z}{w}|\Psi\rangle$)

$$\begin{gathered} \langle\Psi|\mathcal{D}_N|\Psi\rangle = 0 \\ \mathcal{D}_N \equiv D^N - T(w)D^{N-2} - \ldots - \mathcal{W}^{(N)}(w) \end{gathered} \tag{20}$$

One gets the computation of the instanton expansion in four-dimensional theory from two dimensions:

$$Z = \langle \Psi | \Lambda^{2NL_0} | \Psi \rangle \tag{21}$$

e.g. for pure $U(2)$ gauge theory ($\Delta \sim a^2$)

$$\begin{gathered} L_1|\Psi\rangle = |\Psi\rangle, \qquad L_n|\Psi\rangle = 0, \quad n \geq 2 \\ |\Psi\rangle = \sum_{k\geq 0} \sum_{|Y|=k} Q_\Delta^{-1}(1^k, Y)|a; Y\rangle \\ Z = Z_{\text{pert}} \langle \Psi | \Lambda^{4L_0} | \Psi \rangle = Z_{\text{pert}} \sum_{k\geq 0} \Lambda^{4k} Q_\Delta^{-1}(1^k, 1^k) \end{gathered} \tag{22}$$

i.e. the instanton function is expressed through the inverse scalar product $Q(\bullet, \bullet)$ in the Virasoro representation between the states $|1^k\rangle = L_{-1}^k |0_a\rangle = L_{-1}^k |a; \emptyset\rangle$.

Therefore, the contribution of the instanton charge k sector, e.g.

$$Z_{\text{inst}}^{(k)} = Q_{\Delta}^{-1}(1^k, 1^k) \tag{23}$$

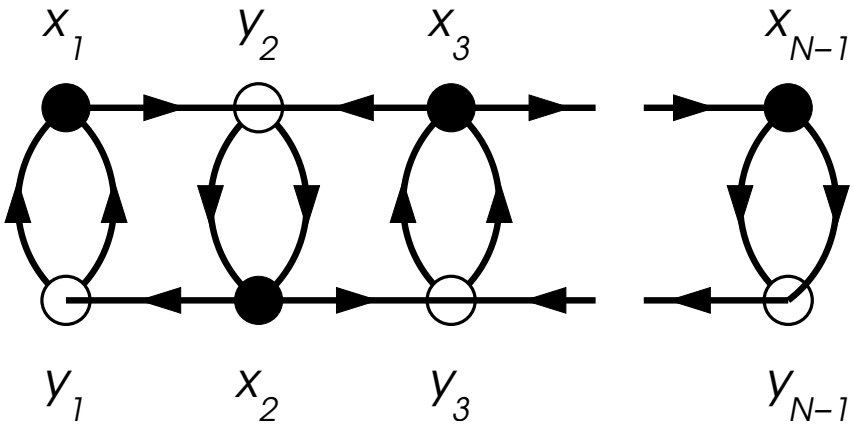
in $U(2)$ case - is totally expressed in terms of (the Virasoro) representation theory in two dimensions. Can be easily performed analytically on computer for any topological charge k ...

Hence, 4d $\longrightarrow$ 2d (holomorphic $\mathcal{N} = 2 \longrightarrow$ CFT) is quite effective ...

More: an *integrable system* 4d $\longrightarrow$ 1d - finite-dimensional integrable models - like Toda theories ...

4d $\to$ 1d

The Toda integrable systems, described by a *quiver*



$$x_k = \exp(-\boldsymbol{\alpha}_k \mathbf{q}), \quad y_k = \exp(\boldsymbol{\alpha}_k(\mathbf{P}+\mathbf{q}))$$
$$\mathbf{P} = \mathbf{p} + \frac{\partial}{\partial \mathbf{q}} \frac{1}{2} \sum_k \mathrm{Li}_2\left(-e^{\boldsymbol{\alpha}_k \mathbf{q}}\right) \tag{24}$$

quiver variables through di-Logarithm $\mathrm{Li}_2'(t) = \log(1+t)$.

Integrals of motion: "good functions" w.r.t. quiver *mutations* - rational transformations of variables preserving the Poisson (symplectic) structure.

The same mutations preserve the Dirac pairing $\gamma_i \circ \gamma_j$, and generate the "full" stable BPS spectrum of $\mathcal{N} = 2$ supersymetric gauge theory:

$$\tilde{\gamma}_i = -\gamma_i$$
$$\tilde{\gamma}_j = \gamma_j + \begin{cases} (\gamma_i \circ \gamma_j)\gamma_i, & \gamma_i \circ \gamma_j > 0 \\ 0, & \gamma_i \circ \gamma_j < 0 \end{cases}, \qquad j \neq i \tag{25}$$

At strong coupling there is always a chamber with finitely many stable BPS states!

E.g. $SU(2)$ pure SYM, at strong coupling the W-boson

$$\gamma = (2,0) = (2,-1) + (0,1) = \gamma_D + \gamma_m \tag{26}$$

is *not* stable, contrary to the weak coupling, separated by wall $\mathrm{Im}(a_D/a) = 0$.

Conclusions

- Exact results in 4d $\mathcal{N} = 2$ supersymmetric QCD are cause by the properties borrowed from low-dimensional theories;

- 2d is almost always easier than 4d, (and 1d - even easier than 2d). It may help more in the future...

Дима,

Будь здоров!