Instantons and NS–conformal field theory

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Alday, Gaiotto and Tachikawa (2009) proposed correspondence

$\mathcal{N}=2$ SUSY $4d$ Gauge theories $\longleftrightarrow$ $2d$ Conformal field theories

$$\sum_{Y,Y'} q^{|Y|+|Y'|} Z_{Y,Y'} = (1-q)^{a_1 a_3} F_\Delta(\Delta_1,\Delta_2,\Delta_3,\Delta_4|q),$$

$\sum_{Y,Y'} q^{|Y|+|Y'|} Z_{Y,Y'}$ — Instanton partition function.

$F_\Delta(\Delta_1,\Delta_2,\Delta_3,\Delta_4|q)$ — Conformal block of Liouville field theory.

Virasoro algebra.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}c\delta_{n,-m}$$

Verma module generated by $|\Delta\rangle$ such that

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_k|\Delta\rangle = 0, \quad \text{for}\, k>0$$

The chain of vectors $|N\rangle_{\Delta_1,\Delta_2,\Delta}$ defined as

$$L_0|N\rangle_{\Delta_1,\Delta_2,\Delta} = (\Delta+N)|N\rangle_{\Delta_1,\Delta_2,\Delta},$$
$$L_k|N\rangle_{\Delta_1,\Delta_2,\Delta} = (\Delta+k\Delta_1-\Delta_2+k)|N-k\rangle_{\Delta_1,\Delta_2,\Delta}, \quad k>0$$

conformal block:

$$F_\Delta(\Delta_1,\Delta_2,\Delta_3,\Delta_4|q) = \sum_N {}_{\Delta_1,\Delta_2,\Delta}\langle N|N\rangle_{\Delta_3,\Delta_4,\Delta}\, q^N$$

ADHM date consist of $N \times N$ matrices B_1, B_2, a $N \times 2$ matrix I and a $2 \times N$ matrix J, which are subject of the following set of conditions:

$$[B_1, B_2] + IJ = 0,$$

The solutions related by $GL(N)$ transformations

$$B_i' = gB_ig^{-1}, \;\; I' = gI, \;\; J' = Jg^{-1}; \;\;\; g \in GL(N)$$

are equivalent.
Among vectors obtained by the repeated action of B_1 and B_2 on $I_{1,2}$ (columns of the matrix I), there exist N linear independent, which span N-dimensional vector space V,a fiber of the N-dimensional fiber bundle, whose base is the moduli space $\mathcal{M}_N$ itself.
The construction of the instanton partition function involves the determinants of the vector field v on $\mathcal{M}_N$, defined by

$$B_l \to t_l B_l; \;\;\; I \to I t_v; \;\;\; J \to t_1 t_2 t_v^{-1} J,$$

where parameters $t_l \equiv \exp \epsilon_l \tau$, $l = 1, 2$ and $t_v = \exp a\sigma_3 \tau$.

Fixed points, labeled by pairs of Young diagrams (Y_1, Y_2) such that the total number of boxes $|Y_1| + |Y_2| = N$. To the cells $(i_1, j_1) \in Y_1$ and $(i_2, j_2) \in Y_2$ correspond vectors $B_1^{i_1} B_2^{j_1} I_1$ and $B_1^{i_2} B_2^{j_2} I_2$ respectively.

$$\det v = \frac{\prod\limits_{s,s'\in\vec{Y}}(\epsilon_1+\phi_{s'}-\phi_s)(\epsilon_2+\phi_{s'}-\phi_s)\prod\limits_{l=1,2;s\in\vec{Y}}(a_l-\phi_s)(\epsilon_1+\epsilon_2-a_l+\phi_s)}{\prod\limits_{s,s'\in\vec{Y}}(\phi_{s'}-\phi_s)(\epsilon_1+\epsilon_2-\phi_{s'}+\phi_s)}$$

$$\det v = \prod_{\alpha,\beta=1}^{2}\prod_{s\in Y_\alpha} E\big(a_\alpha - a_\beta, Y_\alpha, Y_\beta\big|s\big)\big(Q - E\big(a_\alpha - a_\beta, Y_\alpha, Y_\beta\big|s\big)\big),$$

here $E\big(a, Y_1, Y_2\big|s\big)$ are defined as follows

$$E\big(a, Y_1, Y_2\big|s\big) = a + \epsilon_1(L_{Y_1}(s) + 1) - \epsilon_2 A_{Y_2}(s)\ ,$$

where $A_Y(s)$ and $L_Y(s)$ are respectively the arm-length and the leg-length for a cell s in Y.

$$Z^{\mathsf{vec}}(\vec{a}, \vec{Y}) \equiv \frac{1}{\det v}$$

Z_{f} defined in terms of vector bundle V. The answer reads

$$Z_{\mathsf{f}}(\mu_i, \vec{a}, \vec{Y}) = \prod_{i=1}^{4} \prod_{\alpha=1}^{2} \prod_{s \in Y_\alpha} \big(\phi(a_\alpha, s) + \mu_i\big),$$

where

$$\phi(a_\alpha, s) = (i_s - 1)\epsilon_1 + (j_s - 1)\epsilon_2 + a_{p(s)} + a_i - a_\alpha.$$

$$Z_{\vec{Y}} = \frac{Z_{\mathsf{f}}(\mu_i, \vec{a}, \vec{Y})}{Z_{\mathsf{vec}}(\vec{a}, \vec{Y})}$$

Alba, Fateev, Litvinov, Tarnopolsky proved (and may be explained) AGT relation in terms of algebra $\mathcal{A} = \mathrm{Vir} \otimes \mathcal{B}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$
$$[a_n, a_m] = \frac{n}{2}\delta_{n+m,0}, \quad [a_n, L_m] = 0$$
$$c = 1 + 6Q^2, \quad Q = b + b^{-1}$$

There exists a unique orthogonal basis $|P\rangle_{\vec{Y}}$, such that

$$\frac{{}_{\vec{Y}'}\langle P'|V_\alpha(z=1)|P\rangle_{\vec{Y}}}{\langle P'|V_\alpha(z=1)|P\rangle} = \mathcal{F}_{\vec{Y}}^{\vec{Y}'}(\alpha|P,P') = \mathcal{F}_{Y_1 Y_2}^{Y_1' Y_2'}(\alpha|P,P')$$

have a factorized form

$$\mathcal{F}_{\vec{Y}}^{\vec{Y}'}(\alpha|P,P') = \prod_{i,j=1}^{2} \prod_{s\in Y_i}(Q - E_{Y_i,Y_j'}(P_i - P_j'|s) - \alpha)\times$$
$$\times \prod_{t\in Y_j'}(E_{Y_j' Y_i}(P_j' - P_i|t) - \alpha)$$

in particular

$${}_{\vec{Y}'}\langle P|P\rangle_{\vec{Y}} = \delta_{\vec{Y},\vec{Y}'} = Z^{\mathsf{vec}}(\vec{P},\vec{Y})$$

The construction of NS-CFT generalization of AGT is based on the conjectural relation between moduli spaces of $SU(2)$ instantons on $\mathbb{R}^4/\mathbb{Z}_2$ and algebras like $\widehat{gl}(2)_2 \times \mathcal{NSR}$. This conjecture is confirmed by checking the coincidence of number of fixed points on such instanton moduli space with given instanton number N and dimension of subspace degree N in the representation of such algebra.

The subspace of the Moduli space $\mathcal{M}^{\mathbb{Z}_2}$ for $SU(2)$ gauge group is defined by the following additional restriction of $\mathbb{Z}_2$ symmetry

$$B_1 = -PB_1P^{-1}; B_2 = -PB_2P^{-1}; \qquad I = PI; \qquad J = JP^{-1}.$$

where $P \in GL(N)$ is some gauge transformation, obviously $P^2 = 1$. New manifold $\mathcal{M}_{\mathbb{Z}_2}$ is a disjoint union of connected components

$\mathcal{M}_{\mathbb{Z}_2}(N_+, N_-)$, where N_+ and N_- are integers which denote the dimensions of V_+ and V_- (*i.e.* even and odd subspaces of the fiber V) correspondingly, $N_+ + N_- = N$. These numbers are fixed inside given connected component of $\mathcal{M}^{\mathbb{Z}_2}$.

We need only two connected components $\mathcal{M}^{\mathbb{Z}_2}(N,N)$ and $\mathcal{M}^{\mathbb{Z}_2}(N,N-1)$. In this section we calculate the number of fixed points on such components and discuss the result from the $\widehat{gl}(2)_2 \times \mathcal{NSR}$ point of view.
We introduce the generating function

$$\chi(q) = \sum_N |\mathcal{M}^{\mathbb{Z}_2}(N,N)| q^N + \sum_N |\mathcal{M}^{\mathbb{Z}_2}(N,N-1)| q^{N-1/2},$$

where $|\mathcal{M}^{\mathbb{Z}_2}(N_+,N_-)|$ is a number of fixed points on $\mathcal{M}^{\mathbb{Z}_2}(N_+,N_-)$. This number equals to the number of pairs of Young diagrams with N_+ white boxes and N_- black boxes.

$$\chi(q) = \prod_{m\geq 0} \frac{(1+q^{2m+1})^2}{(1-q^{2m+2})^3} = \chi_B(q)^3 \chi_F(q)^2,$$

where

$$\chi_B(q) = \prod_{n\in\mathbb{Z},\, n>0} \frac{1}{(1-q^n)}$$

$$\chi_F(q) = \prod_{r\in\mathbb{Z}+\frac{1}{2},\, r>0} (1+q^r).$$

The $\chi_B(q)\chi_F(q)$ equals to the character of standard representation of the $\mathcal{NSR}$ algebra with generators L_n, G_r. The remaining part is related to the algebra $\mathcal{B} \times \mathcal{B} \times \mathcal{F}$ where $\mathcal{B}$ is the Heisenberg algebra with generators b_n and relations $[b_n, b_m] = n\delta_{n+m}$ and $\mathcal{F}$ is the Clifford algebra with generators f_r and relations $\{f_r, f_s\} = r\delta_{r+s}$.

Thus equation (9) means that the generating function of numbers of fixed points on components $\mathcal{M}^{\mathbb{Z}_2}(N, N)$ and $\mathcal{M}^{\mathbb{Z}_2}(N, N-1)$ equals to the character of representation of the algebra $\mathcal{A} = \mathcal{B} \times \mathcal{B} \times \mathcal{F} \times \mathcal{NSR}$.

This representation theory point of view can be exploit similar to AFLT.

One can consider the whole space $\mathcal{M}^{\mathbb{Z}_2}$. The generating function has the form

$$\chi(q) = \sum_N |\mathcal{M}^{\mathbb{Z}_2}(N)| q^{\frac{N}{2}} = \prod_{n\in\mathbb{Z},\, n>0} \frac{1}{\left(1-q^{\frac{n}{2}}\right)^2} \tag{1}$$

The result equals to the character of the certain representation of $\widehat{gl}(2)_2 \times \mathcal{NSR}$ namely the tensor product of Fock representation of Heisenberg algebra, vacuum representation of $\widehat{sl}(2)_2$ and NS representation of $\mathcal{NSR}$.
In other words the generating function of numbers of fixed points on $\mathcal{M}^{\mathbb{Z}_2}(N)$ equals to the character of representation of the algebra $\mathcal{A} = \widehat{gl}(2)_2 \times \mathcal{NSR}$.

The relation between $U(2)$ $N=2$ SYM on $\mathbb{R}^4/\mathbb{Z}_2$ and algebra $\widehat{gl}(2)_2 \times \mathcal{NSR}$ is a special case of the relation between $U(r)$ $N=2$ SYM on $\mathbb{R}^4/\mathbb{Z}_p$ and conformal field theory based on the coset $\mathcal{A}(r,p) = \widehat{\mathfrak{gl}(n)}_r / \widehat{\mathfrak{gl}(n-p)}_r$. Due to level-rank duality

$$A(r,p) = H \times \widehat{\mathfrak{sl}(p)}_r \times \frac{\widehat{\mathfrak{sl}(r)}_p \times \widehat{\mathfrak{sl}(r)}_{n-p}}{\widehat{\mathfrak{sl}(r)}_n}$$

3	$\dots$	$\dots$	$\dots$
2	$H \times \widehat{\mathfrak{sl}}(2)_1$	$H \times \widehat{\mathfrak{sl}(2)}_2 \times NSR$	$\dots$
1	H	$H \times Vir$	$H \times W_3$
p/r	1	2	3

$$\det{}'v = \frac{\prod\limits_{\substack{s,s'\in\vec{Y}\\ P(s)\neq P(s')}} (\epsilon_1+\phi_{s'}-\phi_s)(\epsilon_2+\phi_{s'}-\phi_s) \prod\limits_{\substack{\alpha=1,2;s\in\vec{Y}\\ P(s)=1}} (a_\alpha-\phi_s)(\epsilon_1+\epsilon_2-a_\alpha+\phi_s)}{\prod\limits_{\substack{s,s'\in\vec{Y}\\ P(s)=P(s')}} (\phi_{s'}-\phi_s)(\epsilon_1+\epsilon_2-\phi_{s'}+\phi_s)}$$

In terms of arm-length and leg-length this expression reads

$$\det{}'v = \prod_{\alpha,\beta=1}^{2} \prod_{s\in {}^{\diamondsuit}Y_\alpha(\beta)} E\big(a_\alpha-a_\beta,Y_\alpha,Y_\beta\big|s\big)\big(Q-E\big(a_\alpha-a_\beta,Y_\alpha,Y_\beta\big|s\big)\big),$$

The region ${}^{\diamondsuit}Y_\alpha(\beta)$ is defined as

$${}^{\diamondsuit}Y_\alpha(\beta) = \Big\{(i,j)\in Y_\alpha\Big|P\big(k'_j(Y_\alpha)\big)\neq P\big(k_i(Y_\beta)\big)\Big\},$$

or, the boxes having different parity of the leg- and arm-factors.

$$Z^{\mathbb{Z}_2}_{\mathsf{vec}}(\vec{a},\vec{Y}) \equiv \frac{1}{\det{}'v}$$

$$Z_{\mathsf{f}}^{(0)}(\mu_i,\vec{a},\vec{Y})=\prod_{i=1}^{4}\prod_{\alpha=1}^{2}\prod_{s\in Y_\alpha,s-\text{white}}\big(\phi(a_\alpha,s)+\mu_i\big),$$

$$Z_{\mathsf{f}}^{(1)}(\mu_i,\vec{a},\vec{Y})=\prod_{i=1}^{4}\prod_{\alpha=1}^{2}\prod_{s\in Y_\alpha,s-\text{black}}\big(\phi(a_\alpha,s)+\mu_i\big),$$

The first expression correspond to the case with even number of instantons ,the second one correspond to the case with odd number of instantons.

AGT type formula for the NS four-point conformal blocks :

$$\sum_{N=0,1,\dots} q^N \sum_{\vec{Y},\substack{N_+(\vec{Y})=N \\ N_-(\vec{Y})=N}} \frac{Z_{\mathsf{f}}^{(0)}(\mu_i,\vec{a},\vec{Y})}{Z_{\mathsf{vec}}^{\mathsf{sym}}(\vec{a},\vec{Y})} = (1-q)^A F_0(\Delta(\lambda_i)|\Delta(a)|q)$$

$$\sum_{N=\frac{1}{2},\frac{3}{2},\dots} q^N \sum_{\vec{Y},\substack{N_+(\vec{Y})=N+\frac{1}{2} \\ N_-(\vec{Y})=N-\frac{1}{2}}} \frac{Z_{\mathsf{f}}^{(1)}(\mu_i,\vec{a},\vec{Y})}{Z_{\mathsf{vec}}^{\mathsf{sym}}(\vec{a},\vec{Y})} = (1-q)^A F_1(\Delta(\lambda_i)|\Delta(a)|q) .$$

The formula is the main result of this talk. The parameters of the conformal block are related to those of the instanton partition function as

$$\mu_1 = \frac{Q}{2} - (\lambda_1+\lambda_2), \qquad \mu_2 = \frac{Q}{2} - (\lambda_1-\lambda_2),$$
$$\mu_3 = \frac{Q}{2} - (\lambda_3+\lambda_4), \qquad \mu_4 = \frac{Q}{2} - (\lambda_3-\lambda_4),$$

and

$$A = \left(\frac{Q}{2}-\lambda_1\right)\left(\frac{Q}{2}-\lambda_3\right).$$