Current Interactions of $4d$ Massless Fields as a Free System in Mixed Dimensions

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Gelfond, MV, arXiv:1012.3143

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Plan

Relativistic fields

$$s = \mathbf{0}, \mathbf{1/2}, \mathbf{1}, \mathbf{3/2}, \mathbf{2}, 5/2, 3 \ldots$$

$$m^2 \geq 0$$

$m = 0, s \geq 1$: **gauge fields**

Fronsdal theory

$\varphi_{n_1 \ldots n_s}$ - **rank s double traceless symmetric tensor**

Gauge transformation:

$$\delta\varphi_{k_1 \ldots k_s} = \partial_{(k_1}\varepsilon_{k_2 \ldots k_s)}\,, \qquad \varepsilon^m{}_{m k_3 \ldots k_{s-1}} = 0$$

$\varepsilon_{k_1 \ldots k_{s-1}}$ **is symmetric traceless** $\qquad \delta\varphi_n{}^n{}_m{}^m{}_{k_5 \ldots k_s} = 0$

HS gauge theory: theory of higher symmetries

Goal: Find a nonlinear HS theory such that

(i) Fronsdal (or Labastida) theory in the free field limit

(ii) HS gauge symmetries related to HS parameters $\varepsilon^{m_1 \dots m_{s-1}}$ deform to non-Abelian

Possibly alternative approach to String Theory as a massive HS theory with spontaneously broken HS symmetries

Nonlinear HS theories involve infinite towers of fields of all spins

Higher Derivatives in HS Interactions

A.Bengtsson, I.Bengtsson, Brink (1983), Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots$$

$$S^3 = \sum_{p,q,r} (D^p\varphi)(D^q\varphi)(D^r\varphi)\rho^{p+q+r+\frac{1}{2}d-3}$$

s derivatives in interactions.

String: $\rho \sim \sqrt{\alpha'}$

HS Gauge Theories ($m = 0$): Fradkin, M.V. (1987)

$$AdS_d : (X^0)^2 + (X^d)^2 - (X^1)^2 - \dots - (X^{d-1})^2 = \rho^2 , \qquad \rho = \lambda^{-1}$$

$$[D_n, D_m] \sim \rho^{-2} = \lambda^2$$

The $\rho \to \infty$ limit is ill-defined at the interaction level both in string theory and in HS gauge theory

AdS/CFT:

Flato, Fronsdal (1978); Klebanov, Polyakov (2002); Giombi, Yin (2009,2010)

Unfolded Dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

DOF = # of dynamical variables

Field theory: infinite number of DOF = spaces of functions

Maxwell $q \sim \vec{A}(x)$, $p \sim \vec{E}(x)$.

Covariant extension $t \to x^n$?

Unfolded dynamics: multidimensional generalization (1988)

$$\frac{\partial}{\partial t} \to d\,, \qquad q^i(t) \to W^\alpha(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W^\alpha_{n_1 \ldots n_p}(x)$$

a set of differential forms

Unfolded equations

$$dW^\alpha(x) = G^\alpha(W(x))\,, \qquad d = dx^n\partial_n$$

$G^\alpha(W)$: **function of "supercoordinates"** W^α

$$G^\alpha(W) = \sum_{n=1}^{\infty} f^\alpha{}_{\beta_1\ldots\beta_n} W^{\beta_1}\wedge\ldots\wedge W^{\beta_n}$$

Covariant first-order differential equations

$d>1$: **Nontrivial compatibility conditions:** $G^\beta(W)\wedge\frac{\partial G^\alpha(W)}{\partial W^\beta}=0$ **equivalent to the generalized Jacobi identities**

$$\sum_{n=0}^{m}(n+1)f^\gamma{}_{[\beta_1\ldots\beta_{m-n}}f^\alpha{}_{\gamma\beta_{m-n+1}\ldots\beta_m\}}=0$$

Any solution to generalized Jacobi identities: FDA **(Sullivan (1968))**

FDA **is universal if the generalized Jacobi identity holds for** W **interpreted as supercoordinates. HS FDAs are universal.**

Every universal FDA = some L_∞ **algebra**

Properties

- General applicability
- Coordinate independence

  Exterior algebra formalism
- Interactions: nonlinear deformation of $G^\alpha(W)$
- Degrees of freedom are in 0-forms $C^i(x_0)$ at any $x = x_0$ (as $q(t_0)$) instead of phase coordinates in the Hamiltonian approach
- Natural realization of infinite symmetries with higher derivatives
- Lie algebra cohomology interpretation
- Covariant twistor transform
- Dynamical origin of space-time

$4d$ Gravity

Gauge Fields

Vierbein one-form $e^{\alpha\alpha'}$ $(\alpha, \beta \ldots = 1, 2,\ \alpha', \beta' \ldots = 1, 2)$

Lorentz connection $\omega^{\alpha\beta}$, $\bar{\omega}^{\alpha'\beta'}$

Torsion

$$R_{\alpha\beta'} = de_{\alpha\beta'} + \omega_\alpha{}^\gamma \wedge e_{\gamma\beta'} + \bar{\omega}_{\beta'}{}^{\delta'} \wedge e_{\alpha\delta'}$$

Lorentz curvature

$$R_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_\alpha{}^\gamma \wedge \omega_{\beta\gamma} + \lambda^2\, e_\alpha{}^{\delta'} \wedge e_{\beta\delta'}$$

$$\overline{R}_{\alpha'\beta'} = d\overline{\omega}_{\alpha'\beta'} + \overline{\omega}_{\alpha'}{}^{\gamma'} \wedge \overline{\omega}_{\beta'\gamma'} + \lambda^2\, e^\gamma{}_{\alpha'} \wedge e_{\gamma\beta'}$$

AdS_4

$$R_{\alpha\beta} = 0\,, \quad \overline{R}_{\alpha'\beta'} = 0\,, \quad R_{\alpha\alpha'} = 0$$

λ^{-1} **is the** AdS_4 **radius.**

Unfolded Einstein equations

$$R_{\alpha\alpha'} = 0\,,$$

$$R_{\alpha_1\alpha_2} = e^{\alpha_3}{}_{\alpha'} \wedge e^{\alpha_4\alpha'} C_{\alpha_1\ldots\alpha_4}\,, \qquad \overline{R}_{\alpha'_1\alpha'_2} = e_\alpha{}^{\alpha'_3} \wedge e^{\alpha\alpha'_4} \overline{C}_{\alpha'_1\ldots\alpha'_4}$$

$C_{\alpha_1\ldots\alpha_4}$ **and** $\overline{C}_{\alpha'_1\ldots\alpha'_4}$: **Weyl tensor**

Bianchi identities + Einstein equations imply

$$D^L C_{\alpha_1\ldots\alpha_{n+4}\,,\alpha'_1\ldots\alpha'_n} + e^{\alpha_{n+5}\alpha'_{n+1}} C_{\alpha_1\ldots\alpha_{n+5}\,,\alpha'_1\ldots\alpha'_{n+1}} = O(C^2)$$

$$D^L \overline{C}_{\alpha_1\ldots\alpha_n\,,\alpha'_1\ldots\alpha'_{n+4}} + e^{\alpha_{n+1}\alpha'_{n+5}} \overline{C}_{\alpha_1\ldots\alpha_{n+1}\,,\alpha'_1\ldots\alpha'_{n+5}} = O(C^2)$$

$C_{\alpha_1\ldots\alpha_{n+4}\,,\alpha'_1\ldots\alpha'_n}$ **and** $\overline{C}_{\alpha_1\ldots\alpha_n\,,\alpha'_1\ldots\alpha'_{n+4}}$: **order-$n$ on-shell nontrivial derivatives of the Weyl tensor**

The equations are unfolded: $dq(x)$ **expresses wedge products of the dynamical fields** $q(x) = (h, \omega, \bar{\omega}, C, \overline{C})$ **themselves**

$4d$ massless fields

HS gauge fields are described by the 1-form connections $(s \geq 1)$

$$\omega_{\alpha_1 \ldots \alpha_n, \alpha'_1 \ldots \alpha'_m}(x)\,, \qquad n + m = 2(s-1)$$

spin one: $\omega(x)$

spin two: $\omega_{\alpha\beta}$, $\omega_{\alpha'\beta'}, \omega_{\alpha\beta'}$

along with the gauge invariant tensors $(s \geq 0)$

$$C_{\alpha_1 \ldots \alpha_n, \alpha'_1 \ldots \alpha'_m}(x)\,, \qquad |n - m| = 2s$$

that describe all on-shell nontrivial derivatives of the generalized Weyl tensors

$$C_{\alpha_1 \ldots \alpha_{2s}}(x) \qquad \overline{C}_{\alpha'_1 \ldots \alpha'_{2s}}(x) \qquad \alpha = 1, 2; \quad \alpha' = 1, 2$$

spin zero: $C(x)$

spin one: $C_{\alpha_1 \alpha_2}(x)$, $\overline{C}_{\alpha'_1 \alpha'_2}(x)$

spin two: $C_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(x)$, $\overline{C}_{\alpha'_1 \alpha'_2 \alpha'_3 \alpha'_4}(x)$

Unfolded Massless Equations

$A = \omega, C, \overline{C}, R$

$$A(y, \bar{y} \mid x) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} y_{\alpha_1} \dots y_{\alpha_n} \bar{y}_{\beta'_1} \dots \bar{y}_{\beta'_m} A^{\alpha_1 \dots \alpha_n}{}_{,}{}^{\beta'_1 \dots \beta'_m}(x)$$

$$D^{ad}\omega = D^L\omega(y, \bar{y} \mid x) - \lambda e^{\alpha\beta'} \Big(y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} + \frac{\partial}{\partial y^\alpha} \bar{y}_{\beta'} \Big) \omega(y, \bar{y} \mid x)$$

$$D^{tw}C = D^L C(y, \bar{y}|x) + e^{\alpha\beta'} \Big(\lambda^2 y_\alpha \bar{y}_{\beta'} + \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^{\beta'}} \Big) C(y, \bar{y}|x)$$

where D^L is the Lorentz covariant derivative

$$D^L A(y, \bar{y}|x) = dA(y, \bar{y}|x) - \Big(w^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \bar{w}^{\alpha'\beta'} \bar{y}_{\alpha'} \frac{\partial}{\partial \bar{y}^{\beta'}} \Big) A(y, \bar{y}|x)$$

$$AdS_4 : \qquad (D^{ad})^2 = 0\,, \qquad (D^{tw})^2 = 0$$

The free unfolded equations $R(y, \bar{y} \mid x) = D^{ad}\omega$

$$R(y, \bar{y} \mid x) = e_\alpha{}^{\alpha'} \wedge e^{\alpha\beta'} \frac{\partial^2}{\partial \bar{y}^{\alpha'} \partial \bar{y}^{\beta'}} \bar{C}(0, \bar{y} \mid x) + e^\alpha{}_{\alpha'} \wedge e^{\beta\alpha'} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y, 0 \mid x)$$

$$D^{tw}C(y, \bar{y} \mid x) = 0\,, \qquad D^{tw}\bar{C}(y, \bar{y} \mid x) = 0$$

Higher-rank equations

Rank r equations

$$D_r^{tw} C(y_i) = 0$$

result from the tensor product

$$y^\alpha \to y_i^\alpha \,, \qquad \overline{y}^{\alpha'} \to \overline{y}_i^{\alpha'} \qquad i,j = 1 \ldots r \,,$$

$$D^{tw} \to D_r^{tw} = D^L + \lambda e^{\alpha\beta'} \left(y_{i\alpha} \overline{y}_{j\beta'} \eta^{ij} + \eta_{ij} \frac{\partial^2}{\partial y_i^\alpha \partial \overline{y}_j^{\beta'}} \right)$$

For diagonal η^{ij} the higher-rank equations are satisfied by the products of rank–one fields

$$C(y_i|X) = C_1(y_1|X) C_2(y_2|X) \ldots C_r(y_r|X) \,, \qquad D^{tw} C(y|x) = 0 \,.$$

Rank two equations and conserved currents

Minkowski reduction of the rank–two field equations gives

$$D^{tw}_{fl\,2} J(y^\pm, \bar{y}^\pm|x) = \Big(D^L + e^{\alpha\beta'} \Big(\frac{\partial^2}{\partial y^{+\alpha} \partial \bar{y}^{-\beta'}} + \frac{\partial^2}{\partial y^{-\alpha} \partial \bar{y}^{+\beta'}} \Big) \Big) J(y^\pm, \bar{y}^\pm|x) = 0$$

Any solution generates a closed three-form

$$\Omega(J) = \mathcal{H}^{\alpha\alpha'} \frac{\partial}{\partial y^{-\alpha}} \frac{\partial}{\partial \bar{y}^{-\alpha'}} J(y^\pm, \bar{y}^\pm|x) \Big|_{y^\pm = \bar{y}^\pm = 0} \qquad \mathcal{H}^{\alpha\delta'} = -\frac{1}{3} e^\alpha{}_{\alpha'} \wedge e^{\beta\alpha'} \wedge e_\beta{}^{\delta'}$$

Since any parameter $\eta(\xi'_{-\beta}, \bar{\xi}'_{-\beta'}, \xi'^{+\alpha}, \bar{\xi}'^{+\alpha'})$ **that depends on**

$$\xi'_{-\alpha} = \frac{\partial}{\partial y^{-\alpha}}, \ \bar{\xi}'_{-\beta'} = \frac{\partial}{\partial \bar{y}^{-\beta'}}, \ \xi'^{+\alpha} = y^{+\alpha} - x^{\alpha\beta'} \frac{\partial}{\partial \bar{y}^{-\beta'}}, \ \bar{\xi}'^{+\alpha'} = \bar{y}^{+\alpha'} - x^{\beta\alpha'} \frac{\partial}{\partial y^{-\beta}}$$

satisfies $\left[D^{tw}_{fl\,2}, \eta \right] = 0$

$$D^{tw}_{fl\,2} J(y^\pm, \bar{y}^\pm|x) = 0 \quad \Rightarrow \quad D^{tw}_{fl\,2}\, \eta \left(J(y^\pm, \bar{y}^\pm|x) \right) = 0$$

Usual currents result from bilinear representation of rank two solution with some parameter η

$$J(y^\pm\, \bar{y}^\pm|x) = \eta C_+(y^+ + y^-, \bar{y}^+ + \bar{y}^-|x) C_-(y^+ - y^-, \bar{y}^+ - \bar{y}^-|x)$$

Unfolded current interactions

In the unfolded dynamics approach current interactions result from a nontrivial mixing between fields of rank one and rank two

$$
\begin{aligned}
D^{ad}\omega(y,\bar{y}|x) &= \overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar{y}^{\alpha'}\partial\bar{y}^{\beta'}}\overline{C}(0,\bar{y}\mid x)+H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}C(y,0\mid x)\;+\\
&+\;\overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar{y}_-{}^{\alpha'}\partial\bar{y}_-{}^{\beta'}}G\Big(N_-\,,\overline{N}_-\Big)\,\mathcal{J}(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0}+\\
&+\;H^{\alpha\beta}\frac{\partial^2}{\partial y_-{}^{\alpha}\partial y_-{}^{\beta}}\overline{G}\Big(N_-\,,\overline{N}_-\Big)\,\overline{\mathcal{J}}(y^{\pm},\bar{y}^{\pm}|x)\,\Big|_{y^{\pm}=\bar{y}^{\pm}=0}\,,
\end{aligned}
$$

$$
\begin{aligned}
D^{tw}C(y,\bar{y}\mid x)+e^{\alpha\alpha'}y_{\alpha}F^{\pm}\Big(N_{\pm}\,,\overline{N}_{\pm}\Big)\frac{\partial}{\partial\bar{y}^{\pm\alpha'}}J(y^{\pm},\bar{y}^{\pm}|x)\,\Big|_{y^{\pm}=\bar{y}^{\pm}=0} &= 0\,,\\
D^{tw}\overline{C}(y,\bar{y}\mid x)+e^{\alpha\alpha'}\bar{y}_{\alpha'}\overline{F}^{\pm}\Big(N_{\pm}\,,\overline{N}_{\pm}\Big)\frac{\partial}{\partial y^{\pm\alpha}}\overline{J}(y^{\pm},\bar{y}^{\pm}|x)\,\Big|_{y^{\pm}=\bar{y}^{\pm}=0} &= 0\,,
\end{aligned}
$$

where

$$
N_{\pm}=y^{\alpha}\frac{\partial}{\partial y^{\pm\alpha}}\,,\qquad\qquad \overline{N}_{\pm}=\bar{y}^{\alpha'}\frac{\partial}{\partial\bar{y}^{\pm\alpha'}}
$$

$$
F^{+}=-\Big(N_-\Big)^{2s}\overline{N}_-\sum_{k\geq 0}\frac{\Big(\overline{N}_+N_-\;+\;\overline{N}_-N_+\Big)^{k}}{k!(k+2s+2)!}\,,
$$

$$F^- = \left(N_-\right)^{2s-1} \sum_{k\geq 0} \frac{\left(\overline{N}_+ N_- + \overline{N}_- N_+\right)^k}{k!(k+2s+1)!} \left(2s + N_- \overline{N}_+ \frac{1}{(k+2s+2)}\right),$$

$$G = \sum_{k=0}^{s-2} \frac{(-1)^k \left(N^-\right)^{s-k-2} \left(\overline{N}^-\right)^{s+k}}{(s+k)!(s-1)!}, \qquad \overline{G} = \sum_{k=0}^{(s-2)} \frac{(-1)^k \left(N^-\right)^{s+k} \left(\overline{N}^-\right)^{s-k-2}}{(s+k)!(s-1)!},$$

$$\mathcal{J} = \sum_{k=0}^{(s-2)} \left(f^-\right)^k \mathcal{J}_0\,, \qquad \overline{\mathcal{J}} = \sum_{k=0}^{(s-2)} \left(f^+\right)^k \mathcal{J}_0$$

$$J = \lambda(-1)^s \frac{1}{2(s-1)!} \left(-f_+\right)^{s-1} \mathcal{J}_0\,, \qquad \overline{J} = \lambda(-1)^s \frac{1}{2(s-1)!} \left(f_-\right)^{s-1} \mathcal{J}_0\,,$$

where $f_\pm$ are the generators of sl_2

$$f_- = \frac{\partial}{\partial y^{+\gamma}} \frac{\partial}{\partial y^-{}_\gamma} - \bar{y}^{+\gamma'} \bar{y}^-{}_{\gamma'}\,, \quad f_+ = y^{+\nu} y^-{}_\nu - \frac{\partial}{\partial \bar{y}^{+\nu'}} \frac{\partial}{\partial \bar{y}^-{}_{\nu'}}\,,$$

$$h = y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha} \frac{\partial}{\partial y^{-\alpha}} - \bar{y}^{+\alpha'} \frac{\partial}{\partial \bar{y}^{+\alpha'}} - \bar{y}^{-\alpha'} \frac{\partial}{\partial \bar{y}^{-\alpha'}}\,,$$

and $\mathcal{J}_0(y^\pm, \bar{y}^\pm|x)$ is any solution of the second rank (current) unfolded equation that satisfies $h\mathcal{J}_0(y^\pm, \bar{y}^\pm|x) = 0$.

Lower-spin examples

Scalar: Yukawa interaction

$$D^L_{\alpha\alpha'} D^{L\alpha\alpha'} C(0\,,0|x) = 4\overline{\mathcal{C}}_{+\alpha'}(x)\overline{\mathcal{C}}_{-}{}^{\alpha'}(x) + 4\mathcal{C}_{+\alpha}(x)\mathcal{C}_{-}{}^{\alpha}(x)\,.$$

Maxwell equation

$$d\left(H^{\alpha\beta}C_{\alpha\beta}(x) - \overline{H}^{\alpha'\beta'}\overline{C}_{\alpha'\beta'}(x)\right) \quad = \mathcal{H}^{\beta\nu'}\Big(\mathcal{C}_{+\beta}(x)\overline{\mathcal{C}}_{-\nu'}(x) - \mathcal{C}_{-}(x)\frac{\partial}{\partial x^{\beta\nu'}}\mathcal{C}_{+}(x) + \mathcal{C}_{+}(x)\frac{\partial}{\partial x^{\beta\nu'}}\mathcal{C}_{-}(x)\Big)$$

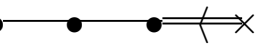
Einstein equation

Stress tensor

$$\begin{aligned} T_{\alpha\alpha,\alpha'\alpha'} \;=\; & \Big(\mathcal{C}^{2,0}_{+}{}_{\alpha\alpha}(0,0\big|x)\overline{\mathcal{C}}^{0,2}_{-}{}_{\alpha'\alpha'}(0,0\big|x) + \mathcal{C}^{1,0}_{+}{}_{\alpha\alpha\alpha'}(0,0\big|x)\overline{\mathcal{C}}^{0,1}_{-}{}_{\alpha'}(0,0\big|x)\Big) \\ + \;& \mathcal{C}^{0,0}_{+}{}_{\alpha\alpha'}(0,0\big|x)\overline{\mathcal{C}}^{0,0}_{-}{}_{\alpha\alpha'}\big(0,0\big|x) + (+\leftrightarrow -)\Big) \end{aligned}$$

Four from Ten

The minimal space where $sp(8|R)$ acts geometrically is $\mathcal{M}_4 \sim Sp(8|R)/P$



$P : (\mathcal{L}, \mathcal{K})$

$\mathcal{M}_4$ is ten-dimensional: local coordinates $X^{AB} = X^{BA}$. **Fronsdal (1985); Bandos, Lukierski, Sorokin (1999)**

$\mathcal{M}_4 : x^{\alpha\alpha'}$ extends to $X^{AB} = (x^{\alpha\alpha'}, x^{\alpha\beta}, \bar{x}^{\alpha'\beta'})$

Rank r unfolded equations in $\mathcal{M}_M$

$$dX^{AB}\left(\frac{\partial}{\partial X^{AB}} + \eta_{ij}\frac{\partial^2}{\partial Y_i^A \partial Y_j^B}\right)C(Y|X) = 0\,, \qquad i,j = 1,\ldots r\,, A,B = 1,\ldots M\,.$$

the equations in $\mathcal{M}_M$ contain the $4d$ equations in the $dx^{\alpha\alpha'}$ sector + equations that determine the dependence on the additional six coordinates $x^{\alpha\beta}$, $\bar{x}^{\alpha'\beta'}$

$4d$ massless field equations from ten-dimensions

rank one: only two dynamical fields:

$C(0|X)$ **describes all** $4d$ **integer spins**

$Y^A C_A(0|X)$ **describes all** $4d$ **half-integer spins**

Nontrivial field equations: $H^1(\sigma_-)$ **(2001)**

$$\textbf{bosons}: \qquad \left(\frac{\partial^2}{\partial X^{AB}\partial X^{CD}} - \frac{\partial^2}{\partial X^{CB}\partial X^{AD}}\right) C(X) = 0$$

$$\textbf{fermions}: \qquad \left(\frac{\partial}{\partial X^{AB}} C_C(X) - \frac{\partial}{\partial X^{CB}} C_A(X)\right) = 0$$

Rank two equations

$$\frac{\partial^3}{\partial X^{[A_1 B_1}\partial X^{A_2 B_2}\partial X^{A_3]_A B_3}} C(X) = 0$$

All $4d$ **conformal conserved currents** **Gelfond, MV (2005)**

Higher rank as higher dimension

A rank r field in $\mathcal{M}_M$ can be interpreted as a rank one field in $\mathcal{M}_{rM}$ with coordinates X_{ij}^{AB}.

$$Y_i^A \to Y^{\widetilde{A}}, \qquad \widetilde{A} = 1 \ldots rM$$

Embedding of $\mathcal{M}_M$ into $\mathcal{M}_{rM}$

$$X_{11}^{AB} = X_{22}^{AB} = \ldots = X_{rr}^{AB} = X^{AB}$$

$4d$ **conformal currents: a rank two field in $\mathcal{M}_4 \sim$ rank one field in $\mathcal{M}_8$.**

A single rank one field in $\mathcal{M}_8$ describes all $6d$ conformal fields.

HS current interactions in four dimensions describe a linear mixing between conformal fields of all spins in four and six dimensions

That a seemingly nonlinear system becomes linear in the unfolded dynamics approach looks like some step towards integrability.

Conclusions

Current interactions from a linear problem via bilinear substitution

Current interactions in $d = 4$ is a mixing of free fields in $d = 4$ and $d = 6$

Unfolding machinery provides a nontrivial integral twistor transform that relates different realizations of the same dynamical system

Riemann θ function as solutions of massless equations

Remarkably unfolded massless equations in $\mathcal{M}_M$ lead to such a fundamental object as Riemann θ–functions **Gelfond, MV, arXiv:0801.2191**

$$C(Y|Z) = \theta(Y,\, Z) = \sum_{n^A \in \mathbb{Z}^M} \exp i\pi(Z^{AB} n_A n_B + 2 n_A Y^A)$$

$$dZ^{AB}(\frac{\partial}{\partial X^{AB}} + \frac{i}{4\pi}\frac{\partial^2}{\partial Y^A \partial Y^B})C(Z|X) = 0$$

Unfolding as a covariant twistor transform

Twistor transform

$$\begin{tikzcd} & C(Y|x) \arrow[dl, "\eta"'] \arrow[dr, "\nu"] & \\ M(x) & & T(Y)\,. \end{tikzcd}$$

$W^\alpha(Y|x)$ are functions on the "correspondence space" C.

Space-time M : coordinates x. Twistor space T : coordinates Y.

Unfolded equations describe the Penrose transform by mapping functions on T to solutions of field equations in M.

Effective (spinorial HS models):

$W^\alpha(Y|x)$ are unrestricted functions on $T = R^n$ or some projective space.

Ineffective (tensorial HS models):

$W^\alpha(Y|x)$ are subject to differential conditions in T. The unfolded field equations are still useful to describe interactions