SUPERCONFORMAL INDICES

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The localization method

The key problem of Quantum Field Theory: compute the path integral

$$Z(J) = \int d\Phi\, e^{-S(\Phi,J)}.$$

Perturbation theory: Feynman diagrams (convergency ?)

Non-perturbative computation ?

Exact computation: topological field theories (no physical excitations): Witten, 1988.

The simplest case: the Witten index in $4d$ SUSY theories

$$I_W = \mathrm{Tr}\,(-1)^{\mathcal{F}} e^{-\beta H} = \int_{PBC} d\phi d\psi e^{-S(\phi,\psi)}.$$

No β-dependence due to SUSY cancellations (only zero modes of the Hamiltonian H contribute).

Computation of the bulk I_W using the localization technique: Moore, Nekrasov, Shatashvili, 1998.

The idea: compute

$$Z = \int d\phi d\psi e^{-S(\phi,\psi)-t\delta\mathcal{O}},$$

where δ is a fermionic symmetry of action S and the measure, $\delta^2 = \delta_b$ is a bosonic symmetry of $\mathcal{O}$ (e.g., $\delta^2 = 0$).

$$\frac{dZ}{dt} = -\int d\phi d\psi \big(\delta\mathcal{O}\big) e^{-S(\phi,\psi)-t\delta\mathcal{O}}$$
$$= -\int d\phi d\psi \delta \left(\mathcal{O} e^{-S(\phi,\psi)-t\delta\mathcal{O}}\right) = 0.$$

$\Rightarrow$ no dependece on t.

$t \to 0$: the needed object, but for $t \to +\infty$: the integral is localized on the zero modes of $\mathcal{O} \Rightarrow$ gaussian path integrals for some fields $\Rightarrow$ finite-dimensional integrals (**matrix models**).

Recent progress of exact computations in realistic field theories on non-trivial manifolds:

• $4d$ superconformal (topological) indices on $S^3 \times S^1$ for SYM theories with $\mathcal{N} = 4$ (Kinney, Maldacena, Minwalla, Raju, 2005) and $\mathcal{N} = 1$ (Romelsberger, 2005).

Dolan, Osborn (2008): identification with the elliptic hypergeometric integrals (V.S., 2000) and confirmation of a number of Seiberg dualities.

V.S., G. Vartanov (2008-2011): a systematic investigation (see below).

• $4d$ partition functions of $\mathcal{N} = 2, 4$ SYM on S^4 (Pestun, 2007)

• $3d$ superconformal indices of SYM and CS on $S^2 \times S^1$ for $\mathcal{N} = 6$ (Kim, 2009) and $\mathcal{N} = 2$ (Imamura, Yokoyama, 2011; a correction, Krattenthaler, V.S., G. Vartanov, 2011).

• $3d$ partition functions of $\mathcal{N} = 2$ SYM and CS on S_b^3 (Kapustin, Willet, Yaakov, 2010; Jafferis, 2010; Hama, Hosomichi, Lee, 2011)

$4d$ **superconformal index**

$SU(2,2|1)$ space-time symmetry group:
$J_i, \overline{J}_i$ ($SU(2)$ subgroups generators, or Lorentz rotations),
$P_\mu, Q_\alpha, \overline{Q}_{\dot\alpha}$ (supertranslations),
$K_\mu, S_\alpha, \overline{S}_{\dot\alpha}$ (special superconformal transformations),
H (dilations) and R ($U(1)_R$-rotations).

Internal symmetries: a local gauge group G_c (generators G^a) and a global flavor group F (generators F_k).

For $Q = \overline{Q}_1$ and $Q^\dagger = -\overline{S}_1$,

$$\{Q, Q^\dagger\} = 2\mathcal{H}, \qquad \mathcal{H} = H - 2\overline{J}_3 - 3R/2.$$

The superconformal index:

$$I(y;p,q) = \mathrm{Tr}\Big((-1)^{\mathcal{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} \prod_k y_k^{F_k}\, e^{-\beta\mathcal{H}}\Big),$$

$\mathcal{R} = H - R/2, \qquad [Q, \mathcal{R}] = [Q, J_3] = [Q, F_k] = 0,$
$\mathcal{F}$ – the fermion number,
p, q, y_k, β are group parameters (fugacities).

It counts BPS states $\mathcal{H}|\psi\rangle = 0$ or cohomology of Q, $Q^\dagger$ operators (hence, no β-dependence).

"Computation" (simple examples, guesswork, plethystic machinery; Römelsberger, 2007) $\Rightarrow$ matrix integral

$$I(y;p,q) \ = \ \int_{G_c} d\mu(z)\, \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} ind\big(p^n,q^n,z^n,y^n\big)\Big),$$

$d\mu(z)$ – the Haar G_c-invariant measure,
ind – the single particle states index,

$$ind(p,q,z,y) = \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj_G}(z)$$
$$+\sum_j \frac{(pq)^{r_j}\chi_{R_F,j}(y)\chi_{R_G,j}(z)-(pq)^{1-r_j}\chi_{\bar{R}_F,j}(y)\chi_{\bar{R}_G,j}(z)}{(1-p)(1-q)}.$$

$\chi_{R_F,j}(y)$ and $\chi_{R_G,j}(z)$ – characters of representations,
y_j and z_a – maximal torus variables of F and G_c,
$2r_j$ – the R-charges.

For the unitary group $SU(N)$, $z=(z_1,\ldots,z_N)$, $\prod_{j=1}^N z_a = 1$,

$$\int_{SU(N)} d\mu(z) \ = \ \frac{1}{N!}\int_{\mathbb{T}^{N-1}} \Delta(z)\Delta(z^{-1}) \prod_{a=1}^{N-1} \frac{dz_a}{2\pi \mathrm{i} z_a},$$

$$\Delta(z) \ = \prod_{1\leq a<b\leq N} (z_a-z_b), \qquad \text{the Vandermonde determinant.}$$

Take $\mathcal{N}=1$ SQCD with $G=SU(2)$, $F=SU(6)$ and the vector and quark superfields:

$$\begin{aligned}
&1)\ (adj,1), \qquad \chi_{SU(2),adj}(z)=z^2+z^{-2}+1,\\
&2)\ (f,f), \qquad \chi_{SU(2),f}(z)=z+z^{-1}, \qquad r_f=1/6,\\
&\chi_{SU(6),f}(y)=\sum_{k=1}^{6} y_k, \quad \chi_{SU(6),\bar f}(y)=\sum_{k=1}^{6} y_k^{-1}, \quad \prod_{k=1}^{6} y_k=1.
\end{aligned}$$

Then the superconformal index (SCI):

$$I_E=\frac{(p;p)_\infty(q;q)_\infty}{4\pi\mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{j=1}^{6}\Gamma(t_j z^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{z},$$

where $\mathbb{T}$ is the unit circle, $(a;q)_\infty=\prod_{k=1}^{\infty}(1-aq^k)$,

$$\Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}, \qquad |p|,|q|<1,$$

is the elliptic gamma function. Conventions: $t_j=(pq)^{1/6}y_j$,

$$\begin{aligned}
&\Gamma(t_1,\ldots,t_k;p,q):=\Gamma(t_1;p,q)\cdots\Gamma(t_k;p,q),\\
&\Gamma(tz^{\pm1};p,q):=\Gamma(tz;p,q)\Gamma(tz^{-1};p,q).
\end{aligned}$$

Seiberg "electric-magnetic" duality (1994): in IR particles confine, the dual theory = a Wess-Zumino model of chiral field T_A :

$\Phi_{ij} = -\Phi_{ji}$, 15-dim irrep. of $F = SU(6)$. Then

$$\chi_{SU(6),T_A}(y) = \sum_{1\le i<j\le 6} y_i y_j, \qquad r_{T_A} = 1/3.$$

and the magnetic SCI is

$$I_M = \prod_{1\le j<k\le 6} \Gamma(t_j t_k; p, q).$$

Theorem (V.S., 2000). Let $|p|, |q|, |t_j| < 1$, $\prod_{j=1}^6 t_j = pq$. Then

$$I_E = I_M.$$

A principally new exactly computable integral:
THE ELLIPTIC BETA INTEGRAL

More generally, a principally new class of special functions = elliptic hypergeometric functions.

Mathematical importance.
Newton (1665): **the binomial theorem**,

$${}_1F_0(a; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n = (1-x)^{-a}, \qquad |x| < 1, \quad a \in \mathbb{C},$$

where

$$(a)_n = a(a+1)\cdots(a+n-1) \qquad \text{the Pochhammer symbol}$$

Euler-Gauss: q**-binomial theorem**,

$$ {}_1\varphi_0(t;q,x) = \sum_{n=0}^{\infty} \frac{(t;q)_n}{(q;q)_n}\, x^n = \frac{(tx;q)_\infty}{(x;q)_\infty}, \quad |x|, |q| < 1. $$

$$ (x;q)_n = \prod_{k=0}^{n-1}(1 - xq^k) \qquad \text{the } q\text{-Pochhammer symbol} $$

These are the simplest representatives of the plain and q-hypergeometric functions. At the elliptic level,

Elliptic beta integral = **elliptic binomial theorem**.

It generalizes the Euler beta integral

$$ \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} $$

and many other integrals (e.g., Askey-Wilson q-beta integral). Many multidimensional analogues, e.g., extending the Selberg integral.

arXiv surveys: math.CA/0511579 and 0805.3135.

Coincidence of SCIs with the elliptic hypergeometric integrals was discovered by **Dolan and Osborn (2008)** $\Rightarrow$ confirmation of some Seiberg dualities.

General Seiberg duality for $SU(N_c)$ gauge group

"Electric" theory:

	$SU(N_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$\tilde{N}_c/N_f$
$\widetilde{Q}$	$\overline{f}$	1	$\overline{f}$	-1	$\tilde{N}_c/N_f$
V	adj	1	1	0	1

"Magnetic" theory:

	$SU(\tilde{N}_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
q	f	$\overline{f}$	1	$N_c/\tilde{N}_c$	N_c/N_f
$\widetilde{q}$	$\overline{f}$	1	f	$-N_c/\tilde{N}_c$	N_c/N_f
M	1	f	$\overline{f}$	0	$2\tilde{N}_c/N_f$
$\tilde{V}$	adj	1	1	0	1

where $\tilde{N}_c = N_f - N_c$ and $3N_c/2 < N_f < 3N_c$ (conformal window).

Seiberg conjecture: these two $\mathcal{N} = 1$ SYM theories have the same physics at their IR fixed points.

Consistency checks:

- The global anomalies match ('t Hooft anomaly matching)
- Matching of the reductions $N_f \to N_f - 1$
- The moduli spaces have the same dimensions and the gauge invariant operators match

The electric theory index:

$$I_E = \kappa_{N_c} \int_{\mathbb{T}^{N_c-1}} \frac{\prod_{i=1}^{N_f}\prod_{j=1}^{N_c}\Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p,q)}{\prod_{1\le i<j\le N_c}\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p,q)} \prod_{j=1}^{N_c-1}\frac{dz_j}{2\pi \mathrm{i} z_j},$$

$$\prod_{j=1}^{N_c} z_j = 1, \qquad \kappa_N = \frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!}.$$

The magnetic theory: $\qquad I_M = \kappa_{\tilde{N}_c}\prod_{i,j=1}^{N_f}\Gamma(s_i t_j^{-1}; p,q)\times$

$$\times \int_{\mathbb{T}^{\tilde{N}_c-1}} \frac{\prod_{i=1}^{N_f}\prod_{j=1}^{\widetilde{N}_c}\Gamma(S^{\frac{1}{\widetilde{N}_c}} s_i^{-1} x_j, T^{-\frac{1}{\widetilde{N}_c}} t_i x_j^{-1}; p,q)}{\prod_{1\le i<j\le \widetilde{N}_c}\Gamma(x_i x_j^{-1}, x_i^{-1} x_j; p,q)} \prod_{j=1}^{\widetilde{N}_c-1}\frac{dx_j}{2\pi \mathrm{i} x_j},$$

where $\prod_{j=1}^{\tilde{N}_c} x_j = 1, \quad \tilde{N}_c = N_f - N_c,$

$S = \prod_{i=1}^{N_f} s_i, \quad T = \prod_{i=1}^{N_f} t_i, \quad ST^{-1} = (pq)^{N_f - N_c}.$

Theorem: If $|s_j|$, $|t_j^{-1}|$, $|t_i/T^{1/\widetilde{N}_c}|$, $|S^{1/\widetilde{N}_c}/s_i| < 1$, then $I_E = I_M$.

For $N_c = 2$, $N_f = 4$ (V.S., 2003), general N_c, N_f (Rains, 2003)

A fundamental physical interpretation:

Explicit computability of the elliptic hypergeometric integrals = confinement in $4d$ $\mathcal{N}=1$ SUSY gauge theories.

The process of integrals' computation = transition from UV (weak coupling) to IR (strong coupling) physics.

Joint work with G.S. Vartanov (Dima's former student) (2008–2011, 7 papers + in preparation)

Summary of the main results:

- 't Hooft anomaly matching $\longleftrightarrow$ the total ellipticity condition for elliptic hypergeometric terms:
  ratios of integral kernels satisfy a set of linear first order q-difference equations with coefficients which are elliptic functions (with modulus p) of all variables z_k, x_l, t_j, s_j, and q

- All known identities lead to totally elliptic hypergeometric terms $\rightarrow$ conjecture: this property is necessary for computability/nice symmetry of integrals

- SCIs are invariants of the conformal manifold against the exactly marginal deformations.
  Non-marginal deformations $\Rightarrow$ special restrictions on fugacities. E.g., $N_f \rightarrow N_f - 1$ reduction by adding mass terms $M_k^k Q^k \widetilde{Q}_k$ with $M_k^k \rightarrow \infty$ in the electric theory leads to Higgsing of the gauge group on the magnetic side, so that $SU(N_f - N_c) \rightarrow SU(N_f - N_c - 1)$. For SCIs: $s_k t_k^{-1} = pq$ (a simple substitution for I_E and a residue calculus for I_M)

- "Vanishing" (delta function behavior) of superconformal indices $\longleftrightarrow$ chiral symmetry breaking

- About 15 new computable elliptic beta integrals on root systems and a similar number of new symmetry transformations for higher order elliptic hypergeometric integrals with $SU(N)$, $SP(2N)$, G_2, E_6, F_4 gauge groups (conjectures)
- About 15 new pairs of $\mathcal{N}=1$ dual field theories (e.g., multiple dualities for $G_c = SP(2N)$ with 8 flavors and $G_c = SU(N)$ with $4+4$ flavors, new confining theories)
- There are non-trivial dualities lying outside the conformal window (for different dual gauge groups)
- For $\mathcal{N}=4$ SYM theories – derivation of explicit forms of indices for all simple gauge groups and exact computation in two important limit. Possible consequences for the AdS/CFT correspondence for finite rank gauge groups.
- Partial confirmation of the equality of indices for $\mathcal{N}=4$ SYM $SP(2N) \longleftrightarrow SO(2N+1)$ duality (GPRR, 2010)
- Discovery of many new relations between dualities (some of them are deducible from the others)
- It is conjectured that there are infinitely (countably) many supersymmetric dualities and corresponding elliptic hypergeometric integral identities.

This year developments:

- The $3d$ supeconformal indices on $S^2 \times S^1$ with $U(1)$ gauge group were computed exactly and their equality for some dual pairs was rigorously proven (Krattenthaler, V.S., Vartanov)
- $3d$ partition functions on S_b^3 are obtained as reductions of $4d$ SCIs on $S^3 \times S^1$, or by reduction of elliptic hypergeometric integrals to the hyperbolic q-hypergeometric integrals (Dolan, V.S., Vartanov; Gadde, Yan; Imamura)
- Analysis/conjectures of about 20 dualities/integral relations for $SO(N)$ gauge group
- State integrals for knots on S^3 (topological invariants, Hikami, 2001) are limits of $4d$ SCIs
- $2d$ vortex partition function is a limit of a particular $3d$ partition function which, in turn, is a limit of a particular $4d$ SCI.

arXiv:1107.5788 (V.S, G. Vartanov), 68 pp.,
dedicated to D.I. Kazakov @ 60.

Further great discoveries to you, Dima !