

# **Effective action for high energy processes in gravity**

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# 1 Gribov Pomeron calculus

Regge asymptotics of elastic amplitudes

$$A(s, t) \approx i s \gamma^2(-q^2) s^{\Delta_P - \alpha'_P q^2}, \quad \sigma = \gamma^2(0) s^{\Delta_P}$$

Mandelstam cut contribution

$$A_M(s, t) \approx -i \int d^2 k \Phi^2(k, q - k) s^{2\Delta - \alpha'_P k^2 - \alpha'_P (q - k)^2}$$

Separation of particles in their rapidities

$$0 < y_1 < \dots < y_n < \ln s, \quad y_k - y_{k-1} \gg 1, \quad y = \frac{1}{2} \ln \frac{E + p}{E - p}$$

Gribov's Pomeron action

$$S = \int dy d^2 \rho \left( \phi^* (\partial_y - \Delta) \phi + \frac{(\vec{\nabla} \phi)^2}{2m} + \lambda (\phi^* \phi^2 + \phi \phi^{*2}) + \dots \right)$$

## 2 Gluon reggeization in QCD

QCD Born amplitude

$$M_{AB}^{A'B'}(s, t)|_{Born} = 2s g T_{A'A}^c \delta_{\lambda_{A'}, \lambda_A} \frac{1}{t} g T_{B'B}^c \delta_{\lambda_{B'}, \lambda_B}$$

Leading Logarithmic Approximation

$$M(s, t) = M_{Born}(s, t) s^{\omega(t)},$$

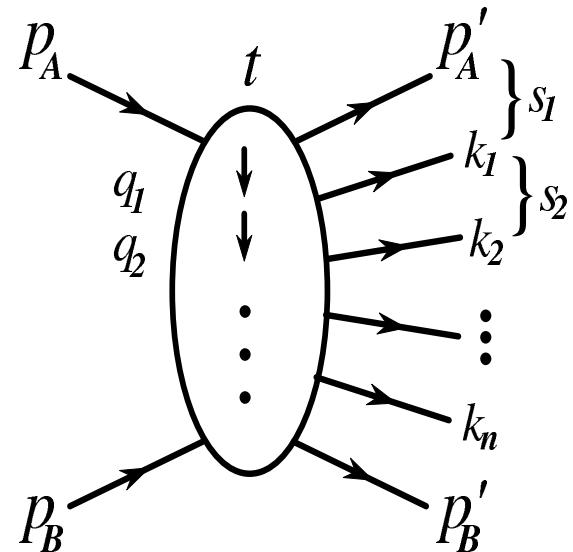
Its region of applicability

$$\alpha_s \ln s \sim 1, \quad \alpha_s = \frac{g^2}{4\pi} \ll 1$$

Gluon trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2 k \frac{|q|^2}{|k|^2 |q - k|^2} \approx -\frac{\alpha_s N_c}{2\pi} \ln \frac{|q^2|}{\lambda^2}$$

### 3 Amplitudes in multi-Regge kinematics



$$M_{2 \rightarrow 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2} g T_{c_2 c_1}^{d_1} C(q_2, q_1) \frac{s_2^{\omega_2}}{|q_2|^2} \dots g T_{c_{n+1} c_n}^{d_n} C(q_{n+1}, q_n) \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2},$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi} \left( \ln \frac{|q_r^2|}{\mu^2} - \frac{1}{\epsilon} \right), \quad C(q_2, q_1) = \frac{q_2 q_1^*}{q_2^* - q_1^*}, \quad \sigma_t = \sum_n \int d\Gamma_n |M_{2 \rightarrow 2+n}|^2$$

## 4 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \sigma_t \sim s^\Delta, \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E_0$$

Hamiltonian for the Pomeron wave function

$$H_{12} = \frac{1}{p_1 p_2^*} (\ln |\rho_{12}|^2) p_1 p_2^* + \frac{1}{p_1^* p_2} (\ln |\rho_{12}|^2) p_1^* p_2 + \ln |p_1 p_2|^2 - 4\psi(1),$$

$$\rho_{12} = \rho_1 - \rho_2, \quad \rho_r = x_r + iy_r, \quad \Delta = 4\alpha N_c \ln 2 / \pi$$

Möbius invariance and Pomeron intercept

$$\rho_k \rightarrow \frac{a\rho_k + b}{c\rho_k + d}, \quad m = \gamma + n/2, \quad \tilde{m} = \gamma - n/2, \quad \gamma = 1/2 + i\nu,$$

$$E = \epsilon_m + \epsilon_{\tilde{m}}, \quad \epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1), \quad \Delta = \frac{g^2 N_c}{\pi^2} \ln 2$$

## 5 Effective action in QCD

Locality of the theory in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}, \quad |y - y_0| < \eta, \quad \eta \ll \ln s$$

Gluon and Reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x), \quad A_\pm(x) = -iT^a A_\pm^a(x), \quad \delta A_\pm(x) = 0$$

Effective action for their interactions (L., 1995)

$$S = \int d^4x \left( L_{QCD} + Tr(V_+ \partial_\mu^2 A_- + V_- \partial_\mu^2 A_+) \right),$$

$$V_+ = -\frac{1}{g} \partial_+ P \exp \left( -g \int_{-\infty}^{x^+} v_+(x') d(x')^+ \right) = v_+ - g v_+ \frac{1}{\partial_+} v_+ + \dots$$

# 6 Production amplitudes in $N = 4$ SUSY

Relative correction  $R$  to the BDS amplitude  $A_{2 \rightarrow 4}$  (F.,L., 2011)

$$R e^{i\pi\delta} = \cos \pi\omega_{ab} + i \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n e^{i\phi n} \int_{-\infty}^{\infty} \frac{|w|^{2i\nu} d\nu}{\nu^2 + \frac{n^2}{4}} \Phi(\nu, n) \left( \frac{1}{u_1 - 1} \right)^{\omega(\nu, n)},$$

$$u_1 = \frac{ss_2}{s_{012}s_{123}}, \quad u_2 = \frac{s_1t_3}{s_{012}t_2}, \quad u_3 = \frac{s_3t_1}{s_{123}t_2}, \quad |w|^2 = \frac{u_2}{u_3},$$

$$\cos \phi = \frac{1 - u_1 - u_2 - u_3}{2\sqrt{u_2 u_3}}, \quad \delta = \frac{\gamma_K}{8} \ln \frac{|w|^2}{|1 + w|^4}, \quad \omega_{ab} = \frac{\gamma_K}{8} \ln |w|^2,$$

$$\omega(\nu, n) = -a E_{\nu, n} - a^2 (\epsilon_{\nu n} + 3\zeta(3)), \quad E_{\nu n} = -\frac{1}{2} \frac{|n|}{\nu^2 + \frac{n^2}{4}} + 2\Re \psi(1 + i\nu + \frac{|n|}{2}) - 2\psi(1),$$

$$\epsilon_{\nu n} = -\frac{\Re}{2} \left( \psi''(1 + i\nu + \frac{|n|}{2}) - \frac{2i\nu\psi'(1 + i\nu + \frac{|n|}{2})}{\nu^2 + \frac{n^2}{4}} \right) - \zeta(2) E_{\nu n} - \frac{1}{4} \frac{|n| \left( \nu^2 - \frac{n^2}{4} \right)}{\left( \nu^2 + \frac{n^2}{4} \right)^3}$$

## 7 Pomeron and graviton in $N = 4$ SUSY

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\nu^2, \quad \gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the energy-momentum conservation

$$D = \Delta$$

AdS/CFT relation for the graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2} t, \quad t = E^2/R^2, \quad \alpha' = \frac{R^2}{2} \Delta$$

Large coupling asymptotics for  $\gamma$  and  $\Delta$  (KLOV, BPST)

$$\gamma = -\Delta \sqrt{j-2+\Delta}, \quad \Delta = \frac{1}{\sqrt{g^2 N_c}}$$

# 8 Perturbation theory in gravity

Einstein-Hilbert action

$$S_{EH} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R, \quad R = R_{\mu\nu} g^{\mu\nu}$$

Riemann tensor

$$R_{\mu\nu} = R_{\mu,\sigma\nu}^\sigma, \quad R_{\mu,\alpha\beta}^\sigma = \partial_\beta \Gamma_{\mu\alpha}^\sigma - \partial_\alpha \Gamma_{\mu\beta}^\sigma + \Gamma_{\mu\alpha}^\rho \Gamma_{\rho\beta}^\sigma - \Gamma_{\mu\beta}^\rho \Gamma_{\rho\alpha}^\sigma$$

Christophel symbol and gravity field

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

General coordinate transformation

$$\delta h_{\mu\nu} = D_\mu \chi_\nu + D_\nu \chi_\mu, \quad D_\mu \chi_\nu = \partial_\mu \chi_\nu - \Gamma_{\mu\nu}^\rho \chi_\rho$$

# 9 High energy amplitudes in gravity

Production amplitudes in LLA (L.L. (1982))

$$A_{2 \rightarrow n} = -s^2 \Gamma_{\mu\nu}^{\mu'\nu'} \frac{s_1^{\omega(q_1^2)}}{q_1^2} \Gamma_{\rho_1\sigma_1} \frac{s_2^{\omega(q_2^2)}}{q_2^2} \Gamma_{\rho_2\sigma_2} \dots \Gamma_{\rho\sigma}^{\rho'\sigma'}$$

Graviton-graviton-reggeon vertex

$$\Gamma_{\mu\nu}^{\mu'\nu'} = \frac{\kappa}{4} (\Gamma_{\mu\mu'} \Gamma_{\nu\nu'} + \Gamma_{\mu\nu'} \Gamma_{\nu\mu'})$$

Gluon-gluon-reggeized gluon vertex

$$\Gamma_{\mu\mu'} = -\delta_{\mu\mu'} + \frac{p_{\mu'}^A p_\mu^B + p_\mu^{A'} p_{\mu'}^B}{p^A p^B} + \frac{q^2}{2} \frac{p_\mu^B p_{\mu'}^B}{(p^A p^B)^2}$$

Reggeon-reggeon-graviton vertex

$$\Gamma_{\rho\sigma} = \frac{\kappa}{4} (C_\rho C_\sigma - N_\rho N_\sigma), \quad N = \sqrt{q_1^2 q_2^2} \left( \frac{p^A}{kp^A} - \frac{p^B}{kp^B} \right)$$

# 10 Graviton trajectory and amplitudes

Graviton Regge trajectory (L. (1982))

$$\omega(q^2) = \frac{a}{\pi} \int \frac{q^2 d^2 k}{k^2 (q - k)^2} f(k, q), \quad a = \frac{\kappa^2}{8\pi^2},$$

$$f(k, q) = (k, q - k)^2 \left( \frac{1}{k^2} + \frac{1}{(q - k)^2} \right) - q^2 + \frac{N}{2}(k, q - k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \int d^4 x \sum_{r=1}^N \bar{\psi}_\mu^r \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma^r$$

Amplitudes in the DL approximation

$$A = A_{Born} s^{-a \ln \frac{q^2}{\lambda^2}} \int \frac{d\omega}{2\pi i} s^\omega \frac{f_\omega}{\omega}, \quad f_\omega = 1 - \frac{N-4}{2} a \frac{f_\omega^2}{\omega^2} - \frac{a}{2\omega} \frac{d}{d\omega} f_\omega$$

# 11 Effective action for gravity

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}, \quad |y - y_0| < \eta, \quad \eta \ll \ln s$$

Reggeized graviton fields

$$\delta A^{++}(x) = \delta A^{--}(x) = 0, \quad \partial_+ A^{++}(x) = \partial_- A^{--}(x) = 0$$

Effective action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa} \int d^4x \left( \sqrt{-g} R + \frac{1}{2} (\partial_+ j^- \partial_\mu^2 A^{++} + \partial_- j_+ \partial_\mu^2 A^{--}) \right)$$

Hamilton-Jacobi equation for effective currents

$$g^{\mu\nu} \partial_\mu j^\pm \partial_\nu j^\pm = 0, \quad \partial_\pm j^\mp = h_{\pm\pm} - \left( h_{\rho\pm} - \frac{1}{2} \frac{\partial_\rho}{\partial_\pm} h_{\pm\pm} \right)^2 + \dots$$

## 12 Effective currents for shock waves

Aichelburg - Sexl metric

$$(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu + a \ln |\vec{x}| \delta(x^-) (dx^-)^2, \quad a = \frac{8}{\sqrt{2}} G \mu, \quad z = a \frac{x^-}{|\vec{x}|^2}$$

Effective current for the shock wave

$$j^+ = -a \mu \left( \ln |\vec{x}| + \ln f(z) - \frac{1}{4} \frac{z}{f^2(z)} \right), \quad f(z) = \frac{1}{2} + \frac{\sqrt{1+2z}}{2}$$

Perturbative expansion

$$j^+ = -a \ln |\vec{x}| + \frac{a^2}{\partial_-} \left( \frac{x_\sigma}{2|\vec{x}|} \right)^2 - \frac{a^3}{\partial_-} \frac{x_\mu}{2|\vec{x}|} \frac{\partial_\mu}{\partial_-} \left( \frac{x_\sigma}{2|\vec{x}|} \right)^2 + \dots$$

Variational principle for  $j^+$

$$j^+ = \int_{-\infty}^{x^-} (g^{++}(y^-, \vec{\rho}(y^-)) + (\partial_- \vec{\rho})^2), \quad \frac{\delta j^+}{\delta \vec{\rho}(y^+)} = 0$$

## 13 Discussion

1. Locality of reggeon interactions in the rapidity space.
2. BFKL equation for Pomeron wave function
3. High energy effective action for gluons in QCD.
4. Pomeron-graviton duality in  $N = 4$  SUSY.
5. Multi-regge processes in gravity.
6. The graviton trajectory and double logarithms.
7. Effective action for the high energy gravity.
8. Hamilton-Jacobi equation for effective currents.