Representation of a Beta-function and Anomalous Dimensions Suitable for Multi-loop Numerical Calculations

L.Ts. Adzhemyan, M.V. Kompaniets

Department of Theoretical Physics,
Saint Petersburg State University

October 5, 2011

The following questions will be the subject of report

- renormalization group and ε-expansion
- numerical calculation
- main objects of interest are anomalous dimensions and beta-function

Global Objective:

- to develop an approach that allows to perform automated multi-loop calculations in a wide range of models

To solve this problem we constructed representation for anomalous dimensions via convergent integrals suitable for numerical calculations.

Current Objectives

- perform 4-loop calculations in ϕ^3-model - completed

  3-loop:
  O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane , J.Phys A: Math Gen. 13 (1980) L245-L251
  O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane , J.Phys A: Math Gen. 14 (1981) 2391-2413

- perform 6-loop calculations in ϕ^4-model - in progress

  5-loop:
  Chetyrkin K.G, Kataev A.L., Tkachev F.V., Phys.Lett., B99, 147 (1981); B101,457(E) (1981)
  Gorihny S.S., Larin S.A.,Tkachev F.V., Phys.Lett., A101, 120 (1984)
  Kazakov D.I., Phys.Lett., B133, 406 (1983)
  Kleinert H., Neu J., Shulte-Frohlinde V., Chetyrkin K.G., Larin S.A., Phys.Lett., B272,39 (1991); Erratum: B319, 545 (1993)

- to generalize the developed approach to models of critical dynamics - planned

A common scheme used for calculations in the framework of ε-expansion:

- calculation of renormalization constants Z (singular functions containing poles on ε)
- using the obtained Z calculation of anomalous dimensions, beta-function, fixed point and other quantities (all poles on ε cancel each other)

$$\gamma_i = \beta \partial_g \ln Z_i$$

L. Ts. Adzhemyan and M. V. Kompaniets, Theoretical and Mathematical Physics, 169(1): 1450–1459 (2011)

On this way we have found

- renormalization scheme suitable for numerical calculation
- representation for residues at poles on ε in the form of well-convergent integrals

Outcome:

- 4-loop corrections for anomalous dimensions in ϕ^3 model calculated.
- The results for anomalous dimensions seems to be much more simple than for renormalization constants.

Main result

representation of anomalous dimensions through renormalized quantities (derivative of 1-irreducible function with respect to mass)

$$\gamma_i = \frac{2f_i}{1+f_2}\,, \qquad i = 2,\,3\,. \tag{1}$$

$$f_i = -R(m^2\partial_{m^2}\bar{\Gamma}_i)|_{m=\mu}\,, \tag{2}$$

Advantages:

- presence of R-operation in (2) guarantees UV finiteness of calculated quantities
- R operation constructed in such a way that **no poles on ϵ arise at any step of calculations** ("Theory without divergences")
- quantities f_i can be expressed in suitable for numerical calculation form.

Model ϕ^3 $(d = 6 - \varepsilon)$

Renormalized action :

$$S = -\frac{1}{2}(m^2 Z_1 + p^2 Z_2 + \delta m^2)\varphi^2 - \frac{1}{6} g \mu^{\varepsilon/2} Z_3 \varphi^3 \,. \tag{3}$$

$$Z_1 = Z_{m^2} Z_\varphi^2 \,, \qquad Z_2 = Z_\varphi^2 \,, \qquad Z_3 = Z_g Z_\varphi^3 \tag{4}$$

bare quantities:

$$m_0^2 = m^2 Z_{m^2} \,, \qquad g_0 = g \mu^{\varepsilon/2} Z_g \,, \qquad \varphi_0 = \varphi Z_\varphi \,. \tag{5}$$

Renormalization constants Z_i eliminate divergences in $\Gamma_2 = \langle \varphi\varphi \rangle_{1-ir}$ and $\Gamma_3 = \langle \varphi\varphi\varphi \rangle_{1-ir}$

R operation

$$\Gamma^R = R\Gamma = (1-K)R'\Gamma, \tag{6}$$

Where R' - incomplete R operation (eliminating divergences in subgraphs)

$1-K$ - eliminates remaining superficial divergence

The choice of a particular operator K amounts to the choice of subtraction scheme

	MS (minimal subtraction)	ZM (zero-momentum)
K	only poles on ε	momentum series
	Anomalous dimensions do not depend on mass, **the RG equations have a simple form**	Renormalized quantities can be represented in the form of a **"theory without divergences"**

R operation in "theory without divergences"

$$\Gamma^R = R\Gamma = \prod_i (1 - K_i)\Gamma, \tag{7}$$

For zero-momentum subtraction scheme (ZM) we may use following representation

$$(1 - K)F(k) = F(k) - \sum_{m=0}^{n} \frac{k^m}{m!} F^{(m)}|_{k=0} = \frac{1}{n!} \int_0^1 da(1-a)^n \partial_a^{n+1} F(ak). \tag{8}$$

$$R\chi = \prod_i \frac{1}{n_i!} \int_0^1 da_i (1 - a_i)^{n_i} \partial_{a_i}^{n_i+1} \chi(\{a\}), \tag{9}$$

product is taken over relevant subgraphs χ_i (including diagram χ as a whole)

n_i - subgraph dimension

a_i -a parameters that stretch moments flowing into subgraph χ_i

- Here subtraction counterterms are replaced with differentiation operation
- There are no poles on ε in the differentiated diagrams in each term

O. I. Zav'yalov, Renormalized Quantum Field Theory [in Russian], Nauka, Moscow (1979); English transl., Kluwer, Dordrecht (1990)

For our purposes subtraction scheme should meet the following requirements:

- (Req. 1) Allow to express anomalous dimensions through quantities, which are possible to calculate using ("theory without divergences")

$$R\chi = \prod_i \frac{1}{n_i!} \int_0^1 da_i (1-a_i)^{n_i} \partial_{a_i}^{n_i+1} \chi(\{a\}), \tag{10}$$

- (Req. 2) Anomalous dimensions do not depend on mass (like in MS)

Req.	MS minimal subtraction	ZM zero-momentum	NP normalization point
1	no	yes	for some quantities
2	yes	no	yes

$$S = -\frac{1}{2}(m^2 Z_1 + p^2 Z_2 + \delta m^2)\varphi^2 - \frac{1}{6} g \mu^{\varepsilon/2} Z_3 \varphi^3 \,. \tag{11}$$

Zero-momentum subtraction scheme (ZM)

Renormalization constants are defined by following conditions:

$$\Gamma_3^R \,|_{\{p\}=0} = -g\mu^{\varepsilon/2} \,, \quad \Gamma_2^R \,|_{p=0} = -m^2 \quad (\partial_{p^2}\Gamma_2^R)\,|_{p=0} = -1 \,. \tag{12}$$

and δm^2 is defined by

$$\Gamma_2^R \,|_{p=0,m=0} = 0. \tag{13}$$

Normalization point subtraction scheme (NP)

$$\Gamma_3^R \,|_{\{p\}=0,m=\mu} = -g\mu^{\varepsilon/2} \,, \quad \Gamma_2^R \,|_{p=0,m=\mu} = -\mu^2 \,, \quad (\partial_{p^2}\Gamma_2^R)\,|_{p=0,m=\mu} = -1 \,, \tag{14}$$

$$\Gamma_2^R \,|_{p=0,m=0} = 0. \tag{15}$$

This modification grants that anomalous dimensions do not depend on mass (like in MS scheme).

Subtraction operation K for NP scheme

$$K\Gamma_3 = \Gamma_3 \mid_{\{p\}=0,m=\mu} , \tag{16}$$

$$K\Gamma_2 = \Gamma_2 \mid_{p=0,m=0} + \frac{m^2}{\mu^2} \left(\Gamma_2 \mid_{p=0,m=\mu} - \Gamma_2 \mid_{p=0,m=0}\right) + \\ + p^2 \cdot \left(\partial_{p^2}\Gamma_2\right) \mid_{p=0,m=\mu} . \tag{17}$$

For quantities in which substitution $m = \mu$ performed, operation (17) becomes:

$$\left(K\Gamma_2\right) \mid_{m=\mu} = \Gamma_2 \mid_{p=0,m=\mu} + p^2 \cdot \left(\partial_{p^2}\Gamma_2\right) \mid_{p=0,m=\mu} . \tag{18}$$

This allows to use for these quantities representation

$$R\chi = \prod_i \frac{1}{n_i!} \int_0^1 da_i (1-a_i)^{n_i} \partial_{a_i}^{n_i+1} \chi(\{a\}) .$$

Anomalous dimensions

Let us consider normalized 1-irreducible Green functions satisfying the conditions:

$$\bar{\Gamma}_i^R|_{m=\mu} = 1\,, \qquad i = 1,\,2,\,3\,. \tag{19}$$

$$\bar{\Gamma}_1 = -\left(\frac{\Gamma_2\,|_{p=0} - \Gamma_2|_{p=0,m=0}}{m^2}\right), \quad \bar{\Gamma}_2 = -(\partial_{p^2}\Gamma_2)\,|_{p=0}\,,$$
$$\bar{\Gamma}_3 = \Gamma_3\,|_{\{p\}=0}\,/(-g\mu^{\varepsilon/2})\,,$$

RG equations have the following form (the same as in MS scheme):

$$\left(\mu\partial_\mu + \beta\partial_g - \gamma_{m^2} m^2 \partial_{m^2} - \gamma_i\right)\bar{\Gamma}_i^R = 0\,, \qquad i = 1,\,2,\,3\,, \tag{20}$$

If we consider RG equations at $m = \mu$ and take into account that $\mu\partial_\mu\bar{\Gamma}_i^R = -2m^2\partial_{m^2}\bar{\Gamma}_i^R$, $\partial_g\bar{\Gamma}_i^R|_{m=\mu} = 0$ we obtain

$$(2 + \gamma_1 - \gamma_2)\,F_i = \gamma_i\,, \qquad i = 1,\,2,\,3\,. \tag{21}$$

where

$$F_i = -(m^2\partial_{m^2}\bar{\Gamma}_i^R)|_{m=\mu}\,. \tag{22}$$

Solving equations (21) we get:

$$\gamma_i = \frac{2F_i}{1 + F_2 - F_1}\,, \qquad i = 1, 2, 3 \tag{23}$$

$$F_i = -(m^2 \partial_{m^2} \bar{\Gamma}_i^R)|_{m=\mu} \,. \tag{24}$$

F_i are finite quantities, but they are not suitable for calculations due to complicated subtraction operation:

$$K\Gamma_2 = \Gamma_2 \,|_{p=0,m=0} + \frac{m^2}{\mu^2} \left(\Gamma_2 \,|_{p=0,m=\mu} - \Gamma_2 \,|_{p=0,m=0}\right) +$$

$$+ p^2 \cdot (\partial_{p^2} \Gamma_2) \,|_{p=0,m=\mu}$$

Thus we cant use representation

$$R\chi = \prod_i \frac{1}{n_i!} \int_0^1 da_i (1-a_i)^{n_i} \partial_{a_i}^{n_i+1} \chi(\{a\}), \tag{25}$$

More suitable quantities are

$$f_i = -R(m^2 \partial_{m^2} \bar{\Gamma}_i)|_{m=\mu} \,,$$

for which subtraction has the simple form:

$$(K\Gamma_2) \,|_{m=\mu} = \Gamma_2 \,|_{p=0,m=\mu} + p^2 \cdot (\partial_{p^2} \Gamma_2) \,|_{p=0,m=\mu} \,, \tag{26}$$

and representation (25) is possible

The main result

It is possible to express quantities F_i in terms of f_i

$$F_i = \frac{f_i}{1 + f_1}\,, \qquad i = 2,\, 3\,, \tag{27}$$

and obtain representation:

$$\gamma_i = \frac{2f_i}{1 + f_2}\,, \qquad i = 2,\, 3\,. \tag{28}$$

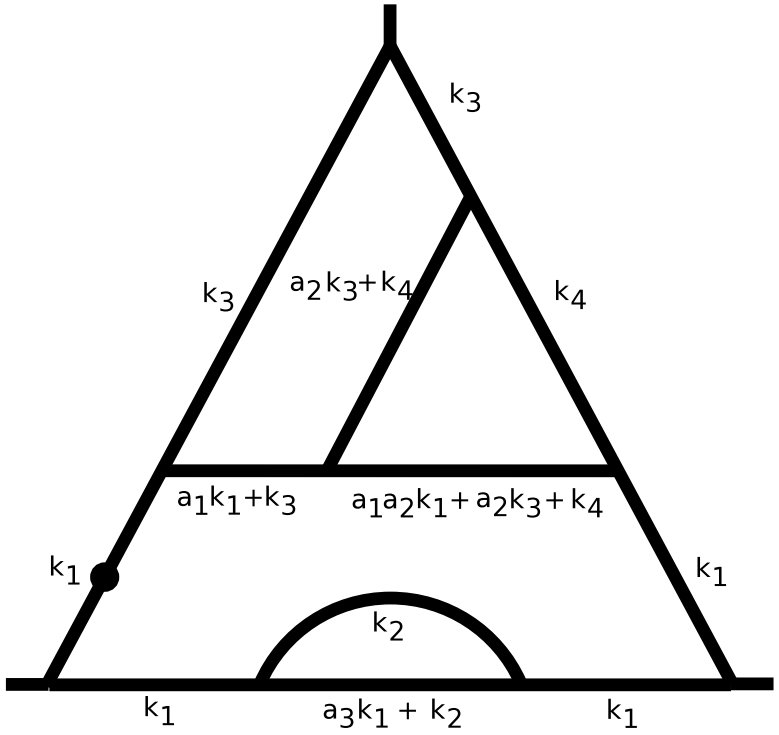
this is desired representation of anomalous dimensions that satisfies our requirements.
Although relation (27) looks very simple, it took much effort to prove it.

Thus,

- anomalous dimensions are calculated **directly from diagrams** of 1-irreducible functions
- the contribution of each diagram is presented in **a form of UV-finite integral**

Example of quantities, which is necessary to calculate

for $f_3 = -R(m^2\partial_{m^2}\bar{\Gamma}_3)$ we need to calculate $R\chi$, where χ is



$$R\chi = \frac{1}{2}\int_0^1 da_1\,\partial_{a_1}\int_0^1 da_2\,\partial_{a_2}\int_0^1 da_3\,(1-a_3)^2\,\partial_{a_3}^3\int d\,\{\mathbf{k}\}\;\chi_1(\{a\},\{\mathbf{k}\}),$$

where

$$\chi_1^{-1}(\{a\},\{\mathbf{k}\}) = (k_1^2+1)^5(k_2^2+1)(k_3^2+1)^2(k_4^2+1)\left[(a_1a_2\mathbf{k_1}+a_2\mathbf{k_3}+\mathbf{k_4})^2+1\right]\cdot$$
$$\cdot\left[(a_3\mathbf{k_1}+\mathbf{k_2})^2+1\right]\left[(a_1\mathbf{k_1}+\mathbf{k_3})^2+1\right]\left[(\mathbf{k_3}+\mathbf{k_4})^2+1\right].$$

This approach allows to calculate 4-loop corrections to critical exponents in model ϕ^3

$$\eta = -0.1111\epsilon - 0.0588\epsilon^2 + 0.0439\epsilon^3 - 0.081\epsilon^4 + O(\epsilon^5)$$

Analytical calculations of the 3rd order of ϵ-expansion:

- O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane, J.Phys A: Math Gen. 13 (1980) L245-L251
- O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane, J.Phys A: Math Gen. 14 (1981) 2391-2413

$$\eta = -\frac{\epsilon}{9} - \frac{43\epsilon^2}{3^6} + \epsilon^3\left(-\frac{8375}{2^2 3^{10}} + \frac{16\zeta(3)}{3^5}\right) + O(\epsilon^4) \approx$$

$$\approx -0.11111\epsilon - 0.05898\epsilon^2 + 0.04367\epsilon^3 + O(\epsilon^4)$$

Our current work is concentrated on 6-loop calculations of index η (critical dimension of field) in model ϕ^4