

# The absence of QCD $\beta$ -function factorization property of the generalized Crewther relation in the 't Hooft scheme

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Among fundamental consequences of the conformal symmetry in the theory of strong interactions is the existence of the quark-parton model Crewther relation [1], namely

$$C_D^{NS} \times C_{Bjp} = 1. \quad (1)$$

The application of renormalization procedure, which result in the appearance of the QCD  $\beta$ -function, leads to the violation of the conformal symmetry in QCD and to the appearance of the additional conformal symmetry breaking contribution  $\Delta_{csb}$  in the r.h.s. of Eq.(1)

$$C_D^{NS}(a_s(Q^2)) \times C_{Bjp}(a_s(Q^2)) = 1 + \sum_{i \geq 1} a_s^{i+1} \lambda_i = 1 + \Delta_{csb}(a_s(Q^2)).$$

The absence of the  $a_s$ -term in the latter equation is the consequence of the validity of (1) in the conformal invariant limit. In this limit the related equation reads

$$C_D^{NS}(a_s) \times C_{Bjp}(a_s)|_{c-i} = 1. \quad (2)$$

## Definitions

$$D^{NS}(Q^2, \alpha_s) = \left( N_c \sum_f Q_f^2 \right) C_D^{NS}(a_s) \approx Q^2 \int_0^\infty \frac{R(s)}{(s + Q^2)^2} ds, \quad (3)$$

where  $R(s)$  is

$$R(s) = \frac{\sigma(e^+e^- \Rightarrow \text{hadrons})}{\sigma_{Born}(e^+e^- \Rightarrow \mu^+\mu^-)}. \quad (4)$$

Here  $C_D^{NS}$  is the normalized Green function, related to the characteristic of the  $e^+e^-$ -annihilation to hadrons process, i.e. Adler D-function, defined as

$$D^{NS} = \left( 3 \sum_f Q_f^2 \right) C_D^{NS},$$

where  $Q_f$  are the electric charges of quarks.

## Definitions

The term  $C_{Bjp}$  in (1) is the Green function, which is proportional to the Bjorken sum rule for the deep-inelastic scattering of polarized leptons on nucleons, namely

$$S_{Bjp}(Q^2, \alpha_s) = \int_0^1 \left[ g_1'^{-p}(x, Q^2) - g_1'^{-n}(x, Q^2) \right] dx = \frac{1}{6} \frac{g_A}{g_V} C_{Bjp}(\alpha_s),$$

where  $g_1'^{-p}, g_1'^{-n}$  - spin-dependent proton and neutron structure functions,  $g_A$  is an axial nucleon decay constant, and  $g_V$  is vector one. They can be derived from the experiments (e.g. in neutron  $\beta$ -decay process.)

Within perturbative QCD both Green functions under consideration can be written as an expansion in the coupling constant as

$$C_D^{NS}(Q^2) = 1 + d_0 a_s + \sum_{i=2}^{\infty} d_0 d_{i-1} a_s^i(Q^2) \quad (5)$$

$$C_{Bjp}(Q^2) = 1 + c_0 a_s + \sum_{i=2}^{\infty} c_0 c_{i-1} a_s^i(Q^2), \quad (6)$$

where  $a_s = \alpha_s/\pi$  obeys the following renormalization group equation

$$\mu^2 \frac{\partial a_s}{\partial \mu^2} \equiv \beta(a_s) = - \sum_{i \geq 0} \beta_i a_s^{i+2}. \quad (7)$$

Since Eq. (2) is true , the following identity holds:

$$d_0 = -c_0$$

It leads to the absence of the  $\alpha_s$ -term in  $C_D^{NS} C_{Bjp}$  OPE.

There are two factorizations of  $\frac{\beta(a_s)}{a_s}$  in  $\overline{MS}$ : factorization  $\frac{\beta(a_s)}{a_s}$ -factorization in expression for  $\Delta_{csb}$  - theorem [11]

$$\Delta_{csb}(a_s) = \left( \frac{\beta(a_s)}{a_s} \right) \sum_{m \geq 1} K_m a_s^m; \quad (8)$$

$\left( \frac{\beta(a_s)}{a_s} \right)^n$ -factorization in  $\Delta_{csb}$  - hypothesis [17]. (which validity in  $\overline{MS}$  in fourth order is proven.)

$$\begin{aligned} \Delta_{csb}(a_s) &= \sum_{n \geq 1} \left( \frac{\beta(a_s)}{a_s} \right)^n \mathcal{P}_n(a_s) \equiv \sum_{n \geq 1} \sum_{r \geq 1} \left( \frac{\beta(a_s)}{a_s} \right)^n P_n^{(r)} a_s^r \equiv \\ &\equiv \sum_{n \geq 1} \sum_{r \geq 1} \left( \frac{\beta(a_s)}{a_s} \right)^n P_n^{(r)}[k, m] C_F^k C_A^m a_s^r, \end{aligned} \quad (9)$$

$P_n^{(r)}[k, m]$  don't depend on Casimir operators and on structure constants in the higher orders of perturbation theory.

## 't Hooft scheme definition

It is known that in gauge theories with single coupling constant first two coefficients of the  $\beta$ -function, defined in Eq.(7), are scheme-independent. In his works of Refs. [18], [19] 't Hooft proposed to use in the theoretical studies the scheme, which is characterised by  $\beta$ -function with two scheme-independent coefficients only:

$$\mu^2 \frac{\partial a_H}{\partial \mu^2} = \beta(a_H) = -\beta_0 a_H^2 - \beta_1 a_H^3 \quad (10)$$

Here and further indexes H label 't Hooft-scheme parameters. The nullification of higher order coefficients of the  $\beta$ -function in another scheme is achieved by finite renormalizations of charge. They are changing the expressions for the coefficients of perturbation theory series for Green functions, evaluated in the concrete renormalization scheme, say  $\overline{MS}$ -scheme.



Green functions are physical values so they shouldn't be dependant on renormalization scheme. Coefficients foregoing the each power of coupling constant should be chosen in order to keep the value of Green function invariant. And, of course, the value of coupling constant changes if the  $\beta$ -function does. It all means that when recomputing the foregoing coefficients coupling constant automatically becomes calculated with 't Hooft  $\beta$ -function.

Vector currents are also being conserved in 't Hooft scheme. It is rather trivial fact. Indeed, in 2-loop calculations 't Hooft-scheme vector current conserves just because it is equal to  $\overline{MS}$ -scheme one, and in higher loops 't Hooft-scheme vector current is identically equal to zero according to  $\beta$ -function definition (7). There just aren't any higher-loop propagator Feynman diagrams.

Within scheme invariants technique one can obtain 't Hooft -scheme coefficient functions:

$$d_2^{t'Hooft} = d_2^{\overline{MS}} + \frac{\beta_2^{\overline{MS}}}{\beta_0} \quad (11)$$

$$d_3^{t'Hooft} = d_3^{\overline{MS}} + \frac{1}{2} \frac{\beta_3^{\overline{MS}}}{\beta_0} + 2d_1 \frac{\beta_2^{\overline{MS}}}{\beta_0} \quad (12)$$

Absolutely identical formulae can be obtained for the coefficients  $c_i$  of Eq.(6) for perturbation theory expansion of the Bjorken polarized sum rule coefficient function.

One can write down Adler and Bjorken coefficient functions up to the fourth order in the coupling constant in 't Hooft scheme, using (11), (12).

**Statement.** In the 't Hooft  $\overline{MS}$ -based scheme there is no explicit factorization of the terms  $(\beta(a_H)/a_H)$  in the QCD generalizations of Crewther relations of (8) and (9). This unexpected feature distinguishes it from  $\overline{MS}$ -scheme (or any other version of MS-scheme) and is raising the questions of applicability of the 't Hooft scheme for revealing theoretical effects, hidden in analytical high-order corrections to Green functions and  $\beta$ -function as well.

It is possible to prove that

$$K_1^H = K_1^{\overline{MS}}$$

$$K_2^H = K_2^{\overline{MS}}$$

$$K_3^H = K_3^{\overline{MS}} + 3K_1 \frac{\beta_2^{\overline{MS}}}{\beta_0} \quad (13)$$

Explicitly,

$$K_1 = \left(-\frac{21}{8} + 3\zeta(3)\right)C_F \quad (14)$$

$$\begin{aligned} K_2 = & \left(\frac{397}{96} + \frac{17}{2}\zeta(3) - 15\zeta(5)\right)C_F^2 + \\ & + \left(-\frac{629}{32} + \frac{221}{12}\zeta(3)\right)C_FC_A + \\ & + \left(\frac{163}{24} - \frac{19}{3}\zeta(3)\right)C_FT_FN_F \end{aligned}$$

$$\begin{aligned}
K_3^{\overline{MS}} = & C_F^3 \left( \frac{2471}{768} + \frac{61}{8} \zeta(3) - \frac{715}{8} \zeta(5) + \frac{315}{4} \zeta(7) \right) + \\
& + C_F^2 C_A \left( \frac{99757}{2304} + \frac{8285}{96} \zeta(3) - \frac{1555}{12} \zeta(5) - \frac{105}{8} \zeta(7) \right) + \\
& + C_F^2 T_F N_F \left( -\frac{7729}{1152} - \frac{917}{16} \zeta(3) + \frac{125}{2} \zeta(5) + 9 \zeta^2(3) \right) + \\
& + C_F T_F^2 N_F^2 \left( -\frac{307}{18} + \frac{203}{18} \zeta(3) + 5 \zeta(5) \right) + \\
& + C_F C_A^2 \left( -\frac{406043}{2304} + \frac{18007}{144} \zeta(3) + \frac{2975}{48} \zeta(5) - \frac{77}{4} \zeta^2(3) \right) + \\
& + C_F C_A T_F N_F \left( \frac{1055}{9} - \frac{2521}{36} \zeta(3) - \frac{125}{3} \zeta(5) - 2 \zeta^2(3) \right)
\end{aligned}$$

$$\begin{aligned}
\beta_2^{\overline{MS}} = & \frac{2857}{3456} C_A^3 - \frac{1415}{1728} C_A^2 T_f N_f - \frac{205}{576} C_A C_F T_f N_f + \frac{79}{864} C_A T_f^2 N_F^2 + \\
& + \frac{1}{32} C_F^2 T_f N_f + \frac{11}{144} C_F T_f^2 N_f^2 \\
\beta_0 = & \frac{11}{12} C_A - \frac{1}{3} T_f N_f
\end{aligned} \tag{15}$$

In 't Hooft scheme conformal symmetry breaking term can be written as:

$$\Delta_{csb}^{t'Hooft} = \left( -\beta_0 a_H - \beta_1 a_H^2 \right) \left( K_1 a_H + K_2 a_H^2 + K_3 a_H^3 \right) - \\ - 3\beta_2 a_H^4 + \mathcal{O}(a_H^5)$$

The latter equation cannot be simplified. There is no explicit  $\beta(a_s)/a_s$ -factorization in 't Hooft renormalization scheme, and the reason is the additive term.

## QED case

Unlike the QCD case, in QED  $\frac{\beta_2}{\beta_0}$  in (13) can be rewritten as

$$\frac{\beta_2}{\beta_0} = -\frac{3}{32}C_F^2 - \frac{11}{48}C_F T_F N_F, \quad (16)$$

hence in QED factorization property exists.



QED case ( $C_F = 1, T_F = 1$ )

$$\begin{aligned}
C_D^{NS} C_{Bjp} \Big|_{t' \text{ Hooft}} &= 1 + \left( \frac{1}{3} T_F N_F a_H + \frac{1}{4} C_F T_F N_F a_H^2 \right) \cdot \\
&\cdot \left( a_H C_F \left( -\frac{21}{8} + 3\zeta(3) \right) + a_H^2 \left[ C_F^2 \left( \frac{397}{96} + \frac{17}{2} \zeta(3) - 15\zeta(5) \right) + \right. \right. \\
&\quad \left. \left. + C_F T_F N_F \left( \frac{163}{24} - \frac{19}{3} \zeta(3) \right) \right] + a_H^3 \cdot K_{3_{t' \text{ Hooft}}} \right) + \mathcal{O}(a_H^5)
\end{aligned}$$

QED case ( $C_F = 1, T_F = 1$ )

$$\begin{aligned}
K_{3_{t'Hooft}} &= K_{3_{\overline{MS}}} + C_F^3 \left( \frac{189}{256} - \frac{27}{32} \zeta(3) \right) + C_F^2 T_F N_F \left( \frac{231}{128} - \frac{33}{16} \zeta(3) \right) = \\
&= C_F^3 \left( \frac{1519}{384} + \frac{217}{32} \zeta(3) - \frac{715}{8} \zeta(5) + \frac{315}{4} \zeta(7) \right) + \\
&+ C_F^2 T_F N_F \left( -\frac{2825}{576} - \frac{475}{8} \zeta(3) + \frac{125}{2} \zeta(5) + 9 \zeta^2(3) \right) + \\
&+ C_F T_F^2 N_F^2 \left( -\frac{307}{18} + \frac{203}{18} \zeta(3) + 5 \zeta(5) \right)
\end{aligned}$$

## Discussions

We found out that in 't Hooft scheme in QCD there is no explicit factorization of  $\beta$ -function in conformal symmetry breaking term  $\Delta_{csb}$ . In  $\Delta_{csb}$ -term there arises an additional factor which is not proportional to 't Hooft  $\beta$ -function.

## Discussions

There are also some observations of pros and cons of using 't Hooft procedure. In  $\mathcal{N} = 1$  Yang-Mills SUSY model (without matter fields) Ward identities and covariant derivative regularization for multiloop calculations lead to that fact that [40][43]  $\beta$ -function can be written as a geometric progression.[44] And if we expand the geometric progression and throw out all the coefficients but the first two, we couldn't see that it is geometric progression. Hence, usage of 't Hooft procedure couldn't help us in finding out peculiar specialities of  $\mathcal{N} = 1$  SUSY QED model. There is one more question to 't Hooft procedure - it cannot be formulated in Feynman diagram language.

## Discussions

But there are also some convincing benefits of using 't Hooft procedure. Sometimes it is much easier to solve 2-loop renormgroup equation instead of 3-loop (in  $\overline{MS}$ -scheme 3-loop renormgroup equation cannot be solved precisely). There is such an approach as analytic perturbation theory.[32] As it is well-known, the exact solution of renormgroup equation for 2-loop  $\beta$ -function is well-known Lambert  $W$ -function. In case of 3-loop  $\beta$ -function the solution can be derived only using numerical methods; also there are some approximative methods of solving.[32]





## Discussions

Analytic perturbation theory method can be supplemented by the following: solution of the 2-loop renormgroup equation gives the explicit expression for '2-loop' coupling constant (depending on  $\frac{\mu^2}{\Lambda_{QCD_{\overline{MS}}}^2}$ ). Such an expression can be substituted into the perturbation theory serie for any Green function (for example, Adler function (with coefficients  $d_i$  calculated after 't Hooft procedure application!)) and, by that, one can get the explicit expression of the Green function in terms of the Lambert functions.

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