

Algebraic Approach to Integrable Spin Chain Models with boundaries

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Plan

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- Braid group \mathcal{B}_{M+1}
- Hecke and Symmetric Group Algebras

2 Algebraic Solutions of Yang-Baxter Equations

- Yang-Baxter equations (YBE)
- Bethe subalgebras

3 Applications: Integrable chain models

- Integrable Hecke chain models
- Hecke chain for corner type and $(n - 2, 2)$ representations.

4 The diagrams \leftrightarrow Perturbative integrals

- Feynman diagrams (F.D.) \leftrightarrow Yang-Baxter Eq.

5 Operator formalism

- Algebraic reformulation of integrals for F.D.: manipulations with integrals \rightarrow manipulations with operators

6 Applications

- Lipatov chain model.
- Ladder diagrams for phi³-theory in $D = 4$.

1. Braid group \mathcal{B}_{M+1}

A braid group \mathcal{B}_{M+1} is generated by invertible elements T_i ($i = 1, \dots, M$) subject relations:

$$\text{Braid : } T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$\text{Locality : } T_i T_j = T_j T_i \quad (|i - j| > 1),$$

We have elements

$$T_i^k \in \mathcal{B}_{M+1} \quad (\forall k = 1, 2, 3, \dots).$$

Thus, it is clear that

$$\dim(\mathcal{B}_{M+1}) = \infty.$$

1. Braid group \mathcal{B}_{M+1}

$$T_i = \begin{array}{ccccccccc} 1 & 2 & \dots & i & i+1 & i+2 & \dots & M & M+1 \\ \bullet & \bullet & & \bullet & \bullet & \bullet & & \bullet & \bullet \\ | & | & & | & \diagdown & | & & | & | \\ & & & & & & & & \\ & & & & & & & & \end{array} = \text{braid of threads}$$

Locality

Braid Relation

aid Relation

$$T_{i+1} T_i T_{i+1} = \begin{array}{c} \text{Diagram showing } T_{i+1} T_i T_{i+1} \text{ as a sequence of three boxes. The first and third boxes have vertical inputs and outputs. The middle box has a vertical input from the top of the first box and a vertical output to the bottom of the third box. The top and bottom horizontal edges of the middle box are crossed by diagonal edges connecting the left and right sides of the boxes. Arrows indicate flow from left to right. Labels } i, i+1, i+2 \text{ are above the boxes.}\end{array} = \begin{array}{c} \text{Diagram showing the result of the multiplication. The boxes are rearranged. The first and third boxes now have horizontal inputs and outputs. The middle box has a horizontal input from the left of the first box and a horizontal output to the right of the third box. The top and bottom vertical edges of the middle box are crossed by diagonal edges connecting the top and bottom of the boxes. Arrows indicate flow from left to right. Labels } i, i+1, i+2 \text{ are above the boxes.}\end{array} = T_i T_{i+1} T_i$$

2. Hecke and Symmetric Group Algebras

Finite-dimensional quotients of braid group algebra:

1. Hecke algebra H_{M+1} :

$$(T_i - q)(T_i + q^{-1}) = 0,$$

$$\dim(H_{M+1}) = (M+1)!$$

2. For $q^2 = 1$ we have $H_{M+1} \rightarrow$ group algebra of S_{M+1} .

$$T_j^2 = 1,$$

$$\rho(T_i) = \dots$$

$$= \delta_{j_1}^{k_1} \dots \delta_{j_{i+1}}^{k_i} \delta_{j_i}^{k_{i+1}} \dots \delta_{j_{M+1}}^{k_{M+1}}$$

Schur-Weyl duality \Rightarrow representations of $SL_q(N)$ and $SL(N)$.

3. Algebraic solutions of Yang-Baxter equations

Let x, y be *spectral* parameters. The Yang-Baxter equation is:

$$T_n(x) T_{n+1}(xy) T_n(y) = T_{n+1}(y) T_n(xy) T_{n+1}(x) \quad (\forall n).$$

where baxterized elements $T_n(x) \in H_{M+1}$ satisfy unitary condition:

$$T_n(x) T_n(x^{-1}) = 1.$$

Baxterized elements, or solutions of YBE, are

(V.Jones (1989), Y.Cheng, M.L.Ge, K.Xue (1991), A.P.I. (1995)):

$$T_n(x) = \frac{T_n - ax}{T_n - ax^{-1}} \in H_{M+1} \text{ or } \in BMW_{M+1},$$

where $a = \pm q^{\pm 1}$. It gives the polynomial solution for H_{M+1}
(M.Jimbo (1986)):

$$T_n(x) = \frac{x}{a - xa^{-1}} (T_n - xT_n^{-1}) \in H_{M+1}.$$

4. Bethe subalgebras

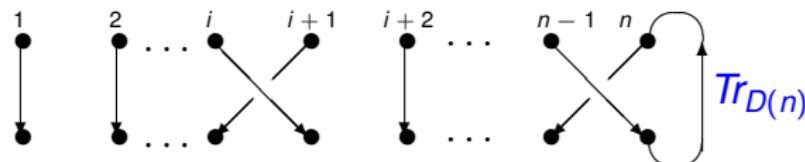
Having local solution $y_1(x)$ of RE one can construct nonlocal solution of RE ($\forall n$)

$$y_n(x) = \sigma_{n-1}(x) \cdots \sigma_1(x) \sigma_1(x) \cdots \sigma_{n-1}(x) ,$$

This is an algebraic analog of Sklyanin's monodromy matrix. Using standard procedure we construct two-row transfer-matrix operators:

$$\tau_{n-1}(x) = \textcolor{red}{Tr}_{D(n)}(y_n(x)) \in \hat{H}_{n-1} ,$$

where $\textcolor{blue}{Tr}_{D(n)}$ is a special map $\hat{H}_n \rightarrow \hat{H}_{n-1}$ and \hat{H}_{n-1} is generated by $\{\sigma_1, \dots, \sigma_{n-2}\}$



4. Bethe subalgebras

Transfer-matrix operators $\tau_{n-1}(x) = \text{Tr}_{D(n)}(y_n(x))$ form a commutative family:

$$[\tau_{n-1}(x), \tau_{n-1}(z)] = 0 \quad (\forall x, z \in \mathbf{C}).$$

The operators $\tau_{n-1}(x)$ are generating functions for **commutative Bethe subalgebras** in \hat{H}_{n-1} , or in $B\hat{M}W_{n-1}$.

From the point of view of integrable systems – $\tau_{n-1}(x)$ are generating functions for "integrals of motion" in integrable chain models.

The special element of the Bethe subalgebra is a Hamiltonians:

$$\mathcal{H} = \partial_x \tau_{n-1}(x)|_{x=1}$$

5. Integrable chain models

for Hecke algebra case we have

$$\mathcal{H} \sim \sum_{m=1}^{n-2} T_m ,$$

The Hamiltonians \mathcal{H} describe chain models with open boundary conditions in the first and last sites of the chain.

5. Integrable chain models

We propose algebraic formulation of integrable chain models.

The usual Hamiltonians of spin chains of $SL_q(N|K)$ type (for Hecke case) are obtained as $\rho(\mathcal{H})$, where ρ – special R -matrix representation of Hecke algebras. Such spin chains were considered by many authors

(P.P. Kulish, E.K. Sklyanin, L. Mezinchescu, R. Nepomechie, D. Arnaudon, E. Ragoucy,
W. L. Yang, Y. Z. Zhang, N. J. MacKay, B. J. Short,...)

Example.

The Hamiltonian of XXZ Heisenberg spin chain is obtained as $\rho_R(\mathcal{H})$ in the case of $SL_q(2)$ type R -matrix representation ρ_R of the Hecke algebra.

Any representation of Hecke algebras gives an integrable chain model.

5. Integrable chain models

We reduce the problem of searching of spectrum of **spin chain Hamiltonians** to the investigation of spectrum of **special elements \mathcal{H} in Bethe subalgebras** in Hecke algebras.

The problem is: how can we find the spectrum of $\rho(\mathcal{H})$ for all irreps ρ of $\hat{\mathcal{H}}_{M+1}$ algebras? This is a part of a more general math. problem: how can we reformulate the representation theory of Hecke algebras (known in the diagonal basis of Jucys - Murphy elements of Gelfand - Zetlin commutative subalgebra) in terms of the diagonal basis for the elements of Bethe commutative subalgebra.

5. Integrable chain models

How it works. The simplest Hamiltonian for Hecke chain with open boundary condition for special irreps $\rho_{k,\ell}$ is

$$\mathcal{H}_{k,\ell}(q) := \rho_{k,\ell} \left(\sum_{i=1}^{k+\ell} T_i \right) - (q k - q^{-1} \ell) \mathbf{1}.$$

We investigate the spectrum of this \mathcal{H} for special irreps $\rho_{k,\ell}$ of $H_{k+\ell+1}$ which correspond to corner (or hook) type Young diagrams ([A.P.I.](#), [O.Ogievetsky, A. Os'kin \(2006\)](#)). Each standard Young tableau for the corner diagram $\{k+1, 1^\ell\}$:

\mathcal{I}	1	j_1	j_2	\dots	j_k
	i_1				
	i_2				
	:				
	i_ℓ				

$$\dim(V_{k,\ell}) = \frac{(k+\ell)!}{k! \ell!}$$

$(j_1, \dots, j_k, i_1, \dots, i_\ell)$ is the special perm. of $(2, 3, \dots, k + \ell + 1)$

corresponds to the basis vector $v_{\mathcal{I}}$ in the space $V_{k,\ell}$ of the irrep $\rho_{k,\ell}$.

5. Integrable chain models

In this case we obtain the spectrum of $\mathcal{H}_{k,\ell}(q)$

$$\text{Spec}(\mathcal{H}_{k,\ell}(q)) = \left\{ \sum_{i=1}^{\ell} 2 \cos \frac{\pi m_i}{k+\ell+1}, \quad 1 \leq m_1 < m_2 < \dots < m_{\ell} \leq k+\ell \right\}.$$

see also

([G.Duchamp, D.Krob, A.Lascoux, B.Leclerc, T.Scharf, J.Thibon, RIMS \(1995\)](#))

5. Integrable chain models

Remark (P.Martin, V.Rittenberg (1992)).

This spectrum is specific for the system of free fermions. Indeed, one can show that the restricted class of representations (corner or hook irreps) of H_{n+1} corresponds to $U_q su(1|1)$ models. These models are related to the model of free fermions.

What can we say about spectrum of $\rho_{(n-2,2)}(\mathcal{H})$ (Hecke chain for representations $(n-2, 2)$)?

1	2	3	\cdots	$n-2$
$n-1$	n			

6. Spectrum of $\rho_{(n-2,2)}(\mathcal{H})$

Using computer simulations we (A.P.I. and S.O.Krivos, 2010) observed that characteristic polynomials for the Hamiltonian

$$x = \sum_{k=1}^{n-1} \rho_{(n-2,2)}(T_k) - \frac{(n-1)}{2}(q - q^{-1})$$

are factorized into two polynomial factors with integer coefficients

$$P_{(n-2,2)}(x, v) = P_{(n-2,2)}^{(1)}(x, v) \cdot P_{(n-2,2)}^{(2)}(x, v)$$

where $v = \frac{q+q^{-1}}{2}$ and $\text{ord}(P^{(1)}) = p_{n-1}$, $\text{ord}(P^{(2)}) = p_n$

$$p_n = \left[\frac{n+1}{2} \right] \left[\frac{n-2}{2} \right],$$

$$p_{n-1} + p_n = \dim \left(\rho_{(n-2,2)} \right) = \frac{n(n-3)}{2}$$

The forms of polynomials $P_{(n-2,2)}^{(1)}(x, v)$, $P_{(n-2,2)}^{(2)}(x, v)$ are very cumbersome

Example. $n = 8 \Leftrightarrow (6, 2)$

$$\begin{aligned} P_{(6,2)}^{(1)} = & (18225v^8 - 43740xv^7 + 45684x^2v^6 - 17010v^6 - 27108x^3v^5 \\ & + 30132xv^5 + 9990x^4v^4 - 22086x^2v^4 + 3528v^4 - 2340x^5v^3 \\ & + 8568x^3v^3 - 4116xv^3 + 340x^6v^2 - 1854x^4v^2 + 1784x^2v^2 - 196v^2 \\ & - 28x^7v + 212x^5v - 340x^3v + 112xv + x^8 - 10x^6 + 24x^4 - 16x^2) \end{aligned}$$

$$\begin{aligned} P_{(6,2)}^{(2)} = & (2460375v^{12} - 8857350xv^{11} + 14565420x^2v^{10} - 4811400v^{10} \\ & - 14464818x^3v^9 + 14244660xv^9 + 9659979x^4v^8 - 18898596x^2v^8 \\ & + 2826090v^8 - 4569372x^5v^7 + 14792544x^3v^7 - 6597288xv^7 \\ & + 1569456x^6v^6 - 7563024x^4v^6 + 6702804x^2v^6 - 610308v^6 \\ & - 394308x^7v^5 + 2638440x^5v^5 - 3869928x^3v^5 + 1049976xv^5 \\ & + 71901x^8v^4 - 635880x^6v^4 + 1388280x^4v^4 - 747648x^2v^4 + 41868v^4 \\ & - 9278x^9v^3 + 104512x^7v^3 - 316760x^5v^3 + 281912x^3v^3 - 47496xv^3 \\ & + 804x^{10}v^2 - 11208x^8v^2 + 44876x^6v^2 - 59340x^4v^2 + 19976x^2v^2 \\ & - 1056v^2 - 42x^{11}v + 708x^9v - 3608x^7v + 6608x^5v - 3688x^3v \\ & + 592xv + x^{12} - 20x^{10} + 126x^8 - 304x^6 + 252x^4 - 80x^2 + 8) \end{aligned}$$

These polynomials are simplified in terms of new variables

$Z = x - \left[\frac{n-2}{2}\right] v$, $Y = x - \left[\frac{n+2}{2}\right] v$ and reveal the "integrable structure":

Example. $n = 8 \Leftrightarrow (6, 2)$

For new variables $Z = x - 3v$, $Y = x - 5v$ we obtain

$$P_{(6,2)}^{(1)} = Y^2 \Pi_2(Z) + Y \Pi_1(Z) + \Pi_0(Z),$$

$$\Pi_2(Z) = Z^6 - 6Z^4 + 5Z^2 - 1, \quad \Pi_1(Z) = -4Z^5 + 16Z^3 - 6Z,$$

$$\Pi_0(Z) = 3Z^4 - 9Z^2$$

$$P_{(6,2)}^{(2)} = Y^3 \Pi_3(Z) + Y^2 \Pi_2(Z) + Y \Pi_1(Z) + \Pi_0(Z),$$

$$\Pi_3(Z) = Z^9 - 14Z^7 + 49Z^5 - 49Z^3,$$

$$\Pi_2(Z) = -6Z^8 + 68Z^6 - 168Z^4 + 98Z^2,$$

$$\Pi_1(Z) = 9Z^7 - 85Z^5 + 136Z^3 - 56Z,$$

$$\Pi_0(Z) = -2Z^6 + 18Z^4 - 24Z^2 + 8,$$

where polynomials $\Pi_n(Z)$ are factorized as char. polynomials for $\rho(H)$ for corner type representations ρ .

We know $P_{(n-2,2)}^{(1)}$ and $P_{(n-2,2)}^{(2)}$ explicitly up to $n = 13$. Integer coefficients are polynomials in n and all our conjectures work perfectly.

$$P_{(n-2,2)}^{(1)} = Z^{\bar{k}_{n-1}} Y^{k_{n-1}} \left(1 - (n-4)Z^{-1}Y^{-1} - \frac{(n-4)(n-5)}{2}Z^{-2} + \right.$$

$$+ \frac{(n-5)(n-6)}{2}Z^{-2}Y^{-2} + \frac{(n-6)(n^2-7n+8)}{2}Z^{-3}Y^{-1} +$$

$$- \frac{(n-6)(n-7)(n^2-5n-4)}{8}Z^{-4} - \frac{(n-6)(n-7)(n-8)}{6}Z^{-3}Y^{-3} -$$

$$\left. - \frac{(n^4-20n^3+137n^2-338n+116)}{4}Z^{-4}Y^{-2} - \dots Z^{-5}Y^{-1} \right) +$$

.....

$$+ (-1)^{[(n-2)/4]} \frac{(1+(-1)^n)}{2} \left\{ \left(\frac{n}{2} - 1 \right) Z + Y \right\}^{k_{n-1}} + (-1)^{[(n-1)/4]} \frac{(1-(-1)^n)}{2} ,$$

$$P_{(n-2,2)}^{(2)} = Z^{\bar{k}_n} Y^{k_n} \left(1 - (n-2)Z^{-1}Y^{-1} - \frac{(n-1)(n-4)}{2}Z^{-2} + \right.$$

$$+ \frac{(n-2)(n-5)}{2}Z^{-2}Y^{-2} + \left\{ \frac{(n-4)(n-5)(n+9)}{3} + \frac{(n-6)(n-7)(n-8)}{6} \right\} Z^{-3}Y^{-1} +$$

$$+ \frac{(n-6)(n-1)(n^2-3n-12)}{8}Z^{-4}$$

$$- \frac{(n-2)(n-6)(n-7)}{6}Z^{-3}Y^{-3} - \frac{(n^4-12n^3+37n^2+18n-124)}{4}Z^{-4}Y^{-2} -$$

$$- \frac{(n^5-12n^4+23n^3+128n^2-252n-224)}{8}Z^{-5}Y^{-1} + \dots \left. \right) +$$

$$+ (-1)^{\left[\frac{(n-1)}{4} \right]} \frac{(1-(-1)^n)}{2} \left\{ (n-2)Z + 2Y \right\}^{k_n} + (-1)^{\left[\frac{n}{4} \right]} \frac{(1+(-1)^n)}{2} 2^{k_n} ,$$

where $k_n = \frac{1}{4}((-1)^n - 5 + 2n)$, $\bar{k}_n = \frac{1}{8}((-1)^n + 7 - 8n + 2n^2)$.

We need additional parameters!

Summary

- Hecke chains (chains based on the Hecke algebras) describe (in the unified way) all spin chains with $U_q(sl(N|K))$ symmetries.

2. Yang-Baxter Equations

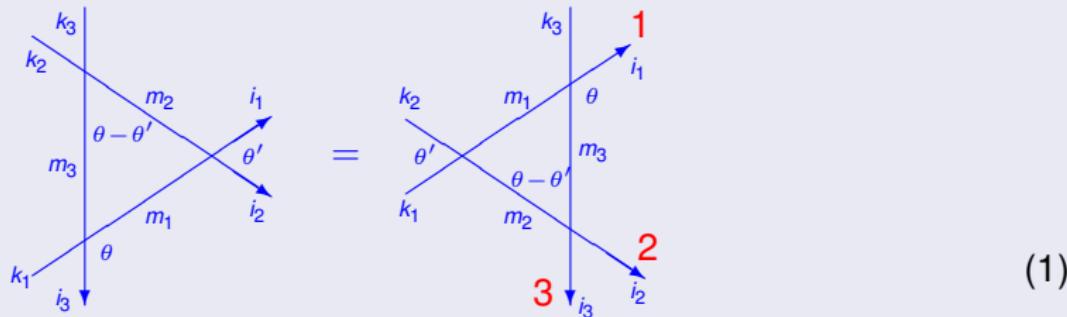
For $d=(1+1)$ quant. integrable field theories one obtains infinite number of IM
 \Rightarrow all particles save their momenta after scattering \Rightarrow factorizing scattering

For two-particle S matrix (a single act of scattering) we have

$$S_{nk}^{ij}(\theta_1 - \theta_2) = \begin{array}{c} i \\ \nearrow \theta_1 - \theta_2 \\ \times \\ \searrow \\ k \end{array} j \equiv T_{nk}^{ji}(x/y), \quad x = e^{\theta_1}, y = e^{\theta_2}$$

where the arrowed lines show trajectories of point particles; $\theta_{1,2}$ - rapidities

For 3-particle S matrix we have



This is a graphical representation of the Yang-Baxter eqs. (triangle eqs.):

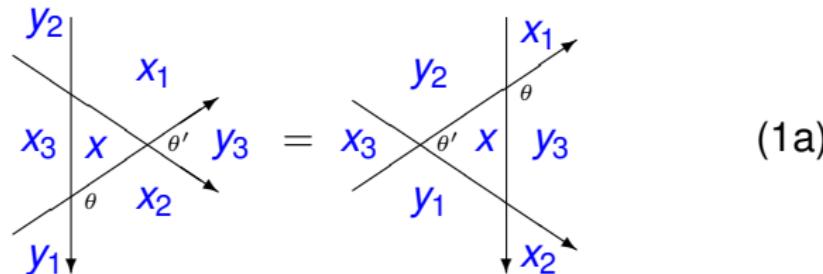
$$S_{k_2 k_3}^{m_2 m_3}(\theta - \theta') S_{k_1 m_3}^{m_1 i_3}(\theta) S_{m_1 m_2}^{i_1 i_2}(\theta') = S_{k_1 k_2}^{m_1 m_2}(\theta') S_{m_1 k_3}^{i_1 m_3}(\theta) S_{m_2 m_3}^{i_2 i_3}(\theta - \theta')$$

2. Yang-Baxter Equations

In concise matrix notations YBE is written as

$$T_{23}(x/y) T_{12}(x) T_{12}(y) = T_{12}(y) T_{13}(x) T_{23}(x/y)$$

Vertex models \Rightarrow Face models. New form of the YBE represented in the form of triangle equation (1), but the indices are not on "lines," but on "faces":



where θ, θ' are angles, and summation is over x . If we denote

$$T_{uz}^{xy}(\theta) = \begin{array}{c} y \\ \diagup \quad \diagdown \\ x \quad z \\ \diagdown \quad \diagup \\ u \end{array}$$

then the analytical form of (1a) is

$$\sum_x T_{y_2 y_3}^{x_1 x}(\theta - \theta') T_{x_3 y_1}^{x x_2}(\theta) T_{x x_2}^{x_1 y_3}(\theta') = \sum_x T_{x_3 y_1}^{y_2 x}(\theta') T_{y_2 x}^{x_1 y_3}(\theta) T_{x y_1}^{y_3 x_2}(\theta - \theta')$$

2. Yang-Baxter Equations

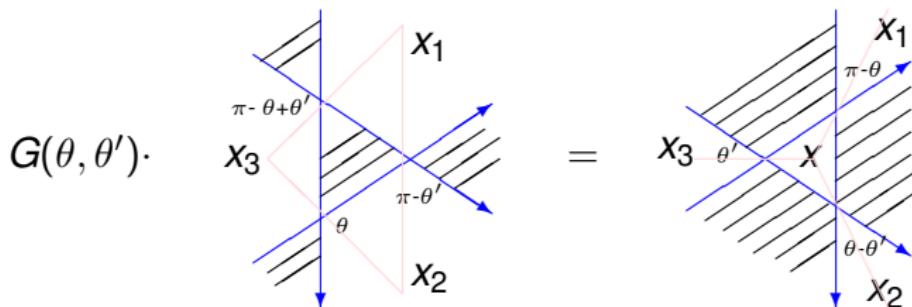
One can try to search a solution of the face YBE (1a) in the form

$$T_{uz}^{xy}(\theta) = D_\theta(y, u) \bar{D}_{\pi-\theta}(x, z)$$

where $D_\theta(y, u)$, $\bar{D}_\theta(x, z)$ satisfy star-triangle relations

$$\begin{aligned} G(\theta, \theta') \cdot \bar{D}_{\pi-\theta+\theta'}(x_1, x_3) D_\theta(x_2, x_3) \bar{D}_{\pi-\theta'}(x_1, x_2) = \\ = \sum_x D_{\theta'}(x, x_3) \bar{D}_{\pi-\theta}(x_1, x) D_{\theta-\theta'}(x_2, x), \end{aligned}$$

where $G(\theta, \theta') = G(\theta, \theta - \theta')$. St.Tr. relations are represented graphically as



Let $\vec{x}, \vec{x}' \in \mathbb{R}^D$, then

$$D_\theta(x, x') = \bar{D}_\theta(x, x') \sim \left((\vec{x} - \vec{x}')^2 \right)^{-\frac{D}{2}(1 - \frac{\theta}{\pi})}, \quad \sum_x \sim \int \frac{d^D x}{\pi^{D/2}}$$

2. Yang-Baxter Equations

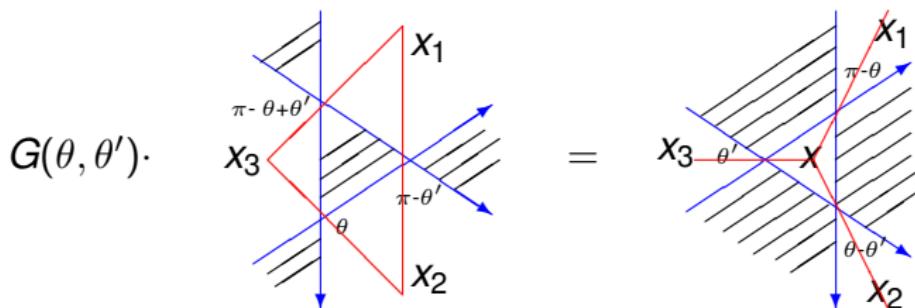
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3. The diagrams

The F.D. (considered here) are graphs with vertices connected by lines labeled by numbers (indices).

To each vertex of the graph we associate the point in D -dimensional Euclidean space \mathbf{R}^D , while the lines (edges) of the graph (with index α) are propagators of massless particles

$$x \xrightarrow{\alpha} y = 1/(x - y)^{2\alpha}$$

where $(x - y)^{2\alpha} := (\sum_{i=1}^D (x_i - y_i)(x_i - y_i))^\alpha$, $\alpha \in \mathbf{C}$, $x, y \in \mathbf{R}^D$. We have 2 types of vertices: the boldface vertices \bullet denote the integration over \mathbf{R}^D . These F.D. are called F.D. in the configuration space.

4. Operator formalism

Let $\hat{q}_i = \hat{q}_i^\dagger$ and $\hat{p}_i = \hat{p}_i^\dagger$ be operators of coordinate and momentum

$$[\hat{q}_k, \hat{p}_j] = i \delta_{kj} .$$

Introduce states $|x\rangle \equiv |\{x_i\}\rangle$, $|k\rangle \equiv |\{k_i\}\rangle$: $\hat{q}_i|x\rangle = x_i|x\rangle$, $\hat{p}_i|k\rangle = k_i|k\rangle$, and normalize these states as:

$$\langle x|k\rangle = \frac{1}{(2\pi)^{D/2}} \exp(i k_j x_j) , \quad \int d^D k |k\rangle \langle k| = \hat{1} = \int d^D x |x\rangle \langle x| .$$

"Matrix representation" of $\hat{p}^{-2\beta}$ (propagator of massless particle) is:

$$\underline{\langle x|\frac{1}{\hat{p}^{2\beta}}|y\rangle = a(\beta) \frac{1}{(x-y)^{2\beta'}}} , \quad \left(a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2} 2^{2\beta} \Gamma(\beta)} \right) .$$

where $\beta' = D/2 - \beta$ and $\Gamma(\beta)$ is the Euler gamma-function.

For $\hat{q}^{2\alpha}$ the "matrix representation" is: $\langle x|\hat{q}^{2\alpha}|y\rangle = x^{2\alpha} \delta^D(x-y)$.

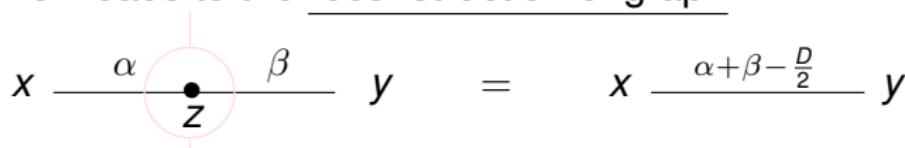
4. Operator formalism

Algebraic relations (a,b,c) which are helpful for analytical calculations of multi-loop perturbative integrals for F.D. \Rightarrow reconstruction of graphs

a. Group relation. Consider a convolution product of two propagators:

$$\int \frac{d^D z}{(x-z)^{2\alpha} (z-y)^{2\beta}} = \frac{G(\alpha', \beta')}{(x-y)^{2(\alpha+\beta-D/2)}}, \quad \left(G(\alpha, \beta) = \frac{a(\alpha+\beta)}{a(\alpha) a(\beta)} \right),$$

which leads to the reconstruction of graph:



This is the "matrix representation" of the operator relation

$$\hat{p}^{-2\alpha'} \hat{p}^{-2\beta'} = \hat{p}^{-2(\alpha'+\beta')}.$$

!!!

Proof.

$$\int d^D z \langle x | \hat{p}^{-2\alpha'} | z \rangle \langle z | \hat{p}^{-2\beta'} | y \rangle = \langle x | \hat{p}^{-2(\alpha'+\beta')} | y \rangle$$



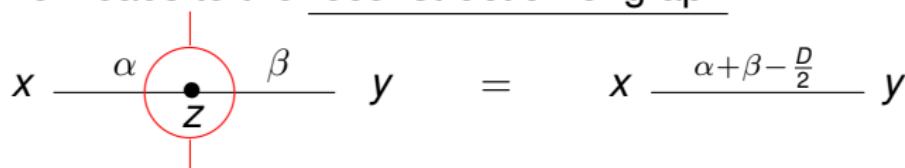
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!!!

Proof.

$$\int d^D z \langle x | \hat{p}^{-2\alpha'} | z \rangle \langle z | \hat{p}^{-2\beta'} | y \rangle = \langle x | \hat{p}^{-2(\alpha'+\beta')} | y \rangle$$

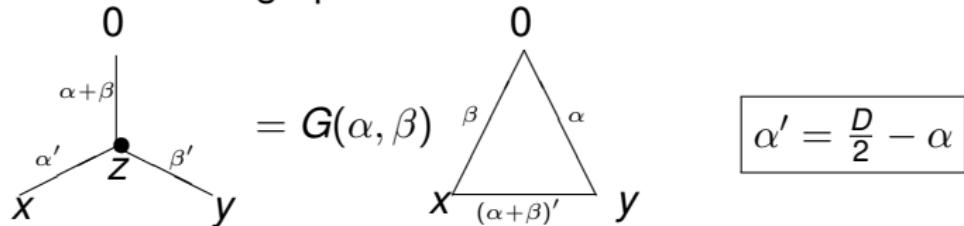
□

b. Star-triangle relation

Firstly considered in CFT (E.S.Fradkin, M.Ya.Palchik, 1978); (A.N.Vasilev, Yu.M.Pis'mak, Yu.R.Khonkonen(1981)); the "Method Of Uniqueness" (D.Kazakov, 1983)

$$\int \frac{d^D z}{(x-z)^{2\alpha'} z^{2(\alpha+\beta)} (z-y)^{2\beta'}} = \frac{G(\alpha, \beta)}{(x)^{2\beta} (x-y)^{2(\frac{D}{2}-\alpha-\beta)} (y)^{2\alpha}}.$$

Reconstruction of graph:



Operator version:

$$\hat{p}^{-2\alpha} \hat{q}^{-2(\alpha+\beta)} \hat{p}^{-2\beta} = \hat{q}^{-2\beta} \hat{p}^{-2(\alpha+\beta)} \hat{q}^{-2\alpha}$$

!!!

Or in the form of Yang-Baxter equation ($T_{ab}(\alpha) = (\hat{q}^{(a)} - \hat{p}^{(b)})^{2\alpha}$):

$$T_{12}(\alpha) T_{23}(\alpha + \beta) T_{12}(\beta) = T_{23}(\beta) T_{12}(\alpha + \beta) T_{23}(\alpha),$$

4. Operator formalism

Remarks on star-triangle relation:

- STR is a commutativity condition for operators $H_\alpha = \hat{p}^{2\alpha} \hat{q}^{2\alpha}$:

$$(\hat{p}^{2\gamma} \hat{q}^{2\gamma}) \hat{p}^{2\alpha} \hat{q}^{2\alpha} = \hat{p}^{2\alpha} \hat{q}^{2\alpha} (\hat{p}^{2\gamma} \hat{q}^{2\gamma}) \Rightarrow$$

$$\hat{p}^{2(\gamma-\alpha)} \hat{q}^{2\gamma} \hat{p}^{2\alpha} = \hat{q}^{2\alpha} \hat{p}^{2\gamma} \hat{q}^{2(\gamma-\alpha)} \Rightarrow \text{STR for } \gamma = \alpha + \beta .$$

- Algebraic proof of the STR. Introduce inversion operator R :

$$R^2 = 1 , \quad \langle x_i | R = \langle \frac{x_i}{x^2} |$$

$$R \hat{q}_i R = \hat{q}_i / \hat{q}^2 , \quad R \hat{p}_i R = \hat{q}^2 \hat{p}_i - 2 \hat{q}_i (\hat{q} \hat{p}) =: K_i ,$$

$$R \hat{p}^{2\beta} R = \hat{q}^{2(\beta+\frac{D}{2})} \hat{p}^{2\beta} \hat{q}^{2(\beta-\frac{D}{2})} .$$

Proof.

$$R \hat{p}^{2\alpha} \hat{p}^{2\beta} R = R \hat{p}^{2(\alpha+\beta)} R \Rightarrow \hat{p}^{2\alpha} \hat{q}^{2(\alpha+\beta)} \hat{p}^{2\beta} = \hat{q}^{2\beta} \hat{p}^{2(\alpha+\beta)} \hat{q}^{2\alpha}$$

\uparrow
 R^2

4. Operator formalism

c. Integration by parts rule. (*F. Tkachov, K. Chetyrkin, 1981*)

(reconstruction of graphs)

$$\begin{array}{c} \text{Diagram 1: } \begin{array}{c} 0 \\ | \\ \bullet \\ / \alpha_1 \backslash \alpha_3 \\ x \quad y \end{array} = \frac{1}{(D-2\alpha_2-\alpha_1-\alpha_3)} \{ \alpha_1 \left(\begin{array}{c} 0 \\ | \\ \bullet \\ / \alpha_2-1 \backslash \alpha_3 \\ x \quad y \end{array} - \begin{array}{c} 0 \\ | \\ \bullet \\ / -1 \backslash \alpha_2 \\ x \quad y \end{array} \right) + \right. \\ \left. + \alpha_3 \left(\begin{array}{c} 0 \\ | \\ \bullet \\ / \alpha_2-1 \backslash \alpha_3+1 \\ x \quad y \end{array} - \begin{array}{c} 0 \\ | \\ \bullet \\ / \alpha_2 \backslash -1 \\ x \quad y \end{array} \right) \right\} \end{array}$$

It can be represented in the operator form:

$$(2\gamma - \alpha - \beta) \hat{p}^{2\alpha} \hat{q}^{2\gamma} \hat{p}^{2\beta} = \frac{[\hat{q}^2, \hat{p}^{2(\alpha+1)}]}{4(\alpha+1)} \hat{q}^{2\gamma} \hat{p}^{2\beta} - \hat{p}^{2\alpha} \hat{q}^{2\gamma} \frac{[\hat{q}^2, \hat{p}^{2(\beta+1)}]}{4(\beta+1)} !!$$

where $\alpha = -\alpha'_1$, $\gamma = -\alpha_2$ and $\beta = -\alpha'_3$.

4. Operator formalism

The integration by parts identity

$$(2\gamma - \alpha - \beta) \hat{p}^{2\alpha} \hat{q}^{2\gamma} \hat{p}^{2\beta} = \frac{[\hat{q}^2, \hat{p}^{2(\alpha+1)}]}{4(\alpha+1)} \hat{q}^{2\gamma} \hat{p}^{2\beta} - \hat{p}^{2\alpha} \hat{q}^{2\gamma} \frac{[\hat{q}^2, \hat{p}^{2(\beta+1)}]}{4(\beta+1)},$$

can be proved by using relations for Heisenberg algebra

$$[\hat{q}^2, \hat{p}^{2(\alpha+1)}] = 4(\alpha+1)(H + \alpha) \hat{p}^{2\alpha},$$

$$H \hat{q}^{2\alpha} = \hat{q}^{2\alpha} (H + 2\alpha), \quad H \hat{p}^{2\alpha} = \hat{p}^{2\alpha} (H - 2\alpha),$$

where $H := \frac{i}{2}(\hat{p}_i \hat{q}_i + \hat{q}_i \hat{p}_i)$ is the dilatation operator.

The set of operators $\{\hat{q}^2, \hat{p}^2, H\}$ generates the algebra $sl(2)$.

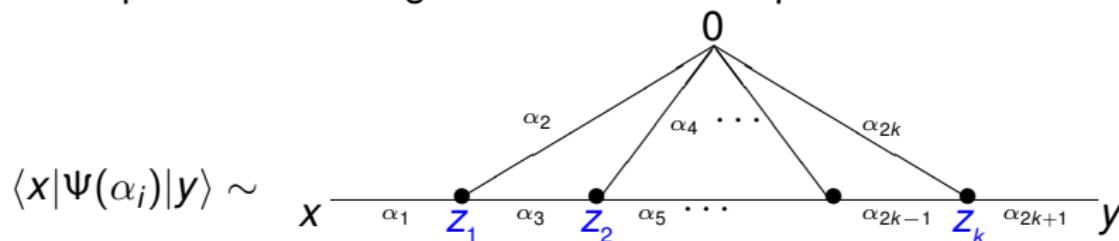
4. Operator formalism

An example of the **operator representation** for F.D.

Consider an operator:

$$\Psi(\alpha_i) = \hat{p}^{-2\alpha'_1} \hat{q}^{-2\alpha_2} \hat{p}^{-2\alpha'_3} \hat{q}^{-2\alpha_4} \hat{p}^{-2\alpha'_5} \dots \hat{q}^{-2\alpha_{2k}} \hat{p}^{-2\alpha'_{2k+1}}.$$

This operator is the algebraic version of 3-point function:



Indeed,

$$\langle x | \Psi(\alpha_i) | y \rangle = \langle x | \hat{p}^{-2\alpha'_1} \hat{q}^{-2\alpha_2} \hat{p}^{-2\alpha'_3} \hat{q}^{-2\alpha_4} \hat{p}^{-2\alpha'_5} \dots \hat{q}^{-2\alpha_{2k}} \hat{p}^{-2\alpha'_{2k+1}} | y \rangle$$

$\int d^D z_1 |z_1\rangle \langle z_1| \quad \int d^D z_2 |z_2\rangle \langle z_2| \quad \int d^D z_k |z_k\rangle \langle z_k|$

Remark. $\langle x | \Psi(\alpha_i) | x \rangle$ represents the propagator-type diagrams.

4. Operator formalism

The advantage: we change the manipulations with integrals by the manipulations with elements of the algebra generated by $\hat{p}^{2\alpha}, \hat{q}^{2\beta}$.

Is it possible to define the trace for this algebra?

$$\text{Tr}(\Psi(\alpha_i)) = \int d^D x \langle x | \frac{1}{\hat{p}^{2\alpha'_1}} \frac{1}{\hat{q}^{2\alpha_2}} \frac{1}{\hat{p}^{2\alpha'_3}} \cdots \frac{1}{\hat{q}^{2\alpha_{2k}}} \frac{1}{\hat{p}^{2\alpha'_{2k+1}}} | x \rangle = c(\alpha_i) \int \frac{d^D x}{x^{2\beta}}.$$

($\beta = \sum_i \alpha_i$; $c(\alpha_i)$ - coeff. function). The dim. reg. procedure requires:

$$\int \frac{d^D x}{x^{2(D/2+\alpha)}} = 0 \quad \forall \alpha \neq 0 .$$

The extension of the definition of this integral is ([S.Gorishnii, A.Isaev, 1985](#))

$$\boxed{\int \frac{d^D x}{x^{2(D/2+\alpha)}} = \pi \Omega_D \delta(\alpha), \quad !!!}$$

where $\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$. Then, the cyclic property of "Tr" can be checked.
"Tr" of propagators \Rightarrow vacuum diagrams \Rightarrow symmetries & identities.

5. Applications

Lipatov chain model.

The Lipatov Hamiltonian which describes bound states of gluons at high energies: $H = \sum_{i=1}^n H_{ii+1}$, where

$$H_{ik} = \hat{p}_i \ln(\rho_{ik}) \hat{p}_i^{-1} + \hat{p}_k \ln(\rho_{ik}) \hat{p}_k^{-1} + \ln(\hat{p}_i \hat{p}_k) - 2\Gamma'(1) = \quad (2)$$

$$= 2 \ln(\rho_{ik}) + \rho_{ik} \ln(\hat{p}_i \hat{p}_k) \rho_{ik}^{-1} - 2\Gamma'(1). \quad (3)$$

Here $\rho_{ik} = q_i - q_k$, where q_i are complex coordinates and $\hat{p}_i = -i \frac{\partial}{\partial q_i}$ are momentum operators. Consider the operator ([S.Derkachev](#)):

$$T_{ik}(\epsilon) := \rho_{ik}^{1+\epsilon} (\hat{p}_i \hat{p}_k)^\epsilon \rho_{ik}^{-1+\epsilon} = 1 + \epsilon \left(2 \ln(\rho_{ik}) + \rho_{ik} \ln(\hat{p}_i \hat{p}_k) \rho_{ik}^{-1} \right) + \epsilon^2 \dots . \quad (4)$$

We use the 1D analog of STR

$$\rho_{ik}^\alpha \hat{p}_i^{\alpha+\beta} \rho_{ik}^\beta = \hat{p}_i^\beta \rho_{ik}^{\alpha+\beta} \hat{p}_i^\alpha \Leftrightarrow \rho_{ki}^\alpha \hat{p}_i^{\alpha+\beta} \rho_{ki}^\beta = \hat{p}_i^\beta \rho_{ki}^{\alpha+\beta} \hat{p}_i^\alpha. \quad (5)$$

5. Applications

First, the "T-operator" $T_{ik}(\epsilon)$ satisfies the Yang-Baxter equation

$$T_{i\,i+1}(\epsilon) T_{i+1\,i+2}(\epsilon + \epsilon') T_{i\,i+1}(\epsilon') = T_{i+1\,i+2}(\epsilon') T_{i\,i+1}(\epsilon + \epsilon') T_{i+1\,i+2}(\epsilon),$$

Second, we have

$$\begin{aligned} T_{ik}(\epsilon) &= \underline{\rho_{ik}^{1+\epsilon} \hat{p}_i^\epsilon \rho_{ik}^{-1}} \underline{\rho_{ik}^1 \hat{p}_k^\epsilon \rho_{ik}^{-1+\epsilon}} = \hat{p}_i^{-1} \rho_{ik}^\epsilon \hat{p}_i^{1+\epsilon} \hat{p}_k^{-1+\epsilon} \rho_{ik}^\epsilon \hat{p}_k^1 = \\ &= 1 + \epsilon \left(\hat{p}_i^{-1} \ln(\rho_{ik}) \hat{p}_i + \hat{p}_k^{-1} \ln(\rho_{ik}) \hat{p}_k + \ln(\hat{p}_i \hat{p}_k) \right) + \epsilon^2 \dots, \end{aligned}$$

and this proves the equivalence of expressions (2) and (3).

Finally, the complete holomorphic Hamiltonian $H = \sum_{i=1}^n H_{ii+1}$ appears in the expansion over ϵ of the monodromy matrix (in the order ϵ^1)

$$T_{(1,2,\dots,n+1)}(\epsilon) = T_{12}(\epsilon) T_{23}(\epsilon) T_{34}(\epsilon) \cdots T_{nn+1}(\epsilon).$$

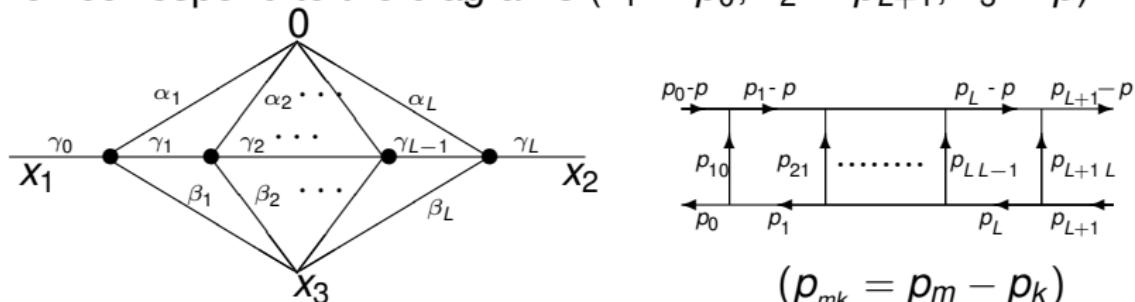
5. Applications

L-loop ladder diagrams for ϕ^3 FT \Leftrightarrow D-dimensional conformal QM

Consider dimensionally and analytically regularized massless integrals

$$D_L(p_0, p_{L+1}, p; \vec{\alpha}, \vec{\beta}, \vec{\gamma}) = \left[\prod_{k=1}^L \int \frac{d^D p_k}{p_k^{2\alpha_k} (p_k - p)^{2\beta_k}} \right] \prod_{m=0}^L \frac{1}{(p_{m+1} - p_m)^{2\gamma_m}}$$

which correspond to the diagrams ($x_1 = p_0$, $x_2 = p_{L+1}$, $x_3 = p$):



The diagrams (in config. and moment. spaces) are dual to each other (the boldface vertices correspond to the loops). The operator version is

$$D_L(x_a; \vec{\alpha}, \vec{\beta}, \vec{\gamma}) \sim \langle x_1 | \hat{p}^{-2\gamma'_0} \left(\prod_{k=1}^L \hat{q}^{-2\alpha_k} (\hat{q} - x_3)^{-2\beta_k} \hat{p}^{-2\gamma'_k} \right) | x_2 \rangle .$$

5. Applications

For simplicity we put $\alpha_i = \alpha, \beta_i = \beta, \gamma_i = \gamma$ and consider the generating function for D_L :

$$D_g(x_a; \alpha, \beta, \gamma) = \sum_{L=0}^{\infty} g^L D_L(x_a; \alpha, \beta, \gamma) \sim \langle x_1 | \left(\hat{p}^{2\gamma'} - \frac{\bar{g}}{\hat{q}^{2\alpha}(\hat{q} - x_3)^{2\beta}} \right)^{-1} | x_2 \rangle$$

where $\bar{g} = g/a(\gamma')$ is the renormalized coupling constant. For the case $\alpha + \beta = 2\gamma'$, using inversions, etc. we obtain

$$D_g \sim \langle u | \left(\hat{p}^{2\gamma'} - \frac{g_x}{\hat{q}^{2\beta}} \right)^{-1} | v \rangle ,$$

where $g_x = \bar{g}(x_3)^{-2\beta}$, $u_i = \frac{(x_1)_i}{(x_1)^2} - \frac{(x_3)_i}{(x_3)^2}$, $v_i = \frac{(x_2)_i}{(x_2)^2} - \frac{(x_3)_i}{(x_3)^2}$.

The ϕ^3 -theory for $D = 4$ is related to $\gamma' = 1 = \beta$ and we obtain the Green's function for conformal QM:

$$D_g \sim \langle u | \left(\hat{p}^2 - \frac{g_x}{\hat{q}^2} \right)^{-1} | v \rangle ,$$

For $D \neq 4$ this GF \Rightarrow ladder diagrams for $\alpha = \beta = 1, \gamma = \frac{D}{2} - 1$.

5. Applications

Our method is based on the identity:

$$\frac{1}{\hat{p}^2 - g/\hat{q}^2} = \sum_{L=0}^{\infty} \left(-\frac{g}{4} \right)^L \left[\hat{q}^{2\alpha} \frac{(H-1)}{(H-1+\alpha)^{L+1}} \frac{1}{\hat{p}^2} \hat{q}^{-2\alpha} \right]_{\alpha^L}$$

where we denote $[\dots]_{\alpha^L} = \frac{1}{L!} (\partial_\alpha^L [\dots])_{\alpha=0}$. Taking into account

$$\frac{(H-1)}{(H-1+\alpha)^{L+1}} = \frac{(-1)^{L+1}}{L!} \int_0^\infty dt t^L e^{t\alpha} \partial_t (e^{t(H-1)})$$

and $e^{t(H+\frac{D}{2})} |x\rangle = |e^{-t}x\rangle$ the Green's function D_g is written in the form

$$\langle u | \frac{1}{(\hat{p}^2 - g_x/\hat{q}^2)} | v \rangle = \sum_{L=0}^{\infty} \frac{1}{L!} \left(\frac{g_x}{4} \right)^L \Phi_L(u, v),$$

$$\Phi_L(u, v) = -a(1) \int_0^\infty dt t^L \left[\left(\frac{u^2}{v^2} \right)^\alpha e^{t\alpha} \right]_{\alpha^L} \partial_t \left(\frac{e^{-t}}{(u - e^{-t}v)^2} \right)^{\left(\frac{D}{2} - 1 \right)}$$

5. Applications

For $D = 4 - 2\epsilon$ one can expand $\Phi_L(u, v)$ over small ϵ :

$$\Phi_L(u, v) = \frac{\Gamma(1 - \epsilon)}{4\pi^{2-\epsilon} u^{2(1-\epsilon)}} \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \Phi_L^{(k)}(z_1, z_2).$$

where $z_1 + z_2 = 2(uv)/u^2$ and $z_1 z_2 = v^2/u^2$. The coeff. functions $\Phi_L^{(k)}$ are expressed in terms of **multiple poly-logarithms**. The first one is
([N.I. Ussyukina and A.I. Davydychev; D.J. Broadhurst; 1993](#))

$$\Phi_L^{(0)}(z_1, z_2) = \frac{1}{z_1 - z_2} \sum_{f=0}^L \frac{(-)^f (2L-f)!}{f! (L-f)!} \ln^f(z_1 z_2) [\text{Li}_{2L-f}(z_1) - \text{Li}_{2L-f}(z_2)].$$

where polylogs are

$$\text{Li}_m(w) = \sum_{n=1}^{\infty} \frac{w^n}{n^m}.$$

5. Applications

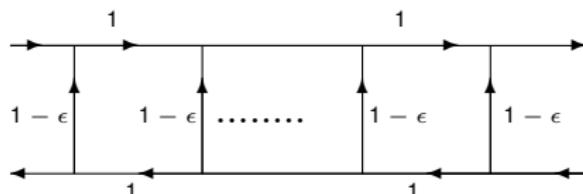
The next coefficient is: $\Phi_L^{(1)}(z_1, z_2) =$

$$= \sum_{n=L}^{2L} \frac{n! \ln^{2L-n}(z_1 z_2) \left[(n \text{Li}_{n+1}(z_1) - \text{Li}_{n,1}(z_1, 1) - \text{Li}_{n,1}(z_1, \frac{z_2}{z_1})) - (z_1 \leftrightarrow z_2) \right]}{(-1)^n (2L-n)! (n-L)! (z_1 - z_2)},$$

where multiple poly-logarithms are

$$\text{Li}_{m_0, m_1, \dots, m_r}(w_0, w_1, \dots, w_r) = \sum_{n_0 > n_1 > \dots > n_r > 0} \frac{w_0^{n_0} w_1^{n_1} \cdots w_r^{n_r}}{n_0^{m_0} n_1^{m_1} \cdots n_r^{m_r}}.$$

The function $\Phi_L^{(1)}(z_1, z_2)$ gives the first term in the expansion over ϵ of the L-loop ladder diagram (with special indices on the lines)



Summary

- Coefficients $\Phi_L(u, v)$ are used for the evaluations of 4-point functions in $N = 4$ SYM theory.
- Lipatov's integrable model – describes high energy scattering of hadrons in QCD.
- Generalizations to massive case and to super-symmetric case. In massive case it is tempting to calculate the Green's function

$$\langle u | \frac{1}{(\hat{p}^2 - g/\hat{q}^2 + m^2)} | v \rangle = \sum_{L=0}^{\infty} g^L \Phi_L(u, v; m^2),$$

- It seems that the approach is not universal even for massless FDs. We should add something new.

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Д.И.Казаков объясняет А.В.Радюшкину, А.П.Исаеву и Д.П.Сорокину как
быстро аналитически вычислять многопетлевые диаграммы Фейнмана с
помощью метода "уникальностей" (1988 год).