On Holographic Entanglement Renyi Entropies in CFT's

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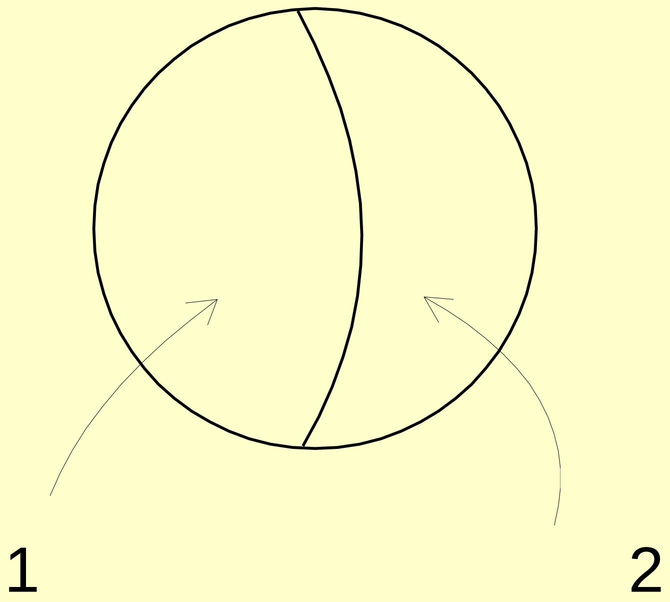
Quantum entanglement

<u>quantum mechanics</u>:
states of subsystems may not be described independently
= states are entangled



<u>importance:</u>
studying correlations of different systems (especially at strong couplings), critical phenomena and etc

Entropy as a measure of entanglement

$$\hat{\rho}_1 = \mathrm{Tr}_2\, \hat{\rho}$$ – reduced density matrix

$$S_1 = -\mathrm{Tr}_1\, \hat{\rho}_1 \ln \hat{\rho}_1$$ – entanglement entropy

$$S_1^{(n)} = \frac{\mathrm{Tr}_1\, \hat{\rho}_1^{\,n}}{1-n}$$ – entanglement Renyi entropy

$$n = 2, 3, 4, \ldots. \qquad S_1^{(n)} \to S_1 \quad , \quad n \to 1$$

Computation of the reduced density matrix and entanglement entropy is a difficult problem, in general

entanglement has to do with quantum gravity:

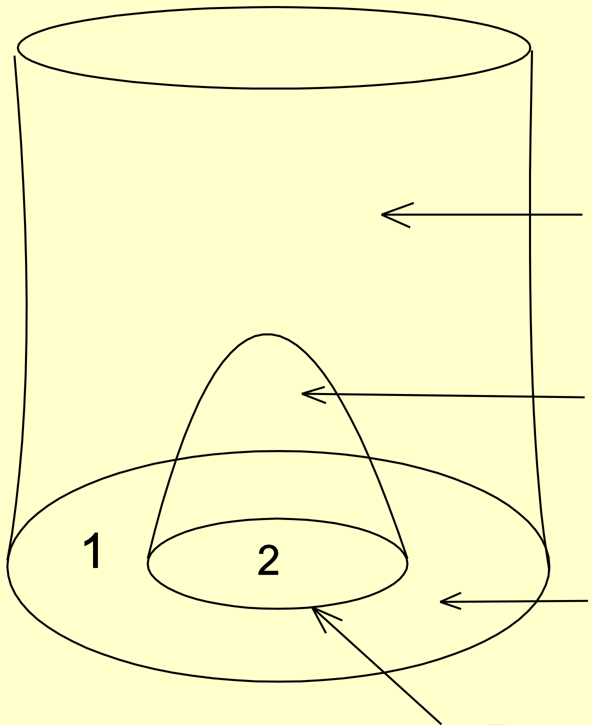
● possible source of the entropy of a black hole (states inside and outside the horizon);

● d=4 supersymmetric BH's are equivalent to 2, 3,… qubit systems

● entanglement entropy allows a *holographic interpretation* for CFT's with AdS duals

Holographic Formula for Entanglement Entropy (n=1)

Ryu and Takayanagi,
hep-th/0603001, 0605073



AdS_{d+1} (bulk space)

$\tilde{B}$ minimal (least area) surface in the bulk

4d space-time manifold (asymptotic boundary of AdS)

B separating surface

entropy of entanglement $\boxed{S = \dfrac{\tilde{A}}{4G^{(d+1)}}}$ is measured in terms of the area of $\tilde{B}$

$G^{(d+1)}$ is the gravity coupling in AdS

Holographic formula enables one to compute entanglement entropy in strongly correlated systems with the help of geometrical methods (the Plateau problem);

Ryu-Takayanagi formula passes several non-trivial tests:

- in 2D and 4D CFT’s (at weak coupling);
-for different quantum states;
-for different shapes and topologies of the separating surface in boundary CFT

Is it possible to find a holographic description of entanglement Renyi entropy?

Plan:

- new result: Renyi entropies in 2D and 4D CFT’s (at weak couplings);

- Difficulties with a holographic description Renyi entropies in CFT’s and a (possible) wayout;

Entanglement Renyi Entropy in CFT’s at weak coupling

1st step: representation in terms of a 'partition function'

$$\hat{\rho} = e^{-\hat{H}/T} / \mathrm{Tr}\, e^{-\hat{H}/T}$$ – thermal density matrix

$$\boxed{Z^{(n)}(T) \equiv \mathrm{Tr}_1 \left(\mathrm{Tr}_2\, e^{-\hat{H}/T}\right)^n}$$ – a partition function

$$Z^{(1)}(T) = Z(T)$$

$$Z(\beta, T) \equiv Z^{(n)}(T) \quad , \quad \beta = 2\pi n$$

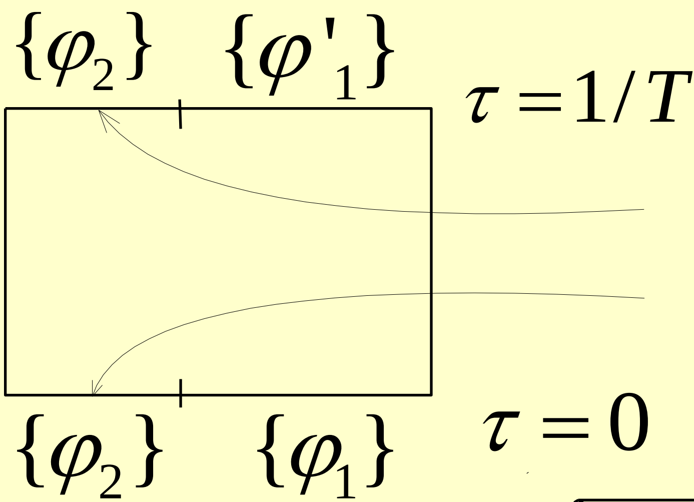
β – "inverse temperature"

$$S_1(T) = -\lim_{\beta \to 1} \left(\beta \partial_\beta - 1\right) \ln Z(\beta, T)$$

$$S_1^{(n)}(T) = \frac{2\pi\, Z(\beta, T) - \beta Z(T)}{2\pi - \beta}$$

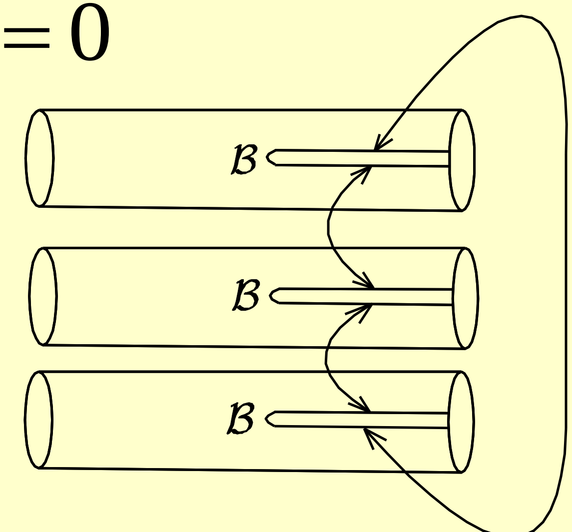
2d step: relation of a ‘partition function’ to an effective action on a ‘curved space’

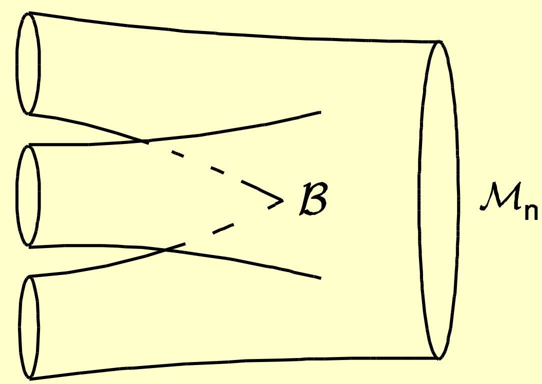
$$W(\beta,T) = -\ln Z(\beta,T) - \text{effective action}$$



these intervals are identified

$$\mathrm{Tr}_1 \rho_1^{\ 3}$$





one glues n (n=3) copies

a ‘curved space’ with conical singularity at the separating point (surface)

3^d step: use results of spectral geometry

$$W = \frac{1}{2}\sum_k \eta_k \ln\det L_k \quad , \quad \eta_k = \pm 1$$

$L_k -$ Laplace operators of different spin fields on M_n

$$W \sim \sum_{p=0}^{d-1} \Lambda^{d-p} \frac{a_p}{p-d} - a_d \ln(\Lambda/\mu) + \ldots$$ for dimension d even,

$$a_p = \sum_k \eta_k a_{k,p}$$, where $a_{k,p}$: $\mathrm{Tr}\, e^{-tL_k} \sim \sum_{p=0} t^{\frac{p-d}{2}} a_{k,p}$;

$\Lambda -$ is a UV cutoff; μ is a physical scale (mass, inverse syze etc)

an example: a scalar Laplacian $L_0 = -\nabla^2$:

$$a_0 \sim n\,,\; a_2 = \frac{1}{24\pi}\int_{M_n} R + \boxed{\frac{1}{12\gamma_n}(\gamma_n^{\,2}-1)\int_B} \quad , \quad \gamma_n = n^{-1}$$

There are non-trivial contributions from conical singularities located at the 'separating' surface B

computations

$$S \sim \sum_{p=2}^{d-2} \Lambda^{d-p} \frac{s_p}{d-p} + s_d \ln(\Lambda/\mu) + \ldots,$$

$$S^{(n)} \sim \sum_{p=2}^{d-2} \Lambda^{d-p} \frac{s^{(n)}{}_p}{d-p} + s^{(n)}{}_d \ln(\Lambda/\mu) + \ldots, - \text{ Renyi entropies}$$

$$s_p \equiv -\lim_{n\to 1}(n\partial_n - 1)a_p(n) \quad , \quad s^{(n)}{}_p \equiv \frac{na_p(1) - a_p(n)}{n-1}$$

$$s_0 = s^{(n)}{}_0 = 0 \quad , \quad s_{2k+1} = s^{(n)}{}_{2k+1} = 0 - \text{(if boundaries are absent)}$$

2D CFT: "c" massless scalars and spinors

$$W = \frac{a}{2} \ln \det \nabla^2 - b \ln \det \gamma^\mu \nabla_\mu$$

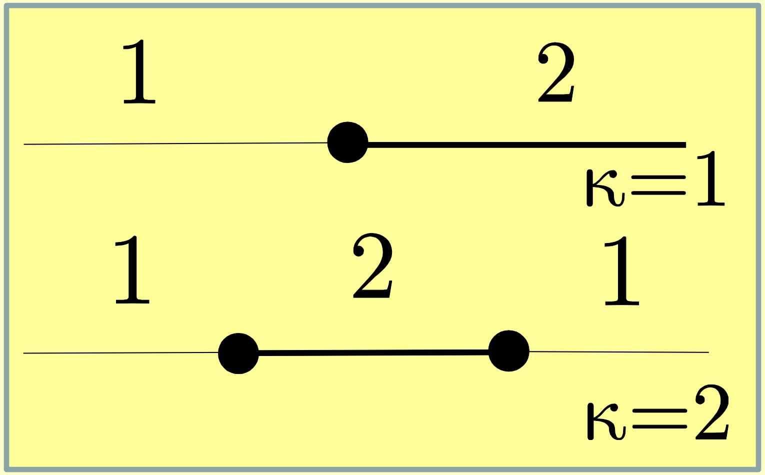
$c = a + b$ − CFT central charge

$$s_2 = \frac{c}{6} k \quad , \quad s^{(n)}{}_2 = \frac{c}{12} k (1 + \gamma_n)$$

$$S = \frac{c}{6} k \ln(L/\varepsilon) \, ,$$

$$S^{(n)} = \frac{c}{12}(1 + \gamma_n) k \ln(L/\varepsilon), -$$ Renyi entropy

$\varepsilon \equiv \Lambda^{-1}$, $L -$ a typical syze of the system,



the result holds for a system on an interval devided into 2 or 3 parts

$k = 1, 2$- the number of separating points (which yield conical singularities)

4D N=4 super SU(N) Yang-Mills theory at weak coup.

6 scalar multiplets, 4 multiplets of Weyl spinors, 1 multiplet of gluon fields

$s^{(n)}{}_2 \sim A(B) -$ area of the separating surface B

$$s^{(n)}{}_4 = c(\gamma_n) F_c + a(\gamma_n) F_a$$

$$c(\gamma_n) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 3\gamma_n + 3)$$

$$F_c = \frac{1}{6\pi} \int_B \sqrt{\gamma} d^2x \left(R_{\mu\nu} n_i^\mu n_i^\nu - 2R_{\mu\nu\lambda\rho} n_i^\mu n_j^\nu n_i^\lambda n_j^\rho - R(B) \right)$$

$$a(\gamma_n) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 7\gamma_n + 15), \quad F_a = \frac{1}{2\pi} \int_B \sqrt{\gamma} d^2x \, R(B)$$

$R(B) -$ scalar curvature of B, $n_i^\mu -$ a pair of unit orthogonal normals to B,

$R_{\mu\nu}, R_{\mu\nu\lambda\rho} -$ Ricci and Rieman tensors of M at B

B has vanishing extrinsic curvatures

Entanglement entropy (n=1)

$$s_4 = \lim_{n\to 1} s^{(n)}{}_4 = cF_c + aF_a$$

$$c = \lim_{n\to 1} c(\gamma_n) = \frac{1}{4}, \quad a = \lim_{n\to 1} c(\gamma_n) = \frac{1}{4}$$

$$a = c$$

relation to the trace anomaly in $D = 4$

$$\left\langle T_\mu^\mu \right\rangle = -aE_4 - cI_4$$

$$E_4 = \frac{1}{64\pi^2}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

$$I_4 = \frac{1}{64\pi^2}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) - (\text{square of Weyl tensor})$$

Cardy's conjecture: "charge" a decreases monotonically along RG flows

Conformal invariance and extrinsic curvatures

$s_4, {s^{(n)}}_4$ should be Weyl invariant functionals (so does a_4 in $D=4$)

F_a are Weyl invariant; F_c should be modified to preserve the invariance

$$F_c = \frac{1}{6\pi}\int_B \sqrt{\gamma}d^2x\left(R_{\mu\nu}n_i^\mu n_i^\nu - 2R_{\mu\nu\lambda\rho}n_i^\mu n_j^\nu n_i^\lambda n_j^\rho \boxed{+\frac{1}{2}\mathrm{Tr}(k_i)\mathrm{Tr}(k_i)} - R(B)\right)$$

$(k_i)_{\mu\nu}$ are extrinsic curvatures of B associated to normals n_i

but there may be an extra piece:

$${s^{(n)}}_4 = c(\gamma_n)F_c + a(\gamma_n)F_a + \boxed{d(\gamma_n)F_d}$$

$$F_d = \frac{1}{2\pi}\int_B \sqrt{\gamma}d^2x\left(\mathrm{Tr}(k_ik_i) - \frac{1}{2}\mathrm{Tr}(k_i)\mathrm{Tr}(k_i)\right)$$

$d(\gamma_n)$ has not been directly computed so far;

$\lim_{n\to 1} d(\gamma_n) = 1,$ this result follows from 'holographic' type arguments

(see S.N. Solodukhin. arXiv:0802.3117)

Toward a holographic description of Entanglement Renyi Entropy in CFT's

A possible structure the holographic formula

$$S^{(n)} = S^{(n)}{}_1 + l^2 S^{(n)}{}_2 + l^4 S^{(n)}{}_3 + \ldots.$$

$S^{(n)}{}_k -$ are local invariant functionals on $\tilde{B}$;

'curvatures' at $\tilde{B}$ are small compared AdS curvature, $\tilde{B} -$ a holographic surface

in the bulk: $\partial\tilde{B} = B$

$$S^{(n)}{}_1 = \frac{1+\gamma_n}{2} S_{RT} = \frac{1+\gamma_n}{8G_N^{(d+1)}} A(\tilde{B})$$

$S^{(n)}{}_1 -$ reproduces Renyi entropy in 2D CFT

$$S = S^{(n)}{}_1 = \frac{c}{12}(1+\gamma_n)k\ln(L/\varepsilon)\,, \quad c = \frac{3l}{G_N^{(3)}}$$

Next order terms and D4 CFT's

$$S^{(n)}{}_1 = \Lambda^2 \frac{s^{(n)}{}_1}{2} + s'^{(n)} \ln(\Lambda/\mu) + \ldots$$

$$s'^{(n)} = \frac{\gamma_n + 1}{8}(F_c + F_a + F_d) \quad , \; s^{(n)}{}_4 = c(\gamma_n)F_c + a(\gamma_n)F_a + d(\gamma_n)F_d$$

therefore, $S^{(n)}{}_2$ must compensate the difference:

$$\bar{s}_2^{(n)} = s^{(n)}{}_4 - s'^{(n)} \equiv \bar{c}(\gamma_n)F_c + \bar{a}(\gamma_n)F_a + \bar{d}(\gamma_n)F_d$$

$$\bar{c}(\gamma_n) = \frac{1}{96}(\gamma_n - 1)^3(\gamma_n + 3), \quad \bar{a}(\gamma_n) = \frac{1}{32}(\gamma_n - 1)(\gamma_n + 1)^3$$

$$S^{(n)}{}_2 = \bar{c}(\gamma_n)\tilde{F}_c + \bar{a}(\gamma_n)\tilde{F}_a + \bar{d}(\gamma_n)\tilde{F}_d$$

$$\tilde{F}_{c,a,d} = \int_{\tilde{B}} \sqrt{\tilde{\gamma}}\, d^{d-1}x \; f_{c,a,d}(\text{curvatures})$$

Summary:

● new result for the entanglement Renyi entropies (ERE) in D=4 CFT’s

● ERE is a local invariant functional which have a structure similar to EE -> possibility to find a holographic description of ERE

● a conjectured approach to a holographic ERE:

-describes ERE in 2D CFT(for a single interval)
- has a potential to reproduce ERE in 4D CFT

The work is in progress

thank you for attention