

KINETIC EQUATION IN DE SITTER SPACE

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TO APPEAR

I. Large IR corrections in dS space:

Typical corrections $\propto [\lambda^2 \log(p \eta)]^n$

λ — coupling constant

p — co-moving momentum

η — conformal time

$p \eta$ — physical momentum

Need to sum them up

Dyson–Schwinger equation reduces in the IR limit to the Boltzman’s kinetic equation

II. dS space:

$$X_0^2 - X_i^2 = -R^2 \rightarrow -1, \quad i = 1, \dots, D$$

$$ds_{D+1}^2 = dX_0^2 - dX_i^2$$

Global coordinates:

$$X_0 = \sinh t$$

$$X_i = \omega_i \cosh t, \quad \omega_i^2 = 1,$$

$$ds^2 = dt^2 - \cosh^2 t \, d\Omega_{D-1}^2$$

Planar coordinates (expanding Poincare patch):

$$ds^2 = dt^2 - e^{2t} d\vec{x}^2 = a(\eta) [d\eta^2 - d\vec{x}^2]$$

$$a(\eta) = \frac{1}{\eta}, \quad \eta = e^{-t}$$

$\eta \rightarrow \infty$ — **past**, $\eta \rightarrow 0$ — **future**

Contracting Poincare patch:

$$ds^2 = dt^2 - e^{-2t} d\vec{x}^2 = a(\eta) [d\eta^2 - d\vec{x}^2]$$

$$a(\eta) = \frac{1}{\eta}, \quad \eta = e^t$$

$\eta \rightarrow 0$ — **past**, $\eta \rightarrow \infty$ — **future**

III. Interacting fields in dS

We are going to study 4D real scalar minimally

coupled $\lambda\phi^3$ theory with $m^2 > 9/4$

Klein–Gordon equation:

$$\left[a^{-4} \partial_\eta a^2 \partial_\eta - \frac{\Delta}{a^2} + m^2 \right] \phi(\eta, x) = 0.$$

$$g_k(\eta) = \eta^{3/2} h(k\eta) / \sqrt{2}$$

$h(k\eta)$ — solution o the Bessel equation

$$\phi(\eta, \vec{x}) = \int d^3k \left[a_k g_k(\eta) e^{-i \vec{k} \vec{x}} + a_k^+ g_k^*(\eta) e^{i \vec{k} \vec{x}} \right],$$

IV Tree-level two-point function:

Free Hamiltonian $H(\eta) = a^2(\eta) \int d^3x T_{00}(\eta)$

$$\begin{aligned}
 H_0(\eta) &= \int d^3k \left[a_k^+ a_k A_k(\eta) + a_k a_{-k} B_k(\eta) + c.c. \right], \\
 A_k(\eta) &= \frac{a^2(\eta)}{2} \left\{ \left| \frac{dg_k}{d\eta} \right|^2 + [k^2 + a^2(\eta) m^2] |g_k|^2 \right\}, \\
 B_k(\eta) &= \frac{a^2(\eta)}{2} \left\{ \left(\frac{dg_k}{d\eta} \right)^2 + [k^2 + a^2(\eta) m^2] g_k^2 \right\}.
 \end{aligned}$$

Solution of the KG equation and of $B_k = 0$ do not coincide.

Hence, H_0 is not diagonal.

E.g. Bunch–Davies harmonics

$$g_k(\eta) \equiv \frac{\sqrt{\pi} \eta^{\frac{3}{2}} e^{-\frac{\pi\mu}{2}}}{2} \mathcal{H}_{i\mu}^{(1)}(k\eta), \quad \mu = \sqrt{m^2 - \frac{9}{4}},$$

$\mathcal{H}_{i\mu}^{(1)}(x)$ is the Hankel function. For them $B_k(\eta \rightarrow \infty) \rightarrow 0$

Nonstationary situation. Then Retarded propagator:

$$D^R(\eta_1, \eta_2 | \vec{r}) = \theta(\eta_1 - \eta_2) \langle [\phi(\eta_1, 0), \phi(\eta_2, \vec{r})] \rangle$$

And Keldysh propagator

$$D^K(\eta_1, \eta_2 | \vec{r}) = \frac{1}{2} \langle \{ \phi(\eta_1, 0), \phi(\eta_2, \vec{r}) \} \rangle$$

We define

$$d^K(\eta_1, \eta_2 | \vec{p}) = (\eta_1 \eta_2)^{-\frac{3}{2}} \int d^3r D^K(\eta_1, \eta_2 | \vec{r}) e^{-i \vec{p} \vec{r}}$$

Tree-level

$$\begin{aligned} d_0^K(p\eta_1, p\eta_2) = \frac{1}{2} [h(p\eta_1) h^*(p\eta_2) + h^*(p\eta_1) h(p\eta_2)] \langle a_p a_p^+ \rangle + \\ + h(p\eta_1) h(p\eta_2) \langle a_p a_{-p} \rangle + h.c., \end{aligned}$$

If we average wrt the “vacuum” state for the given choice of harmonics:

$$\langle a_p^+ a_p \rangle = 0, \quad \langle a_p a_{-p} \rangle = 0, \quad \langle a_p^+ a_{-p}^+ \rangle = 0$$

And

$$D^K(\eta_1, \eta_2, |\vec{x} - \vec{y}|) = C_1 (z^2 - 1)^{-\frac{1}{2}} P_{-\frac{1}{2}+i\mu}^1(z) + \\ + C_2 (z^2 - 1)^{-\frac{1}{2}} Q_{-\frac{1}{2}+i\mu}^1(z),$$

P_ν^1 and Q_ν^1 — associated Legendre functions

$z = 1 + \frac{(\eta_1 - \eta_2)^2 + |\vec{x} - \vec{y}|^2}{2\eta_1 \eta_2}$ — hyperbolic distance between (η_1, \vec{x}) and (η_2, \vec{y})

$C_{1,2}$ — complex constants. Depend on the particular choice of the Harmonics.

E.g. for the BD harmonics $C_2 = 0$ — analytical continuation from the sphere

One-loop two-point function:

$$d_1^K(p\eta_1, p\eta_2) = \frac{1}{2} [h(p\eta_1) h^*(p\eta_2) + h^*(p\eta_1) h(p\eta_2)] 2 n_p + \\ + h(p\eta_1) h(p\eta_2) \kappa_p + h.c.$$

$$n_p(\eta) \equiv \langle a_p^+ a_p \rangle = \frac{\lambda^2}{4\pi^2} \int_p^{1/\eta} \frac{dk}{k} \int_{-\infty}^0 dx_1 dx_2 (x_1 x_2)^{\frac{1}{2}} \times \\ \times h \left[\frac{p}{k} x_1 \right] h^* \left[\frac{p}{k} x_2 \right] h^2(x_1) [h^*(x_2)]^2$$

$$\kappa_p \equiv \langle a_p a_{-p} \rangle = \frac{\lambda^2}{2\pi^2} \int_p^{1/\eta} \frac{dk}{k} \int_{-\infty}^0 dx_1 \int_{-\infty}^{x_1} dx_2 (x_1 x_2)^{\frac{1}{2}} \times \\ \times h^* \left[\frac{p}{k} x_1 \right] h^* \left[\frac{p}{k} x_2 \right] h^2(x_1) [h^*(x_2)]^2.$$

Even if we start with $n_p = 0$, $\kappa_p = 0$. They are generated at loops — pair creation.

Feynman diagrammatic technic does not lead to terms $\propto h(p\eta_1) h(p\eta_2)$ and c.c..

We are interested in the leading IR terms $p\eta_{1,2} \rightarrow 0$

For BD harmonics:

$$h(x) \approx A_+ x^{i\mu} + A_- x^{-i\mu}, \quad \text{as } x \rightarrow 0$$

And

$$\begin{aligned} d_{0+1}^K(p\eta_1, p\eta_2) &\approx \frac{\coth(\pi\mu)}{2\mu} s^{i\mu} + A_+ A_-^* (p\eta)^{2i\mu} \times \\ &\times \left\{ 1 + \frac{\lambda^2}{2\pi^2\mu} \log\left(\frac{1}{p\eta}\right) \iint_{\infty}^0 dx_1 dx_2 (x_1 x_2)^{\frac{1}{2}} h^2(x_1) [h^*(x_2)]^2 \times \right. \\ &\quad \left. \times \left[\theta(x_1 - x_2) \left(\frac{x_1}{x_2}\right)^{i\mu} - \theta(x_2 - x_1) \left(\frac{x_1}{x_2}\right)^{-i\mu} \right] \right\} + c.c. \end{aligned}$$

$$s = \frac{\eta_1}{\eta_2}, \quad \eta = \sqrt{\eta_1 \eta_2}$$

From this answer we see that $n_p \sim |\kappa_p|$. Hence, BD harmonics are not suitable for the kinetic equation.

Side remark:

Analytical continuation from the sphere — renormalization of the mass only

$\mu \rightarrow \mu + \Delta\mu$, $\Delta\mu$ is complex

Then the corrections would have had the form:

$$d_1^K(p\eta_1, p\eta_2) = \frac{\coth(\pi\mu)}{2\mu} s^{i\mu} i \Delta\mu \log(s) + \\ + A_+ A_-^* (p\eta)^{2i\mu} 2i \Delta\mu \log(p\eta) + c.c.$$

Does not coincide with that what we actually get.

One loop result for the out-Jost functions:

$$h(x) = \sqrt{\frac{\pi}{\sinh(\pi\mu)}} J_{i\mu}(x).$$

$$h(x) \sim x^{i\mu}, \quad x \rightarrow 0$$

$$h(x) = \sqrt{\frac{\pi}{4 \sinh(\pi\mu) x}} \left[e^{ix} + e^{-\pi\mu - ix} \right], \quad x \rightarrow \infty$$

The leading IR contribution to the two-point function:

$$d_{0+1}^K(p\eta_1, p\eta_2) \approx \frac{1}{2\mu} \left[s^{i\mu} + s^{-i\mu} \right] \times \\ \times \left\{ 1 + \frac{\lambda^2}{2\pi^2\mu} \log\left(\frac{1}{p\eta}\right) \left| \int_{\infty}^0 dx x^{\frac{1}{2}+i\mu} \left[h^2(x) - \frac{\pi e^{-\pi\mu}}{4 \sinh(\pi\mu) x} \right] \right|^2 \right\}$$

The κ_p is suppressed in comparison with n_p in the IR limit. Suitable for the kinetic equation.

V. The kinetic equation:

$$\begin{aligned} \frac{dn_p(\eta)}{d \log(\eta)} &= \frac{\lambda^2}{\pi^2} \int_0^\infty dk \, \eta (k\eta)^{\frac{1}{2}} \times \\ &\times \left\{ F_1 \times \left[(1 + n_p) n_k n_{p-k} - n_p (1 + n_k) (1 + n_{p-k}) \right] (\eta') + \right. \\ &+ 2 F_2 \times \left[n_k (1 + n_{k-p}) (1 + n_p) - (1 + n_k) n_{k-p} n_p \right] (\eta') + \\ &\left. + F_3 \times \left[(1 + n_k) (1 + n_{p+k}) (1 + n_p) - n_k n_{p+k} n_p \right] (\eta') \right\} \end{aligned}$$

$$\begin{aligned} F_1 &= \text{Re} \left(C^* \left[\frac{p}{k} k\eta, k\eta, \left(\frac{p}{k} - 1 \right) k\eta \right] \times \right. \\ &\times \left. \int_{k\eta_0}^{k\eta} dy' (y')^{\frac{1}{2}} C \left[\frac{p}{k} y', y', \left(\frac{p}{k} - 1 \right) y' \right] \right) \end{aligned}$$

$$\begin{aligned} F_2 &= \text{Re} \left(C^* \left[k\eta, \left(1 - \frac{p}{k} \right) k\eta, \frac{p}{k} k\eta \right] \times \right. \\ &\times \left. \int_{k\eta_0}^{k\eta} dy' (y')^{\frac{1}{2}} C \left[y', \left(1 - \frac{p}{k} \right) y', \frac{p}{k} y' \right] \right) \end{aligned}$$

$$\begin{aligned} F_3 &= \text{Re} \left(D^* \left[k\eta, \left(\frac{p}{k} + 1 \right) k\eta, \frac{p}{k} k\eta \right] \times \right. \\ &\times \left. \int_{k\eta_0}^{k\eta} dy' (y')^{\frac{1}{2}} D \left[y', \left(\frac{p}{k} + 1 \right) y', \frac{p}{k} y' \right] \right) \end{aligned}$$

η_0 is the moment of time when we switch on the interactions; $C[x, y, z] = h^*(x) h(y) h(z)$, $D[x, y, z] = h(x) h(y) h(z)$

Solution in the expanding patch

In the limit $p\eta \rightarrow 0$ $n(p\eta) \rightarrow 0$. Then

$$(1 + n_p) n_k n_{p-k} - n_p (1 + n_k) (1 + n_{p-k}) \approx -n(p\eta)$$

$$n_k (1 + n_{k-p}) (1 + n_p) - (1 + n_k) n_{k-p} n_p \approx n(k\eta)$$

$$(1 + n_k) (1 + n_{p+k}) (1 + n_p) - n_k n_{p+k} n_p \approx 1$$

Furthermore, $n(x) \gg n(y)$ for $y \gg x$

$$\begin{aligned} \frac{dn(x)}{d \log(x)} &= \Gamma n(x) - \Gamma', \\ \Gamma &= \frac{\lambda^2}{2\pi^2 \mu} \left| \int_{\infty}^0 dy y^{\frac{1}{2}-i\mu} \left[h^2(y) - \frac{\pi e^{-\pi\mu}}{4 \sinh(\pi\mu) |y|} \right] \right|^2, \\ \Gamma' &= \frac{\lambda^2}{2\pi^2 \mu} \left| \int_{\infty}^0 dy y^{\frac{1}{2}+i\mu} \left[h^2(y) - \frac{\pi e^{-\pi\mu}}{4 \sinh(\pi\mu) |y|} \right] \right|^2 \end{aligned}$$

$$\text{I.e. } n(p\eta) = \frac{\Gamma'}{\Gamma} \left[C (p\eta)^{\Gamma} + 1 \right], \quad \frac{\Gamma'}{\Gamma} \approx e^{-2\pi\mu} \ll 1$$

C — integration constant. Depends on the initial conditions. $C = -1$ — above one-loop result.

Solution in the contracting patch

The same equation as in expanding patch with the opposite relative sign between RHS and LHS.

$\eta \rightarrow \infty$ — future.

As $p\eta \rightarrow 0$ we expect $n_p(\eta)$ to be independent of p and $n(\eta) \gg 1$

$$(1 + n_p) n_k n_{p-k} - n_p (1 + n_k) (1 + n_{p-k}) \approx -n^2(\eta)$$

$$n_k (1 + n_{k-p}) (1 + n_p) - (1 + n_k) n_{k-p} n_p \approx n^2(\eta)$$

$$(1 + n_k) (1 + n_{p+k}) (1 + n_p) - n_k n_{p+k} n_p \approx n^2(\eta)$$

$$\frac{dn(\eta)}{d \log(\eta)} = \bar{\Gamma} n^2(\eta), \quad \text{where} \quad \bar{\Gamma} \approx \frac{\lambda^2 \mu^2}{2 \pi^2 m^2 (m^2 - \frac{3}{2})} > 0$$

$$n(\eta) \sim \frac{1}{\bar{\Gamma} \log \frac{\eta_0}{\eta}}$$

$\eta < \eta_0 = e^{const/\lambda^2} \gg 1$. A — integration constant.