

# Playing with numbers

V.A. Smirnov

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*in collaboration with R.N. Lee and A.V. Smirnov*



N=3/2

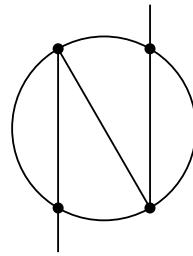
$$S = \int d^4x \left[ \frac{1}{2} \partial^\mu \bar{\phi} \partial_\mu \phi - \frac{1}{2} M^2 \bar{\phi} \phi + \dots \right]$$

$$N = \frac{4\pi}{3} \frac{3}{2}$$

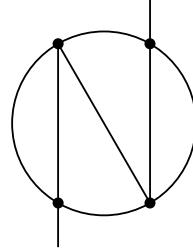
$$+ \int d^4p \quad [W \theta W + h.c.]$$

Fini!!!

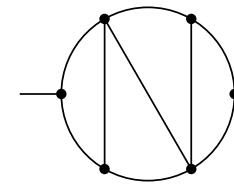
y =



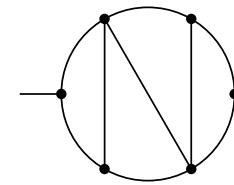
$$= \frac{441}{8} \zeta(7) + O(\varepsilon)$$



$$\neq \frac{441}{8}\zeta(7) + O(\varepsilon)$$

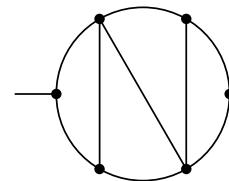

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$$\varepsilon = (4 - d)/2$$

A diagram showing a circle with four external lines extending from its circumference. Inside the circle, there are two vertices connected by a diagonal line. There are also two vertical lines connecting the top and bottom vertices to the left and right vertices respectively.
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[D.I. Kazakov'83]

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## Five-loop renormalization group calculations in the $\phi^4$ theory

[S.G. Gorishny, S.A. Larin, F.V. Tkachov & K.G. Chetyrkin'83, D.I. Kazakov'83]

$$\begin{aligned}
&= (1 - 2\epsilon)^3 \left( \frac{\Gamma(1 - \varepsilon)^2 \Gamma(1 + \varepsilon)}{\Gamma(2 - 2\varepsilon)} \right)^3 \left\{ \frac{441\zeta_7}{8} + \epsilon \left( -216\zeta_3\zeta_5 + \frac{5733\zeta_8}{16} - \frac{81\zeta_{2,6}}{2} \right) \right. \\
&\quad + \left( -267\zeta_3^3 - 81\zeta_4\zeta_5 - \frac{675\zeta_3\zeta_6}{2} + \frac{4583\zeta_9}{2} \right) \epsilon^2 + \left( -\frac{2403}{2}\zeta_3^2\zeta_4 - \frac{502287\zeta_5^2}{56} - \frac{7731\zeta_3\zeta_7}{56} \right. \\
&\quad \left. + \frac{1324935\zeta_{10}}{112} + \frac{18441\zeta_{3,7}}{56} \right) \epsilon^3 + \left( -\frac{24315}{2}\zeta_3^2\zeta_5 - \frac{358023\zeta_5\zeta_6}{8} + \frac{139401\zeta_4\zeta_7}{8} \right. \\
&\quad \left. - \frac{59895\zeta_3\zeta_8}{4} + \frac{232767\zeta_2\zeta_9}{4} - \frac{402081\zeta_{11}}{32} - \frac{621}{2}\zeta_3\zeta_{2,6} + \frac{6291}{2}\zeta_{2,1,8} \right) \epsilon^4 \\
&\quad - \left( -6023\zeta_3^4 + 6660\zeta_3\zeta_4\zeta_5 - \frac{650997}{7}\zeta_2\zeta_5^2 + 40507\zeta_3^2\zeta_6 - \frac{1323426}{7}\zeta_2\zeta_3\zeta_7 \right. \\
&\quad \left. + \frac{1750957\zeta_5\zeta_7}{2} + \frac{964778\zeta_3\zeta_9}{3} - \frac{104287641323\zeta_{12}}{132672} - 2853\zeta_4\zeta_{2,6} + \frac{48222}{7}\zeta_2\zeta_{3,7} \right. \\
&\quad \left. - \frac{190175\zeta_{3,9}}{6} - 10716\zeta_{2,1,1,8} \right) \epsilon^5 + O(\epsilon^6) \Big\}
\end{aligned}$$

[R. Lee, A. and V. Smirnov's'11]

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**Multiple zeta values**  $\zeta_m = \zeta(m)$ ,  $\zeta_{m_1, m_2} = \zeta(m_1, m_2), \dots$

$$\zeta(m_1, \dots, m_k) = \sum_{i_1=1}^{\infty} \sum_1^{i_1-1} \cdots \sum_1^{i_{k-1}-1} \prod_{j=1}^k \frac{\text{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}.$$

# Applications of the Roman Lee's method based on dimensional recurrence relations

[R. Lee'09]

[O. Tarasov'96]

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- Conclusion

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The whole problem of evaluation→

- constructing a reduction procedure
- evaluating master integrals

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- Gegenbauer polynomials, gluing,

[K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov'79–81]

mapping Feynman integrals to series and summing  
them up, [M. Kalmykov, O. Veretin, *Sigma by C. Schneider*]

in particular using expansions of hypergeometric  
functions at (half)-integer indices [A. Davydychev, M. Kalmykov]

...

A new method of evaluating master integrals is based on the use of dimensional recurrence relations [O. Tarasov'96] and analytic properties of Feynman integrals as functions of  $d$ .

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The missing finite parts of  $A_{9,2}$  and  $A_{9,4}$

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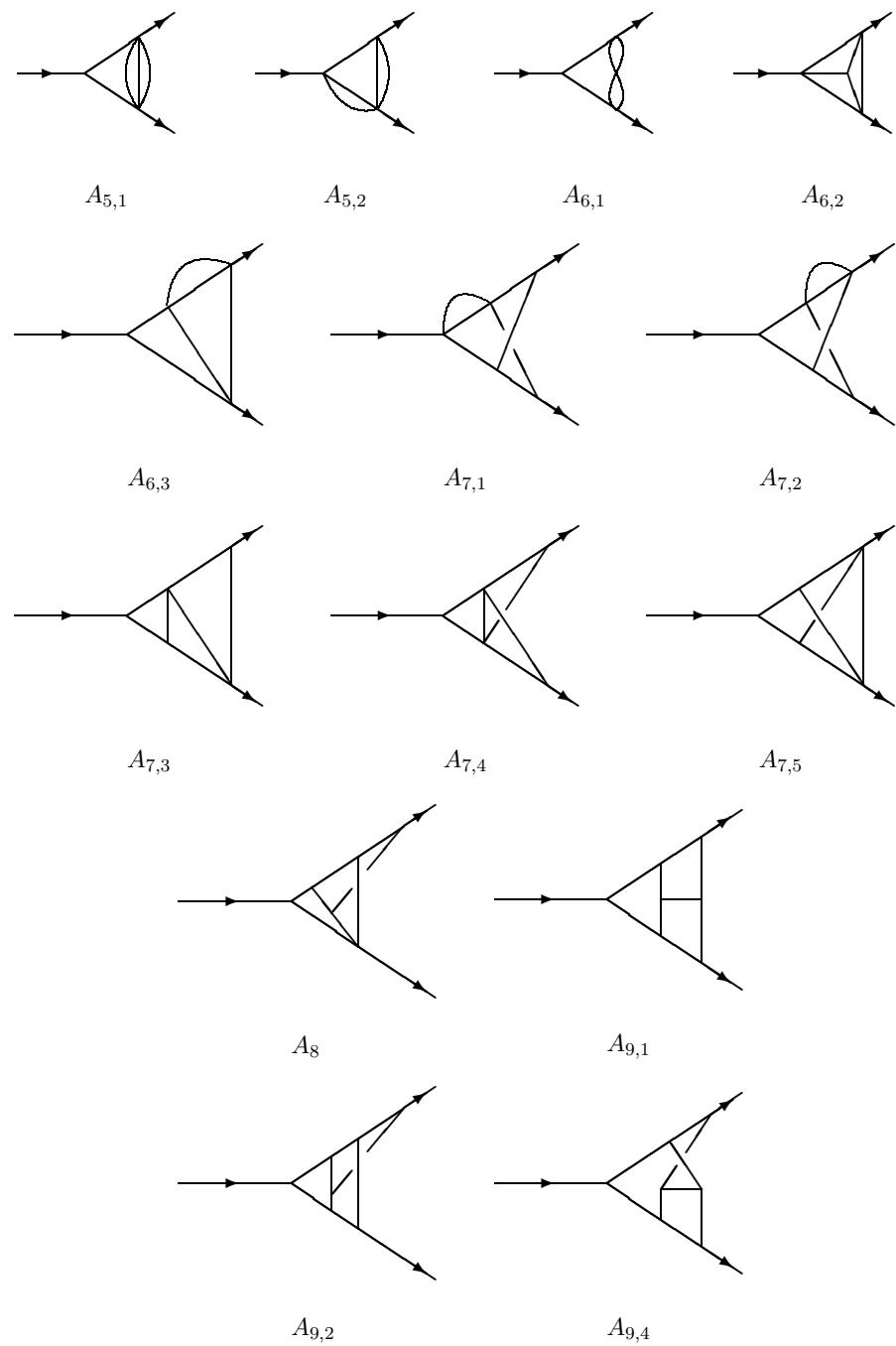
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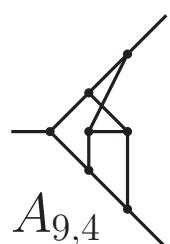
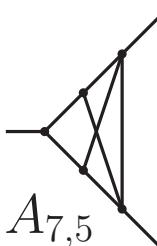
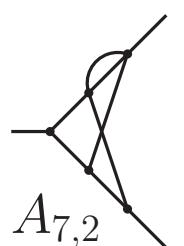
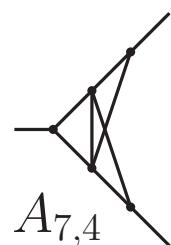
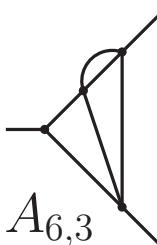
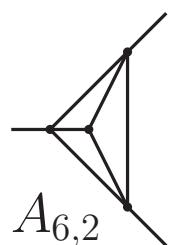
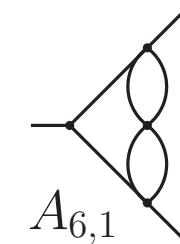
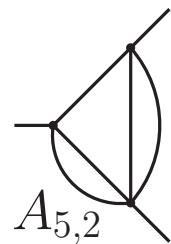
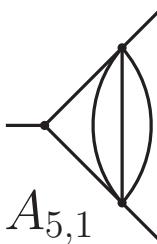
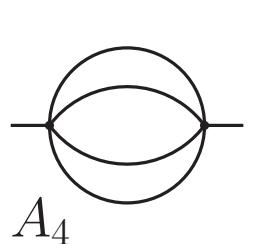
Also the subleading  $O(\varepsilon)$  terms for the fermion-loop type contributions were calculated.

The evaluation of terms of transcendentality weight 8 characteristic for four loops:  $\pi^8, \zeta(5)\zeta(3), \zeta(3)^2\pi^2, \zeta(-6, -2)$ , i.e. up to  $\varepsilon^2$  for  $A_{9,2}$  and  $A_{9,4}$  and  $\varepsilon^3$  for  $A_{9,1}$ .

[R. Lee, and V. Smirnov'10]



## $A_{9,4}$ and lower master integrals



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$A_4$ ,  $A_{5,1}$ ,  $A_{5,2}$  are of complexity level 0.

Graph  $\Gamma \rightarrow$  dimensionally regularized Feynman integral

$$F_\Gamma(a_1 \dots, a_L; d) = \frac{e^{i\pi(a+h(1-d/2))/2} \pi^{hd/2}}{\prod_l \Gamma(a_l)} \\ \times \int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} e^{i\mathcal{V}/\mathcal{U} - i \sum m_l^2 \alpha_l},$$

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For a Feynman integral with  $1/(m^2 - k^2 - i0)^{a_l}$  propagators,

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{l \notin T} \alpha_l,$$

$$\mathcal{V} = \sum_{2-\text{trees } T} \prod_{l \notin T} \alpha_l \left( q^T \right)^2.$$

## Dimensional recurrence relation

[O. Tarasov'96]

$$d \rightarrow d - 2$$

$$\mathcal{U}^{-d/2} \rightarrow \mathcal{U}^{-(d-2)/2} = \mathcal{U}\mathcal{U}^{-d/2}$$

Inserting  $\alpha_l \rightarrow (-ia_l)\mathbf{l}^+$

$$F_\Gamma(a_1 \dots, a_L; d-2) = \frac{1}{\pi} \sum_T \prod_{l \notin T} a_l \mathbf{l}^+ F_\Gamma(a_1 \dots, a_L; d)$$

For  $F = A_{6,3}$ :

$$\begin{aligned} F(1, 1, 1, 1, 1, 1; d-2) &= F(1, 1, 1, 2, 2, 2; d) + F(1, 1, 2, 1, 2, 2; d) \\ &+ F(1, 1, 2, 2, 1, 2; d) + F(1, 2, 1, 2, 1, 2; d) + F(1, 2, 1, 2, 2, 1; d) \\ &+ F(1, 2, 2, 1, 1, 2; d) + F(1, 2, 2, 1, 2, 1; d) + F(1, 2, 2, 2, 1, 1; d) \\ &+ F(2, 1, 1, 2, 1, 2; d) + F(2, 1, 1, 2, 2, 1; d) + F(2, 1, 2, 1, 1, 2; d) \\ &+ F(2, 1, 2, 1, 2, 1; d) + F(2, 1, 2, 2, 1, 1; d) \end{aligned}$$

The integrals with raised indices are reduced to master integrals by IBP relations

**FIRE** [A. Smirnov] and R.Lee's code

$A_{6,3}$  is of complexity level 1.

$$\begin{aligned} A_{6,3}(d-2) &= \frac{8(-3+d)(-9+2d)(-7+2d)(-10+3d)}{-16+3d} A_{6,3}(d) \\ &+ \frac{32(-3+d)(-9+2d)(-7+2d)(-5+2d)(-10+3d)}{(-5+d)(-4+d)^2(-16+3d)} \\ &\times (-8+3d)(-32+7d) A_4(d) \end{aligned}$$

To solve this difference relation turn to  $g(d) = \Sigma(d)A_{6,3}(d)$ , where

$$\begin{aligned}\Sigma(d) &= \frac{1}{\sqrt{\pi}} 32^{4d-\frac{5}{2}} \left(\frac{d}{2} - \frac{5}{3}\right) \sin\left(\frac{1}{2}\pi(d-5)\right) \sin\left(\frac{1}{2}\pi\left(d - \frac{14}{3}\right)\right) \\ &\times \sin^2\left(\frac{\pi d}{2}\right) \Gamma\left(\frac{d}{2} - \frac{5}{4}\right) \Gamma\left(\frac{d}{2} - \frac{3}{4}\right) \Gamma\left(\frac{d}{2} - \frac{1}{2}\right)\end{aligned}$$

Equation for  $g$ :

$$g(d-2) = g(d) + h(d)$$

with

$$\begin{aligned}h(d) &= \frac{1}{3(d-5)} 1024\pi(32-7d) \sin\left(\frac{\pi d}{2}\right) \sin(\pi d) \cos\left(\frac{1}{6}(3\pi d + \pi)\right) \\ &\times \Gamma\left(7 - \frac{3d}{2}\right) \Gamma\left(\frac{d}{2} - 2\right)^3\end{aligned}$$

$-\sum_{k=0}^{\infty} h(d - 2k)$  and  $+\sum_{k=1}^{\infty} h(d + 2k)$  are solutions

General solution:

$$g(d) = \sum_{k=1}^{\infty} h(d + 2k) + \omega(d)$$

where  $\omega(d) = \Omega(e^{i\pi d})$  is a periodic function of period 2.

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The analytic structure of  $g(d)$  follows from the analytic properties of  $A_{6,3}(d)$ . They can be obtained by

**Fiesta** [A. Smirnov and M. Tentyukov'08, A. and V. Smirnovs and M. Tentyukov'09]

# Sector decompositions

[T. Binoth & G. Heinrich'00,04; C. Bogner & S. Weinzierl'07]

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SDAnalyze [U, F, h, degrees, dmin, dmax]

where U and F are the basic functions, h is the number of loops, degrees are the indices, and dmin and dmax are values of the real part of  $d$  that determine the basic stripe.

For  $A_{6,3}$ , Fiesta says that in the band  $(3, 5]$  there can be simple poles at  $d = 10/3, 4, 14/3, 5$ .

Analyzing behaviour at infinity ( $\text{Im}d \rightarrow \pm\infty$ ) with the help of the alpha representation provides the following Ansatz for  $\Omega(z)$ , with  $z = e^{i\pi d}$ :

$$a_0 + \frac{a_1}{z - e^{-2i\pi/3}} + \frac{a_2}{z - 1}$$

which corresponds to the following Ansatz for  $\omega(d)$ :

$$b_0 + b_1 \cot\left(\frac{1}{2}\pi(d - 4)\right) + b_2 \cot\left(\frac{1}{2}\pi\left(d - \frac{10}{3}\right)\right)$$

The constants  $b_i$  are fixed by using an MB representation for  $A_{6,3}$

$$512\pi^4 3^{-1/2} \cot\left(\frac{1}{2}\pi\left(d - \frac{10}{3}\right)\right) - 512\pi^4 3^{-1/2} \cot\left(\frac{1}{2}\pi(d - 4)\right)$$

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MB tools at <http://projects.hepforge.org/mbtools/>

[MB.m](#) [M. Czakon'05]

[MBresolve.m](#) [A. Smirnov'09]

[barnesroutines.m](#) [D. Kosower'08]

## Result:

$$A_{6,3}(d) = S(d) \sum_{k=0}^{\infty} h(d + 2k) + g(d),$$

$$S(d) = -\sin(\pi d)g(d) = \frac{\pi^4 2^{11-3d} \csc\left(\frac{3\pi d}{2}\right) \csc\left(\frac{\pi d}{2}\right)}{(3d-10)\Gamma\left(d-\frac{5}{2}\right)\Gamma\left(\frac{d-1}{2}\right)},$$

$$h(d) = \frac{(7d-18) \sin\left(\frac{\pi d}{2}\right) \Gamma\left(\frac{d}{2}-1\right)^3}{3\pi^2(d-3)\Gamma\left(\frac{3d}{2}-3\right)}.$$

PSLQ

[H.R.P. Ferguson & D.H. Bailey'91]

is an algorithm that provides the best approximation of a given numerical number as a linear combination, with rational coefficients, of a given set of transcendental numbers.

Typically, an accuracy of at least seven digits per transcendental number is needed.

For example,  $54 \times 7 = 378$  digits.

PSLQ gives

$$\begin{aligned}
A_{6,3}(4 - 2\epsilon) = & \frac{e^{-3\gamma\epsilon}}{(1 - 4\epsilon)(1 - 3\epsilon)(1 - 2\epsilon)} \left\{ \frac{1}{6\epsilon^3} + \frac{\pi^2}{8\epsilon} - \frac{35\zeta_3}{6} - \frac{77\pi^4\epsilon}{2880} \right. \\
& - \epsilon^2 \left( \frac{49\pi^2\zeta_3}{24} + \frac{651\zeta_5}{10} \right) + \epsilon^3 \left( \frac{1141\zeta_3^2}{12} - \frac{93451\pi^6}{725760} \right) \\
& - \epsilon^4 \left( \frac{713\pi^2\zeta_5}{40} - \frac{511}{320}\pi^4\zeta_3 + \frac{9017\zeta_7}{14} \right) + \epsilon^5 \left( \frac{623}{48}\pi^2\zeta_3^2 \right. \\
& \left. - \frac{544}{9}\zeta_{-6,-2} + \frac{11195\zeta_5\zeta_3}{6} - \frac{2022493\pi^8}{11612160} \right) + O(\epsilon^6) \left. \right\},
\end{aligned}$$

$$\begin{aligned}
A_{9,4}(4 - 2\varepsilon) = & e^{-3\gamma_E \varepsilon} \left\{ -\frac{1}{9\varepsilon^6} - \frac{8}{9\varepsilon^5} + \left[ 1 + \frac{43\pi^2}{108} \right] \frac{1}{\varepsilon^4} \right. \\
& + \left[ \frac{109\zeta(3)}{9} + \frac{14}{9} + \frac{53\pi^2}{27} \right] \frac{1}{\varepsilon^3} \\
& + \left[ \frac{608\zeta(3)}{9} - 17 - \frac{311\pi^2}{108} - \frac{481\pi^4}{12960} \right] \frac{1}{\varepsilon^2} \\
& \left. + \left[ -\frac{949\zeta(3)}{9} - \frac{2975\pi^2\zeta(3)}{108} + \frac{3463\zeta(5)}{45} + 84 + \frac{11\pi^2}{18} + \frac{85\pi^4}{108} \right] \frac{1}{\varepsilon} \right\}
\end{aligned}$$

[P. Baikov, K. Chetyrkin, A. and V. Smirnovs, & M. Steinhauser'09]

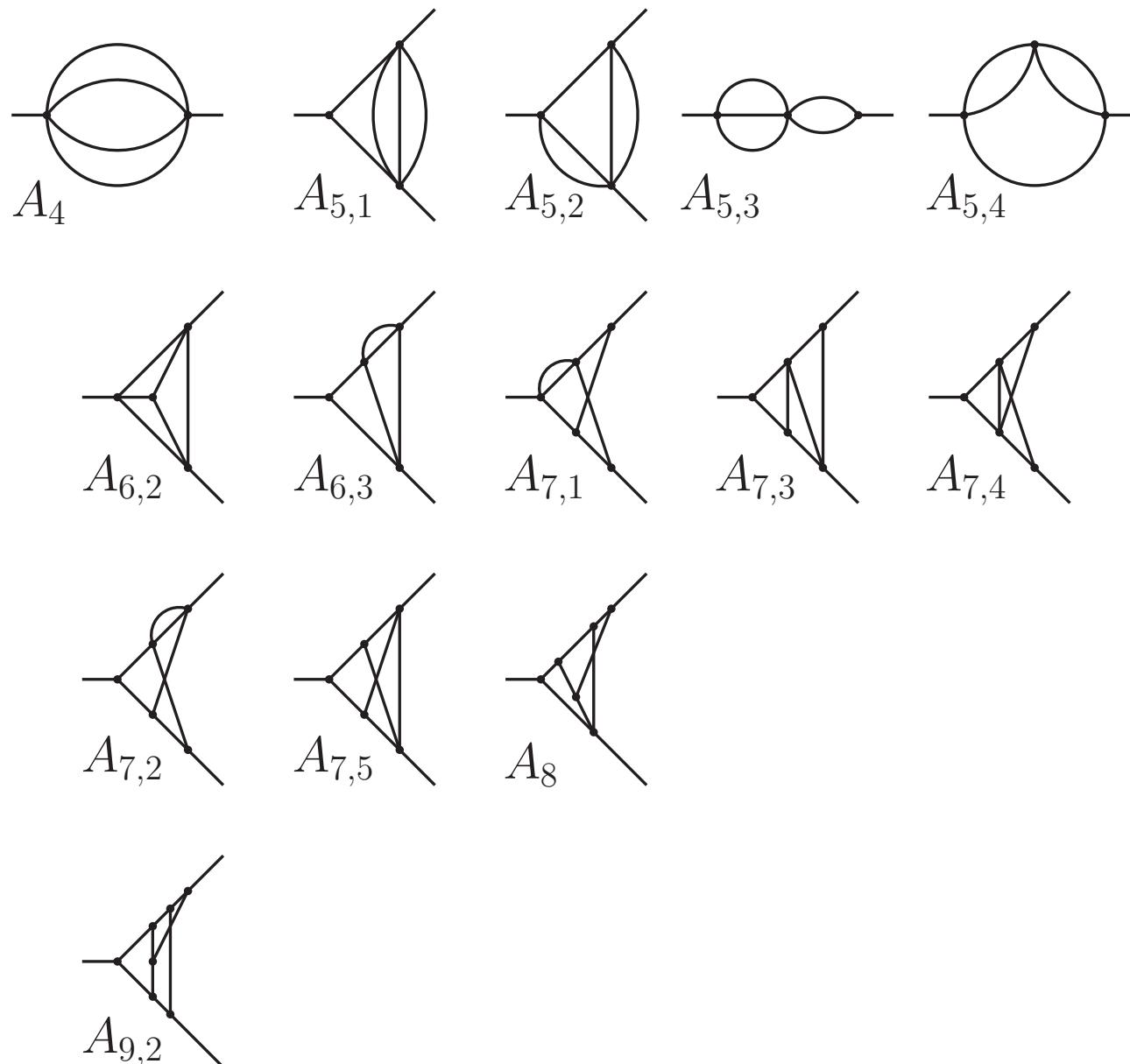
[ G. Heinrich, T. Huber, D. Kosower and V. Smirnov'09]

$$\begin{aligned}
& + \left[ \frac{434\zeta(3)}{9} - \frac{299\pi^2\zeta(3)}{3} - \frac{3115\zeta_3^2}{6} + \frac{7868\zeta(5)}{15} - 339 \right. \\
& \left. + \frac{77\pi^2}{4} - \frac{2539\pi^4}{2592} - \frac{247613\pi^6}{466560} \right] [R. Lee, A. and V. Smirnovs'10]
\end{aligned}$$

$$\begin{aligned}
& + \left[ -1242 + 112\pi^2 - 589\zeta_3 + \frac{487\pi^4}{432} - \frac{19499\pi^2\zeta_3}{108} + \frac{30067\zeta_5}{45} + \frac{25567\pi^6}{30240} \right. \\
& + \frac{18512\zeta_3^2}{9} + \frac{38903\pi^4\zeta_3}{2592} + \frac{113629\pi^2\zeta_5}{540} - \frac{8564\zeta_7}{63} \Big] \varepsilon \\
& + \left[ 4293 - \frac{1887\pi^2}{4} + 3756\zeta_3 - \frac{491\pi^4}{32} + \frac{4019\pi^2\zeta_3}{18} + \frac{7874\zeta_5}{15} \right. \\
& - \frac{9901847\pi^6}{3265920} - \frac{26291\zeta_3^2}{6} + \frac{9037\pi^4\zeta_3}{135} + \frac{35728\pi^2\zeta_5}{45} - \frac{72537\zeta_7}{14} \\
& \left. + \frac{30535087\pi^8}{31352832} - \frac{152299}{216}\pi^2\zeta_3^2 + \frac{730841\zeta_3\zeta_5}{135} - \frac{76288}{81}\zeta_{-6,-2} \right] \varepsilon^2 + O(\varepsilon^3) \Big\}
\end{aligned}$$

[R. Lee and V. Smirnov'10]

## $A_{9,2}$ and lower master integrals



$$\begin{aligned}
A_{9,2}(4 - 2\varepsilon) = & e^{-3\gamma_E \varepsilon} \left\{ -\frac{2}{9\varepsilon^6} - \frac{5}{6\varepsilon^5} + \left[ \frac{20}{9} + \frac{17\pi^2}{54} \right] \frac{1}{\varepsilon^4} \right. \\
& + \left[ \frac{31\zeta(3)}{3} - \frac{50}{9} + \frac{181\pi^2}{216} \right] \frac{1}{\varepsilon^3} \\
& + \left[ \frac{347\zeta(3)}{18} + \frac{110}{9} - \frac{17\pi^2}{9} + \frac{119\pi^4}{432} \right] \frac{1}{\varepsilon^2} \\
& \left. + \left[ -\frac{514\zeta(3)}{9} - \frac{341\pi^2\zeta(3)}{36} + \frac{2507\zeta_5}{15} - \frac{170}{9} + \frac{19\pi^2}{6} + \frac{163\pi^4}{960} \right] \frac{1}{\varepsilon} \right\}
\end{aligned}$$

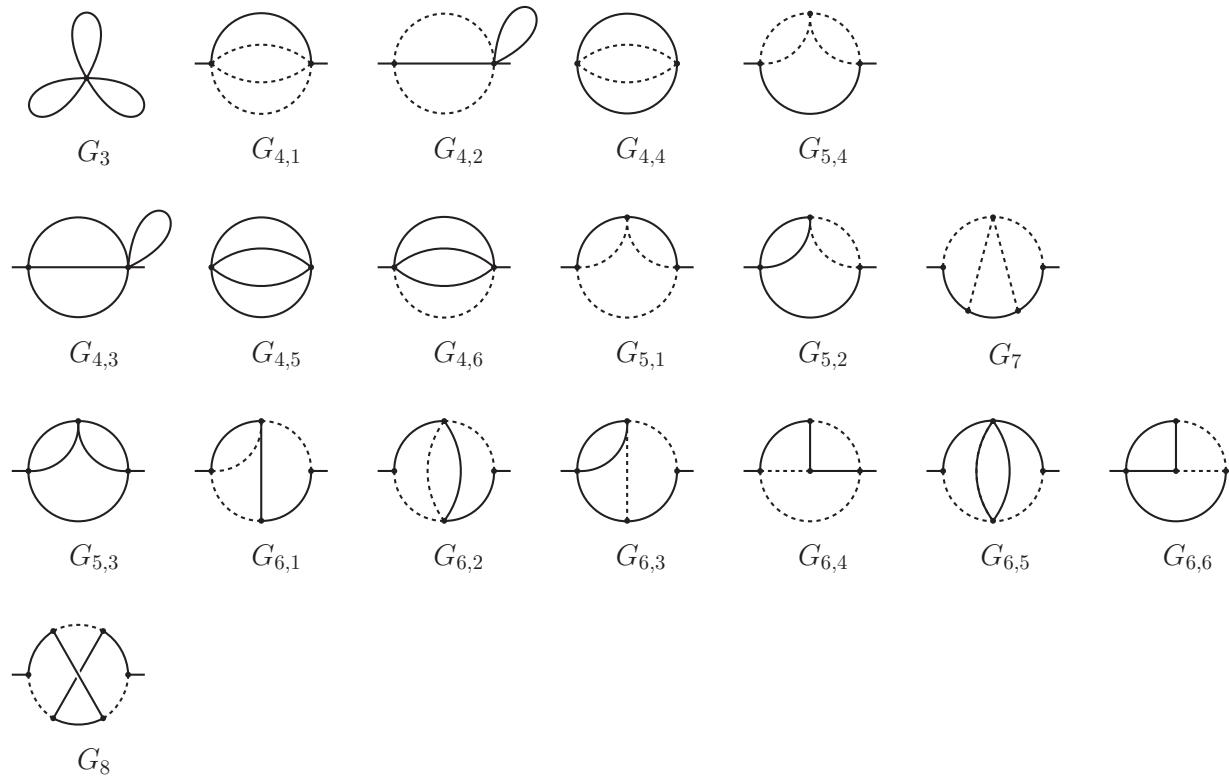
[P. Baikov, K. Chetyrkin, A. and V. Smirnovs, & M. Steinhauser'09]

[ G. Heinrich, T. Huber, D. Kosower and V. Smirnov'09]

$$\begin{aligned}
& + \left[ \frac{1516\zeta(3)}{9} - \frac{737\pi^2\zeta(3)}{24} - 29\zeta(3)^2 + \frac{2783\zeta(5)}{6} - \frac{130}{9} \right. \\
& \left. + \frac{\pi^2}{2} - \frac{943\pi^4}{1080} + \frac{195551\pi^6}{544320} \right] [R. Lee, A. and V. Smirnovs'10]
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{2950}{9} - \frac{83\pi^2}{2} - \frac{4444\zeta_3}{9} + \frac{8801\pi^4}{2160} + \frac{1357\pi^2\zeta_3}{18} - \frac{2830\zeta_5}{3} \right. \\
& + \frac{2416889\pi^6}{2177280} + \frac{19169\zeta_3^2}{36} + \frac{5963\pi^4\zeta_3}{4320} - \frac{8183\pi^2\zeta_5}{60} + \frac{43329\zeta_7}{14} \Big] \varepsilon \\
& + \left[ -\frac{19090}{9} + \frac{569\pi^2}{2} + \frac{12916\zeta_3}{9} - \frac{7795\pi^4}{432} - \frac{1433\pi^2\zeta_3}{9} + \frac{3112\zeta_5}{3} \right. \\
& - \frac{64733\pi^6}{54432} - \frac{1214\zeta_3^2}{9} + \frac{58517\pi^4\zeta_3}{1728} - \frac{146521\pi^2\zeta_5}{360} + \frac{580805\zeta_7}{42} \\
& \left. + \frac{24178127\pi^8}{14515200} + \frac{6419}{36}\pi^2\zeta_3^2 - \frac{101288\zeta_3\zeta_5}{15} - \frac{20752}{9}\zeta_{-6,-2} \right] \varepsilon^2 + O(\varepsilon^3) \Big\}
\end{aligned}$$

## master integrals for 3-loop $g - 2$



[S. Laporta and E. Remiddi'96-97]

[K. Melnikov and T. van Ritbergen'00]

A pure numerical approach to evaluate 4-loop  $g - 2$ :

[T. Kinoshita et al.'06-10]

IBP reduction to master integrals and numerical evaluation  
of the master integrals:

[S. Laporta and E. Remiddi'08]

A warm-up before 4-loop calculation: numerical evaluation  
of 3-loop master integrals to higher orders in  $\varepsilon$ :

[S. Laporta'01]

Analytic evaluation of 3-loop master integrals up to  
transcendentality weight 7:

[R. Lee and V. Smirnov'10]

$$\begin{aligned}
G_8(4 - 2\epsilon) = & \Gamma(1 + \epsilon)^3 \left\{ \left( -\frac{\pi^4}{6} + 4\pi^2 \ln^2 2 \right) \right. \\
& + \epsilon \left( \frac{2\pi^4}{3} - 16\pi^2 \ln^2 2 + \frac{166}{45}\pi^4 \ln 2 - \frac{208}{9}\pi^2 \ln^3 2 - \frac{32 \ln^5 2}{15} \right. \\
& \left. + 256a_5 + \frac{17\pi^2 \zeta_3}{6} - 291\zeta_5 \right) \\
& + \epsilon^2 \left( -\frac{14\pi^4}{3} + 112\pi^2 \ln^2 2 - \frac{664}{45}\pi^4 \ln 2 + \frac{832}{9}\pi^2 \ln^3 2 + \frac{128 \ln^5 2}{15} \right. \\
& - 1024a_5 - \frac{34\pi^2 \zeta_3}{3} + 1164\zeta_5 - \frac{21743\pi^6}{11340} - \frac{197}{9}\pi^4 \ln^2 2 + \frac{713}{9}\pi^2 \ln^4 2 \\
& \left. + \frac{64 \ln^6 2}{9} - 104\pi^2 a_4 + 5120a_6 + 2688s_6 - 51\pi^2 \ln 2 \zeta_3 - 953\zeta_3^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + \epsilon^3 \left( \frac{80\pi^4}{3} - 640\pi^2 \ln^2 2 + \frac{4648}{45}\pi^4 \ln 2 - \frac{5824}{9}\pi^2 \ln^3 2 - \frac{896 \ln^5 2}{15} + 7168a_5 \right. \\
& + \frac{238\pi^2 \zeta_3}{3} - 8148\zeta_5 + \frac{21743\pi^6}{2835} + \frac{788}{9}\pi^4 \ln^2 2 - \frac{2852}{9}\pi^2 \ln^4 2 \\
& - \frac{256 \ln^6 2}{9} + 416\pi^2 a_4 - 20480a_6 - 10752s_6 + 204\pi^2 \ln 2 \zeta_3 + 3812\zeta_3^2 \\
& + \frac{4868}{189}\pi^6 \ln 2 + \frac{8492}{135}\pi^4 \ln^3 2 - \frac{7288}{45}\pi^2 \ln^5 2 - \frac{4864 \ln^7 2}{315} \\
& + 1776\pi^2 \ln 2 a_4 + 1520\pi^2 a_5 + 77824a_7 - \frac{106880}{7} \ln 2 s_6 + \frac{4003\pi^4 \zeta_3}{21} \\
& + \frac{2167}{3}\pi^2 \ln^2 2 \zeta_3 - \frac{316}{3}\ln^4 2 \zeta_3 - 2528a_4 \zeta_3 + \frac{133600}{7} \ln 2 \zeta_3^2 \\
& + \frac{875561\pi^2 \zeta_5}{84} + 37200 \ln^2 2 \zeta_5 - \frac{1325727\zeta_7}{7} + \frac{106880s_{7,a}}{7} \\
& \left. - \frac{161920s_{7,b}}{7} \right) + O(\epsilon^4) \Bigg\}
\end{aligned}$$

$$a_i = \text{Li}_i\left(\frac{1}{2}\right),$$

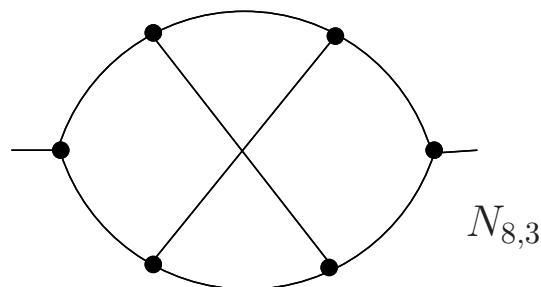
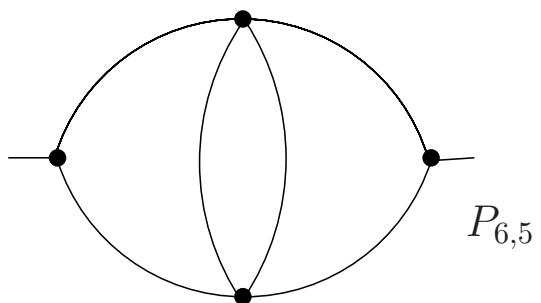
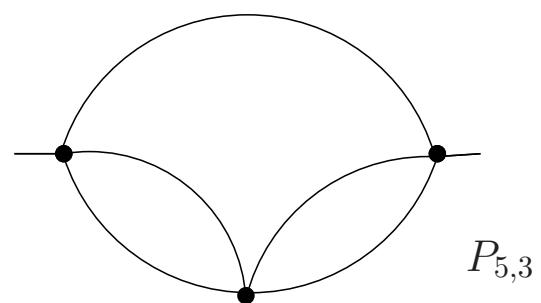
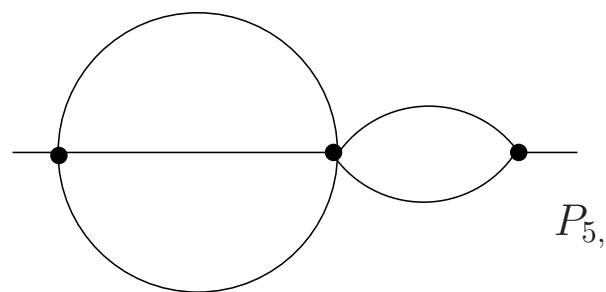
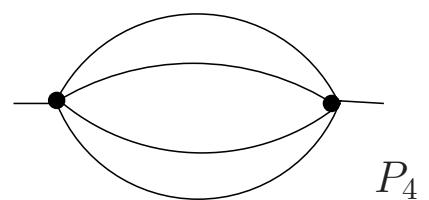
$$s_6 = \zeta(-5, -1) + \zeta(6),$$

$$s_{7a} = \zeta(-5, 1, 1) + \zeta(-6, 1) + \zeta(-5, 2) + \zeta(-7),$$

$$s_{7b} = \zeta(7) + \zeta(5, 2) + \zeta(-6, -1) + \zeta(5, -1, -1)$$

The coefficients in the  $\varepsilon$ -expansion of planar massless propagator diagrams up to five loops should be expressed in terms of multiple zeta values, while the non-planar graphs may contain, in addition, multiple sums with 6th roots of unity.

[Brown'08]

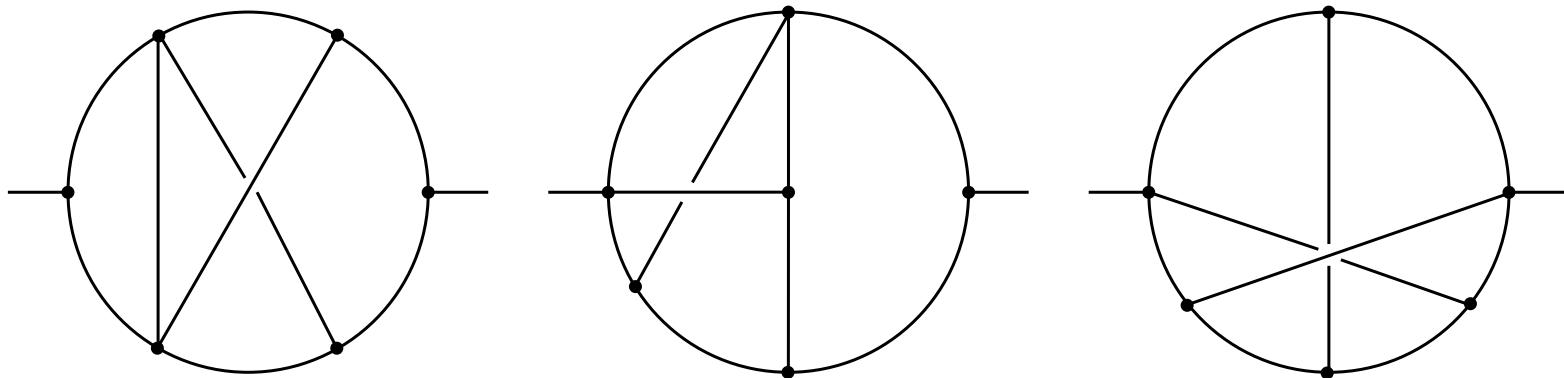
 $N_{8,3}$  $P_{6,5}$  $P_{5,3}$  $P_{5,4}$  $P_4$

$$\begin{aligned}
N_{8,3}(d+2) &= \frac{(d-4)}{8(d-2)(d-1)(2d-7)(2d-5)} N_{8,3} \\
&+ \frac{4(5d^2 - 28d + 38)}{(d-4)^2(d-2)(d-1)(2d-5)} P_{5,3} \\
&\left[ 4(d-4)(d-2)(d-1)(2d-7)(2d-5)(3d-8) \right]^{-1} \\
&\times (37d^3 - 313d^2 + 858d - 752) P_{6,5} \\
&- \left[ 2(d-4)^2(d-3)(d-2)(d-1)(2d-7)(2d-5) \right]^{-1} \\
&\times (43d^4 - 478d^3 + 1963d^2 - 3530d + 2352) P_{5,4} \\
&- \left[ (d-4)^3(d-3)^2(d-2)(d-1)(2d-7)(3d-8) \right]^{-1} \\
&\times (401d^6 - 7251d^5 + 54491d^4 - 217784d^3 \\
&+ 489064d^2 - 581248d + 287232) P_4
\end{aligned}$$

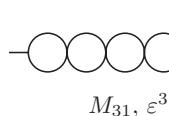
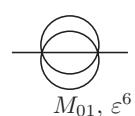
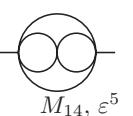
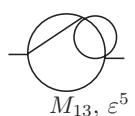
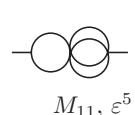
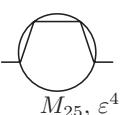
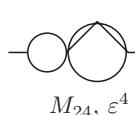
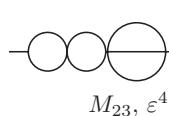
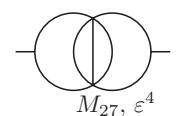
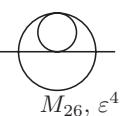
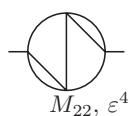
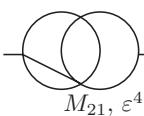
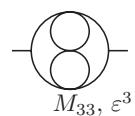
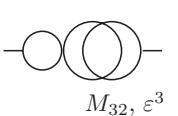
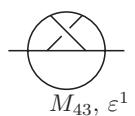
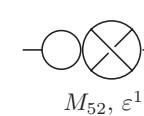
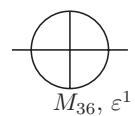
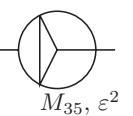
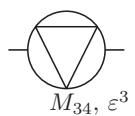
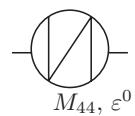
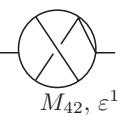
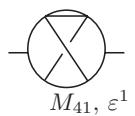
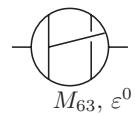
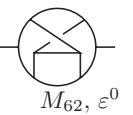
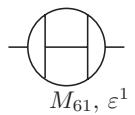
$$\begin{aligned}
& \frac{e^{-3\gamma_E \varepsilon}}{1 - 2\varepsilon} \left\{ 20\zeta(5) + \varepsilon \left( 68\zeta(3)^2 + \frac{10\pi^6}{189} \right) \right. && [\text{Chetyrkin, Kataev \& Tkachov'80, Kazakov'84}] \\
& + \varepsilon^2 \left( \frac{34\pi^4\zeta(3)}{15} - 5\pi^2\zeta_5 + 450\zeta(7) \right) && [\text{Becavac'06}] \\
& + \varepsilon^3 \left( -\frac{9072}{5}\zeta(5,3) - 2588\zeta(3)\zeta(5) - 17\pi^2\zeta(3)^2 + \frac{6487\pi^8}{10500} \right) \\
& + \varepsilon^4 \left( -\frac{4897\pi^6\zeta_3}{630} - \frac{6068\zeta(3)^3}{3} + \frac{13063\pi^4\zeta_5}{120} - \frac{225\pi^2\zeta(7)}{2} + \frac{88036\zeta(9)}{9} \right) \\
& + \varepsilon^5 \left( \frac{2268}{5}\pi^2\zeta(5,3) + 42513\zeta(8,2) - 145328\zeta(3)\zeta(7) \right. \\
& \left. - 73394\zeta(5)^2 + 647\pi^2\zeta(3)\zeta(5) - \frac{11813\pi^4\zeta(3)^2}{120} + \frac{28138577\pi^{10}}{9355500} \right) + \dots \left. \right\}
\end{aligned}$$

F. Brown (2008):

*For the following three diagrams, every coefficient in the Taylor expansion in  $\varepsilon$  is a rational linear combination of multiple zeta values and Goncharov's polylogarithms with 6<sup>th</sup> roots of unity as arguments.*



$$\text{Li}_{n_1, \dots, n_r}(x_1, \dots, x_r) = \sum_{0 < k_1 < \dots < k_r} \frac{x_1^{k_1} \dots x_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}}$$



$$\begin{aligned}
M_{4,5} = & \frac{(1-2\epsilon)^3}{1-6\epsilon} \left( \frac{\Gamma(1-\varepsilon)^2 \Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)} \right)^3 \left\{ 36\zeta_3^2 - \left( -108\zeta_3\zeta_4 + 378\zeta_7 \right) \epsilon + \left( 5868\zeta_3\zeta_5 \right. \right. \\
& \left. \left. - \frac{14805\zeta_8}{2} + 1512\zeta_{2,6} \right) \epsilon^2 - \left( 732\zeta_3^3 + 270\zeta_4\zeta_5 - 6930\zeta_3\zeta_6 + \frac{42458\zeta_9}{3} \right) \epsilon^3 \right. \\
& \left. + \left( -3294\zeta_3^2\zeta_4 + \frac{202569\zeta_5^2}{7} + \frac{697887\zeta_3\zeta_7}{7} - \frac{895521\zeta_{10}}{7} - \frac{58563\zeta_{3,7}}{7} \right) \epsilon^4 \right. \\
& \left. - \left( 223152\zeta_3^2\zeta_5 - 105454\zeta_5\zeta_6 + 214815\zeta_4\zeta_7 - \frac{1404105\zeta_3\zeta_8}{2} + 1002552\zeta_2\zeta_9 \right. \right. \\
& \left. \left. - \frac{4736453\zeta_{11}}{4} + 33504\zeta_3\zeta_{2,6} + 54192\zeta_{2,1,8} \right) \epsilon^5 + \left( -\frac{22816\zeta_3^4}{3} - 611040\zeta_3\zeta_4\zeta_5 \right. \right. \\
& \left. \left. - 1041984\zeta_2\zeta_5^2 - \frac{553066}{3}\zeta_3^2\zeta_6 - 2118272\zeta_2\zeta_3\zeta_7 + \frac{17993629\zeta_5\zeta_7}{3} \right. \right. \\
& \left. \left. + \frac{60130438\zeta_3\zeta_9}{9} - \frac{659340762619\zeta_{12}}{99504} - 59016\zeta_4\zeta_{2,6} + 77184\zeta_2\zeta_{3,7} \right. \right. \\
& \left. \left. - \frac{3234680\zeta_{3,9}}{9} - 120064\zeta_{2,1,1,8} \right) \epsilon^6 + O(\epsilon^7) \right\}
\end{aligned}$$

$$\begin{aligned}
M_{6,3} = & \left( \frac{\Gamma(1-\varepsilon)^2 \Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)} \right)^3 \frac{(1-2\epsilon)^3}{(1+3\epsilon)(1+4\epsilon)} \left\{ -\frac{5\zeta_5}{\epsilon} - \left( 20\zeta_5 + 41\zeta_3^2 + \frac{25\zeta_6}{2} - \frac{161\zeta_7}{2} \right) \right. \\
& + \left( -308\zeta_3^2 - 50\zeta_6 - 123\zeta_3\zeta_4 + 514\zeta_7 + 4862\zeta_3\zeta_5 - \frac{24451\zeta_8}{4} + 1566\zeta_{2,6} \right) \epsilon \\
& + \left( -924\zeta_3\zeta_4 - 1500\zeta_7 + 68636\zeta_3\zeta_5 - \frac{744639\zeta_8}{8} + 23220\zeta_{2,6} + \frac{1526\zeta_3^3}{3} \right. \\
& \left. - 2103\zeta_4\zeta_5 + 4325\zeta_3\zeta_6 + \frac{111709\zeta_9}{36} \right) \epsilon^2 + \left( 235200\zeta_3\zeta_5 - \frac{710311\zeta_8}{2} \right. \\
& \left. + 85536\zeta_{2,6} + \frac{22048\zeta_3^3}{3} - 36366\zeta_4\zeta_5 + 55695\zeta_3\zeta_6 + \frac{237103\zeta_9}{12} + 2289\zeta_3^2\zeta_4 \right. \\
& \left. + \frac{1341143\zeta_5^2}{56} + \frac{3816969\zeta_3\zeta_7}{56} - \frac{7815019\zeta_{10}}{112} - \frac{500565\zeta_{3,7}}{56} \right) \epsilon^3 + \left( \frac{61040\zeta_3^3}{3} \right. \\
& \left. - 160416\zeta_4\zeta_5 + 161860\zeta_3\zeta_6 - \frac{460411\zeta_9}{9} + 33072\zeta_3^2\zeta_4 + \frac{60035137\zeta_3\zeta_7}{56} \right. \\
& \left. + \frac{12859479\zeta_5^2}{56} - \frac{134815227\zeta_{10}}{112} - \frac{7724781\zeta_{3,7}}{56} - 453668\zeta_3^2\zeta_5 + \frac{280574047\zeta_{11}}{64} \right. \\
& \left. + \frac{1346777\zeta_5\zeta_6}{6} - \frac{4654793\zeta_4\zeta_7}{8} + 1309878\zeta_3\zeta_8 - 2749211\zeta_2\zeta_9 - 87752\zeta_3\zeta_{2,6} \right. \\
& \left. - 148606\zeta_{2,1,8} \right) \epsilon^4
\end{aligned}$$

$$\begin{aligned}
& + \left( 91560\zeta_3^2\zeta_4 + \frac{5319415\zeta_5^2}{14} + \frac{55472425\zeta_3\zeta_7}{14} - \frac{705626\zeta_3^4}{9} \right. \\
& - \frac{7239597\zeta_{3,7}}{14} - 5183674\zeta_3^2\zeta_5 + \frac{3338505\zeta_5\zeta_6}{2} - 1597650\zeta_{2,1,8} - \frac{1752859}{18}\zeta_3^2\zeta_6 \\
& + \frac{62741559\zeta_3\zeta_8}{4} - 29556525\zeta_2\zeta_9 + \frac{2990096591\zeta_{11}}{64} - \frac{29828681054659\zeta_{12}}{2388096} \\
& - \frac{53503881\zeta_4\zeta_7}{8} - \frac{142150835\zeta_{10}}{28} - 1729457\zeta_3\zeta_4\zeta_5 - \frac{25832277}{7}\zeta_2\zeta_5^2 \\
& - \frac{157544998}{21}\zeta_2\zeta_3\zeta_7 + \frac{311533051\zeta_5\zeta_7}{18} + \frac{1792012205\zeta_3\zeta_9}{108} - 1085340\zeta_3\zeta_{2,6} \\
& \left. - 227636\zeta_4\zeta_{2,6} + \frac{1913502}{7}\zeta_2\zeta_{3,7} - \frac{105698899\zeta_{3,9}}{108} - \frac{1275668}{3}\zeta_{2,1,1,8} \right) \epsilon^5 \\
& + O(\epsilon^6) \Bigg\}
\end{aligned}$$

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# extra slides

The full color dependence of the 4-loop 4-gluon amplitude  
in N=4 SUSY YM in terms of 50 4-loop 4-point integrals.

[Bern, Carrasco, Johansson & Roiban'10]

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We have

–6.1983992267494959168200925479819368763478987989679152...

# Most complicated master integrals for the three-loop static quark potential

[A. and V. Smirnovs & M. Steinhauser'09]

