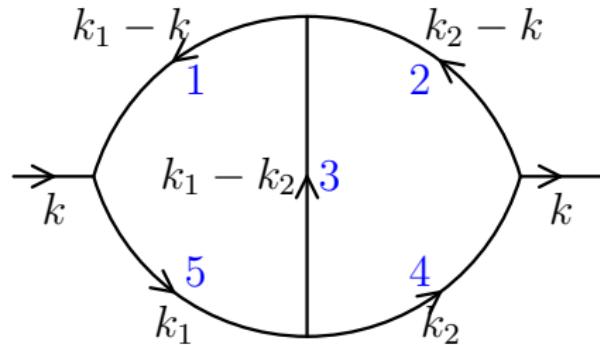


Andrey Grozin

Momentum space



$$I(a_1, a_2, a_2, a_4, a_5) = \frac{(k^2)^{\sum a_i - d}}{\pi^d} \times \int \frac{d^d k_1 d^d k_2}{[(k_1 - k)^2]^{a_1} [(k_2 - k)^2]^{a_2} [(k_1 - k_2)^2]^{a_3} (k_2^2)^{a_4} (k_1^2)^{a_5}}$$

Motivation

At 3 loops, only 1 master integral (nonplanar) does not reduce to I

Gluing: its value at $\varepsilon = 0$ is equal to the ladder integral which reduces to $I(1, 1, 1 + \varepsilon, 1, 1)$

Motivation

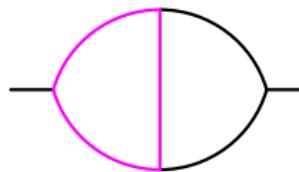
At 3 loops, only 1 master integral (nonplanar) does not reduce to I

Gluing: its value at $\varepsilon = 0$ is equal to the ladder integral which reduces to $I(1, 1, 1 + \varepsilon, 1, 1)$

At 4 loops, 15 master integrals (of 28) reduce to I , and thus can be easily expanded in ε up to high powers

Early history

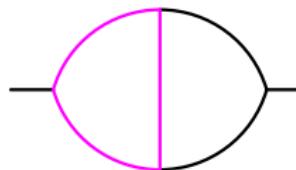
K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, Nucl. Phys.
B 174 (1980) 345



via Gegenbauer polynomials

Early history

K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, Nucl. Phys. **B 174** (1980) 345



via Gegenbauer polynomials

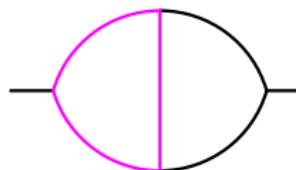
S. J. Hathrell, Ann. Phys. **139** (1982) 136; **142** (1982) 34



Im; 4-particle phase space integral ${}_3F_2(1)$)

Early history

K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, Nucl. Phys. **B 174** (1980) 345



via Gegenbauer polynomials

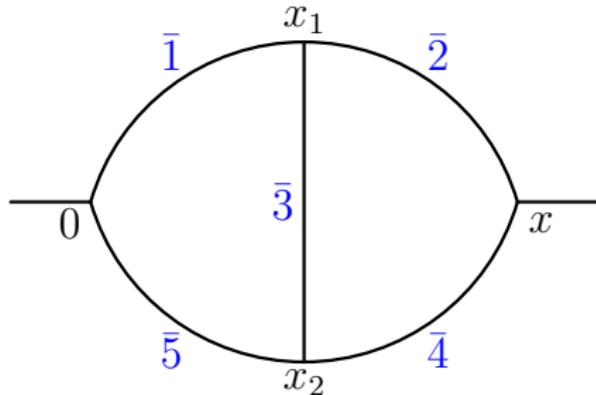
S. J. Hathrell, Ann. Phys. **139** (1982) 136; **142** (1982) 34



Im; 4-particle phase space integral ${}_3F_2(1)$)

A. N. Vasiliev, Yu. M. Pismak, Yu. R. Khonkonen,
TMΦ **47** (1981) 291 [Theor. Math. Phys. **47** (1981) 465]
duality, inversion, star-triangle, IBP

Coordinate space



$$I(a_1, a_2, a_2, a_4, a_5) \sim$$

$$\int \frac{d^d x_1 d^d x_2}{(x_1^2)^{\bar{a}_1} [(x_1 - x)^2]^{\bar{a}_2} [(x_1 - x_2)^2]^{\bar{a}_3} [(x_2 - x)^2]^{\bar{a}_4} (x_2^2)^{\bar{a}_5}}$$

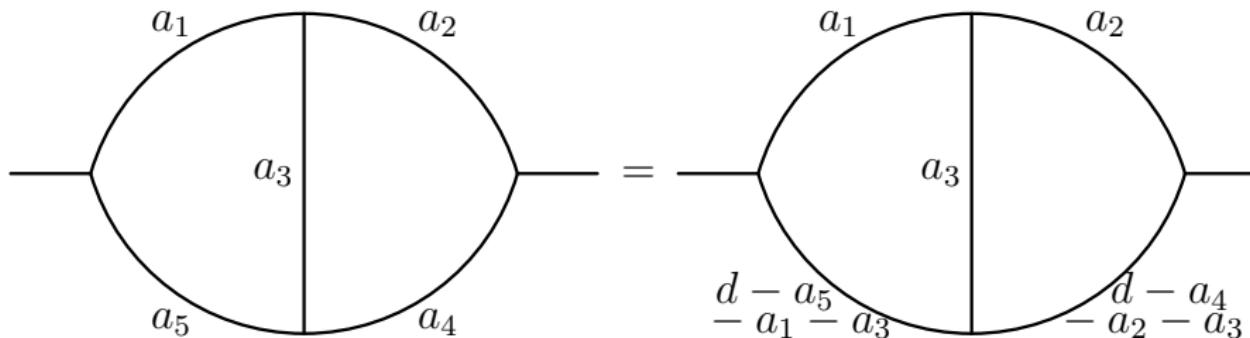
$$\sim I(\bar{a}_2, \bar{a}_4, \bar{a}_3, \bar{a}_5, \bar{a}_1)$$

$$\bar{a}_i = \frac{d}{2} - a_i$$

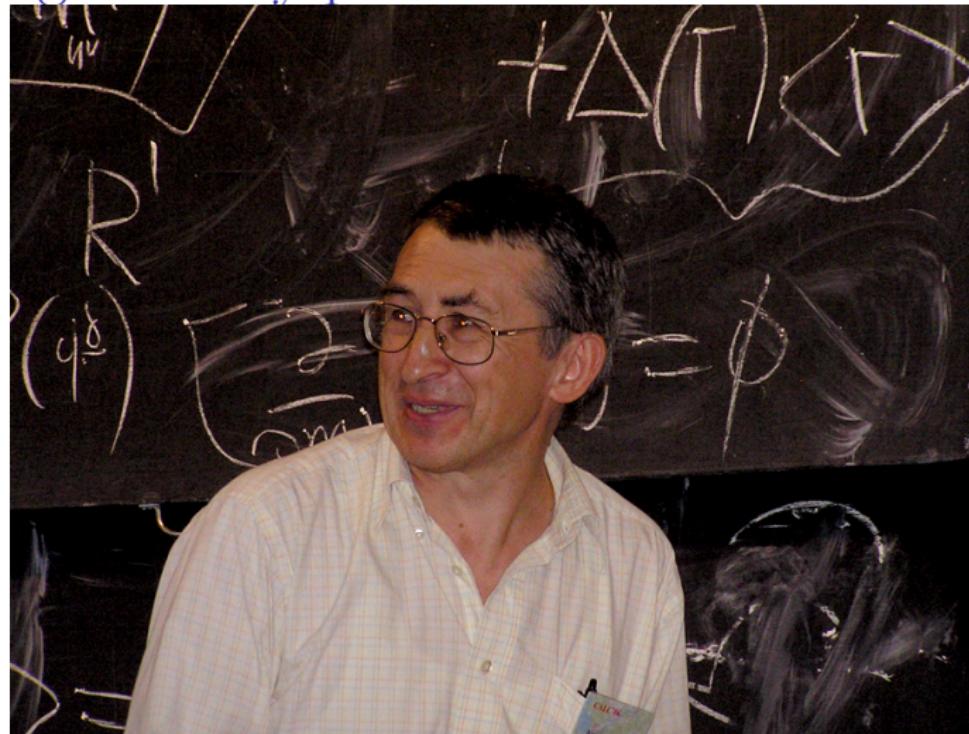
Inversion

$$k_i = \frac{k'_i}{k'^2_i} \quad k_i^2 = \frac{1}{k'^2_i} \quad d^d k_i = \frac{d^d k'_i}{(k'^2_i)^d}$$

$$(k_1 - k_2)^2 = \frac{(k'_1 - k'_2)^2}{k'^2_1 k'^2_2}$$



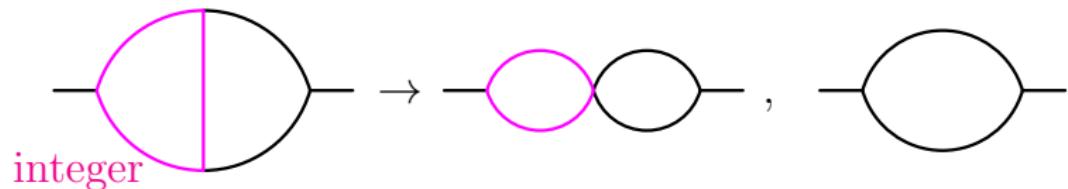
Integration by parts



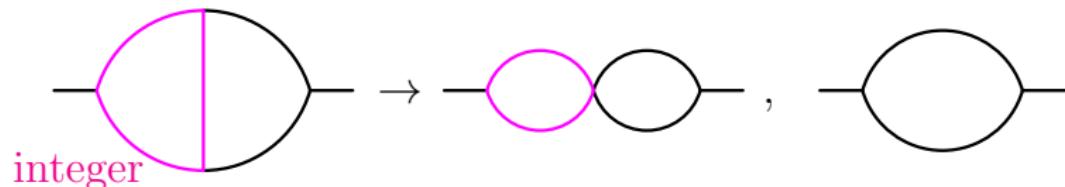
K. G. Chetyrkin, F. V. Tkachov,
Nucl. Phys. **B** 192 (1981) 159

F. V. Tkachov, Phys. Lett. **B** 100 (1981) 65

Integration by parts



Integration by parts



$$\begin{aligned} & [(d - 2a_3 - 4)\mathbf{3}^+ + 2(d - a_3 - 3)] \text{---} \bigg|_{a_3} \\ & = 2\mathbf{1}^+(\mathbf{5}^- - \mathbf{2}^-\mathbf{3}^-) \text{---} \bigg|_{a_3} \end{aligned}$$

Uniqueness



D. I. Kazakov,
TMΦ **58** (1984) 343

[Theor. Math. Phys. **58**
(1984) 223]

Phys. Lett. **B 133** (1983)
406

TMΦ **62** (1985) 127

[Theor. Math. Phys. **62**
(1985) 84]

Preprint JINR E2-84-
410, Dubna (1984)

Uniqueness

Coordinate space

$$\begin{array}{c} a_1 \\ \text{---} \\ \text{---} \\ a_2 \end{array} = \bullet \overbrace{\hspace{1cm}}^{a_1 + a_2}$$
$$\bullet \overbrace{\hspace{1cm}}^{a_1} \bullet \overbrace{\hspace{1cm}}^{a_2} \sim \bullet \overbrace{\hspace{1cm}}^{a_1 + a_1 - \frac{d}{2}}$$

Uniqueness

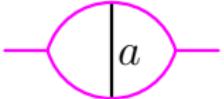
Coordinate space

$$\begin{array}{c} a_1 \\ \text{---} \\ \text{---} \\ a_2 \end{array} = \bullet - a_1 + a_2$$
$$\bullet - a_1 - a_2 \sim \bullet - a_1 - \frac{d}{2}$$

Star–triangle relation

$$\begin{array}{c} \bullet \\ | \\ a_1 \\ \backslash \quad / \\ a_2 \quad a_3 \end{array} = \begin{array}{c} \bullet \\ \backslash \quad / \\ \bar{a}_3 \quad \bar{a}_2 \\ \backslash \quad / \\ \bar{a}_1 \end{array}$$
$$a_1 + a_2 + a_3 = d \qquad \bar{a}_i = \frac{d}{2} - a_i$$

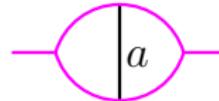
$$I(1, 1, a, 1, 1)$$


$$= I(a)$$

$$I(1+a) = I(1-a-3\varepsilon) \quad I(0) = \text{known}$$

$$\begin{aligned} I(1+a) &= \frac{1-a-2\varepsilon}{a+\varepsilon} I(a) \\ &\quad - 2 \frac{(1-2a-3\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)}{(a+\varepsilon)\Gamma(1+a)\Gamma(2-a-3\varepsilon)} \end{aligned}$$

$$I(1, 1, a, 1, 1)$$


$$= I(a)$$

$$I(1+a) = I(1-a-3\varepsilon) \quad I(0) = \text{known}$$

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$$I(1+a) = 2 \frac{\Gamma^2(1-\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma(1+a)\Gamma(1-a-3\varepsilon)} G(1+a)$$

$$\begin{aligned} G(1+a) &= \frac{a}{1-a-3\varepsilon} G(a) \\ &\quad + \frac{1}{a-1+3\varepsilon} \left(\frac{1}{a+\varepsilon} + \frac{1}{a-1+2\varepsilon} \right) \end{aligned}$$

$$I(1, 1, a, 1, 1)$$

$$G(1+a) = \sum_{n=1}^{\infty} f_n^{(1)} \left(\frac{1}{n+a+\varepsilon} + \frac{1}{n-a-2\varepsilon} \right)$$

$$+ \sum_{n=1}^{\infty} f_n^{(2)} \left(\frac{1}{n+a} + \frac{1}{n-a-3\varepsilon} \right)$$

$$f_n^{(1)} = -\frac{n+\varepsilon}{n+1-2\varepsilon} f_{n+1}^{(1)} \quad f_n^{(2)} = -\frac{n}{n+1-3\varepsilon} f_{n+1}^{(2)}$$

$$f_n^{(1)} = (-1)^n \frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)} c_1(\varepsilon)$$

$$f_n^{(2)} = (-1)^n \frac{\Gamma(n+1-3\varepsilon)}{\Gamma(n)} c_2(\varepsilon)$$

$$c_1(\varepsilon) = \frac{\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} \quad c_2(\varepsilon) = -\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)}$$

$I(1, 1, a, 1, 1)$

$$\begin{aligned} F(1+a) &= 2 \frac{\Gamma^2(1-\varepsilon)\Gamma(-a-\varepsilon)\Gamma(\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma(1+a)\Gamma(1-a-3\varepsilon)\Gamma(2-2\varepsilon)} \\ &\left[\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)} \left(\frac{1}{n+a+\varepsilon} + \frac{1}{n-a-2\varepsilon} \right) \right. \\ &- \frac{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)} \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n+1-3\varepsilon)}{\Gamma(n)} \\ &\left. \left(\frac{1}{n+a} + \frac{1}{n-a-3\varepsilon} \right) \right] \end{aligned}$$

$$I(1, 1, a, 1, 1)$$

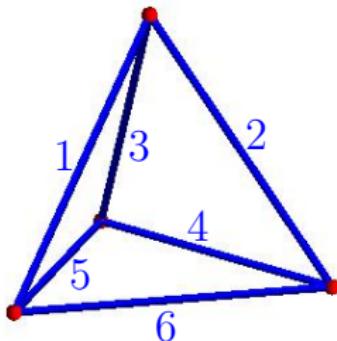
$$\begin{aligned} F(1+a) &= 2 \frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)} \\ &\left\{ \frac{\Gamma(-a-\varepsilon)\Gamma(2\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma(1+a)\Gamma(1-a-3\varepsilon)\Gamma(-1+\varepsilon)} \right. \\ &\left[\frac{1}{1-a-\varepsilon} {}_3F_2 \left(\begin{array}{c} 1, -2\varepsilon, -1+a+\varepsilon \\ a+\varepsilon, -1+\varepsilon \end{array} \middle| -1 \right) \right. \\ &\quad \left. + \frac{1}{1+a+2\varepsilon} {}_3F_2 \left(\begin{array}{c} 1, -2\varepsilon, -1-a-2\varepsilon \\ -a-2\varepsilon, -1+\varepsilon \end{array} \middle| -1 \right) \right] \\ &\left. + \frac{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(-a-2\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)} \right\} \end{aligned}$$

Symmetry



S. G. Gorishnii, A. P. Isaev, TMF **58** (1984) 343
[Theor. Math. Phys. **58** (1984) 232]

Tetrahedron group S_4



All lines with mass m

$$I = \frac{1}{\pi^{d/2}} \int \frac{F(k^2) d^d k}{(k^2 + m^2)^{a_6}}$$

$$F(k^2 \rightarrow \infty) \rightarrow \frac{I(a_1, a_2, a_3, a_4, a_5)}{(k^2)^{a_1+a_2+a_3+a_4+a_5-d}}$$

$$I_{\text{UV}} = \frac{1}{\Gamma(d/2)} \frac{I(a_1, a_2, a_3, a_4, a_5)}{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 - \frac{3}{2}d}$$

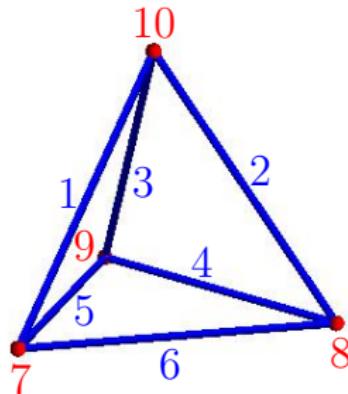
Symmetry



D. J. Broadhurst, Z. Phys. **C 32** (1986) 249

D. T. Barfoot, D. J. Broadhurst, Z. Phys. **C 41** (1988) 81

Notation



$$a_7 = a_1 + a_5 + a_6 - \frac{d}{2}$$

$$a_8 = a_2 + a_4 + a_6 - \frac{d}{2}$$

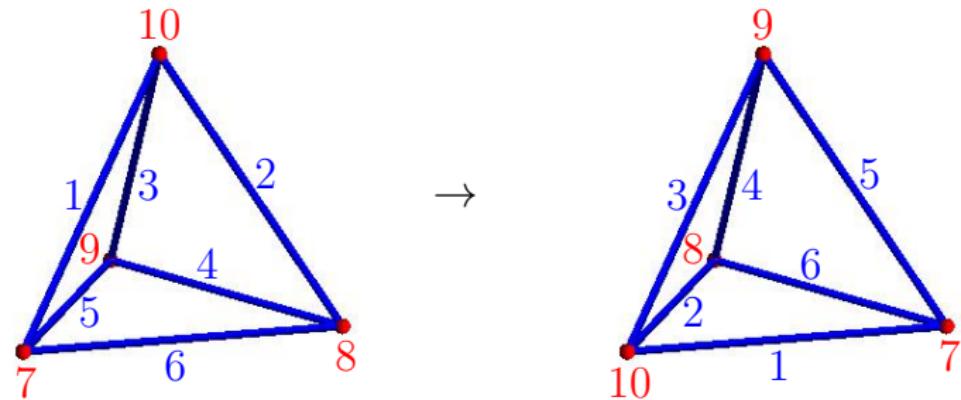
$$a_9 = a_3 + a_4 + a_5 - \frac{d}{2}$$

$$a_{10} = a_1 + a_2 + a_3 - \frac{d}{2}$$

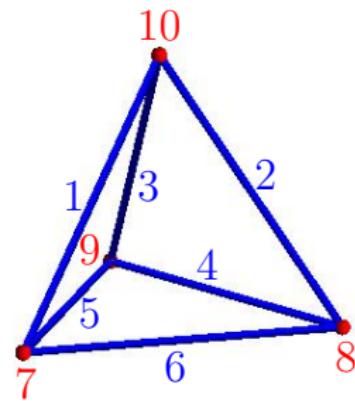
$$I(a_1, a_2, a_3, a_4, a_5) = \left[\prod_{i=1}^{10} G(a_i) \right]^{1/2} \frac{f(a_1, a_2, a_3, a_4, a_5, a_6)}{(d-3)\Gamma^2\left(\frac{d}{2}-1\right)}$$

$$a_6 = \frac{3}{2}d - \sum_{i=1}^5 a_i \quad \bar{a}_i = \frac{d}{2} - a_i \quad G(a) = \frac{\Gamma(\bar{a})}{\Gamma(a)}$$

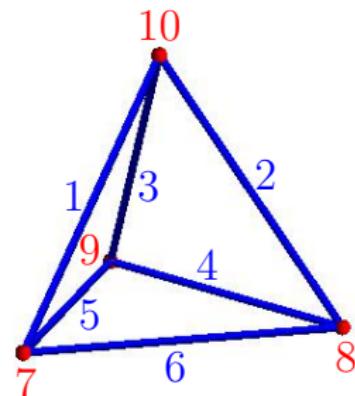
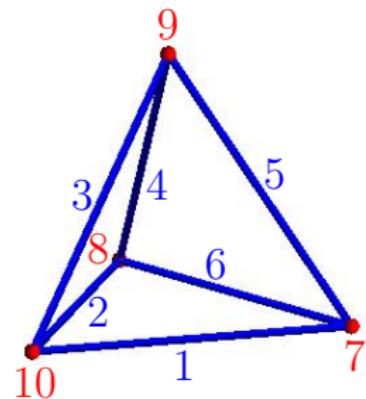
Generators 1, 2



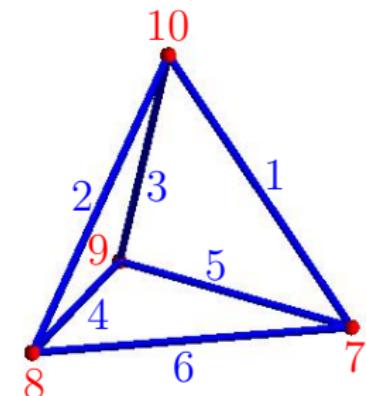
Generators 1, 2



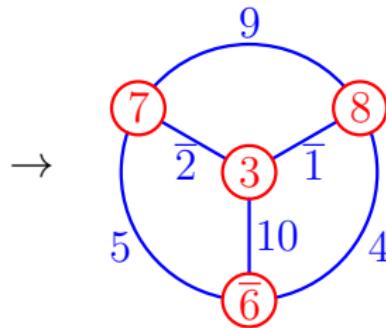
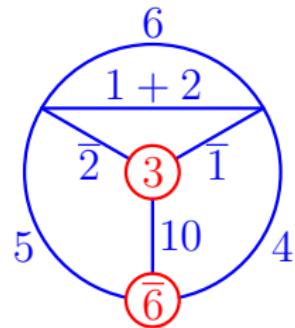
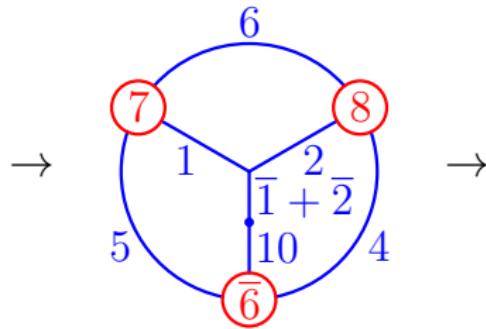
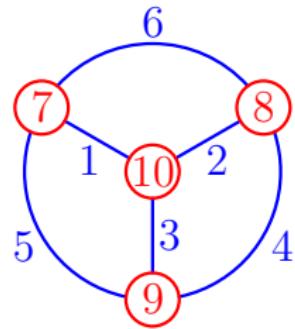
→



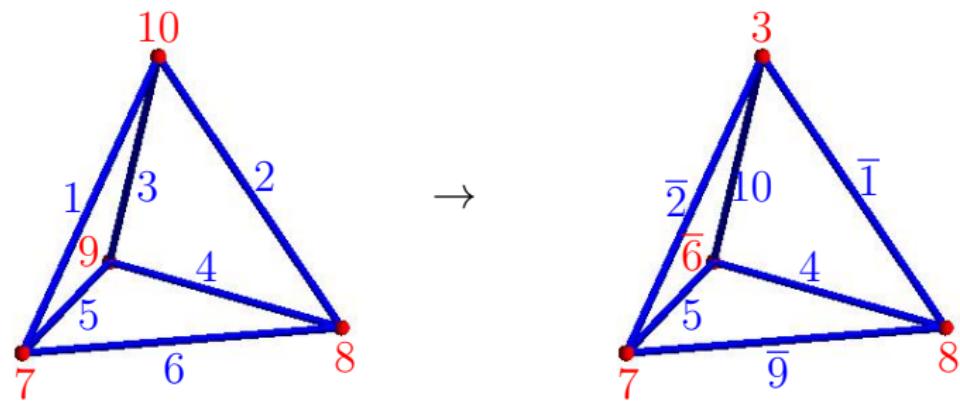
→



Generator 3



Generator 3



$Z_2 \times S_6$

$$f(a_1, a_2, a_3, a_4, a_5, a_6) = g(b_1, b_2, b_3, b_4, b_5, b_6)$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ -1 & 0 & 1 & 2 & 0 & 1 \\ 1 & -1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 2 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

$$g(b_1, b_2, b_3, b_4, b_5, b_6) \rightarrow$$

- ▶ $\hat{P}_1 g(b_1, b_2, b_3, b_4, b_5, b_6) = g(\bar{b}_1, \bar{b}_6, \bar{b}_2, \bar{b}_4, \bar{b}_3, \bar{b}_5)$
- ▶ $\hat{P}_2 g(b_1, b_2, b_3, b_4, b_5, b_6) = g(\bar{b}_3, \bar{b}_5, \bar{b}_1, \bar{b}_6, \bar{b}_2, \bar{b}_4)$
- ▶ $\hat{P}_3 g(b_1, b_2, b_3, b_4, b_5, b_6) = g(b_3, b_2, b_1, b_4, b_5, b_6)$

$$Z_2 \times S_6$$

$$\hat{Q}_1 = (\hat{P}_3 \hat{P}_1^2)^2 \hat{P}_3 \hat{P}_2 \hat{P}_1 \quad \hat{Q}_2 = \hat{P}_1^3 \hat{P}_3 \hat{P}_1 \quad \hat{Q}_3 = (\hat{P}_3 \hat{P}_2 \hat{P}_1^2)^2$$

$g(b_1, b_2, b_3, b_4, b_5, b_6) \rightarrow$

- ▶ $\hat{Q}_1 g(b_1, b_2, b_3, b_4, b_5, b_6) = g(b_2, b_3, b_4, b_5, b_6, b_1)$
- ▶ $\hat{Q}_2 g(b_1, b_2, b_3, b_4, b_5, b_6) = g(b_2, b_1, b_3, b_4, b_5, b_6)$
- ▶ $\hat{Q}_3 g(b_1, b_2, b_3, b_4, b_5, b_6) = g(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{b}_5, \bar{b}_6)$

Expansion in invariants

$$I_1 = 1 - \frac{d}{4} = 1 - \frac{1}{6} \sum_{i=1}^6 b_i$$

$$I_n = \sum_{i=1}^6 \left(b_i - \frac{d}{4} \right)^n \quad n = 2, 3, 4, 5, 6$$

$$f(a_1, a_2, a_3, a_4, a_5, a_6) = \sum_{i_3+i_4 \text{ even}} C_{i_1 i_2 i_3 i_4 i_5 i_6} I_1^{i_1} I_2^{i_2} I_3^{i_3} I_4^{i_4} I_5^{i_5} I_6^{i_6}$$

Expansion to ε^4 (from IBP)

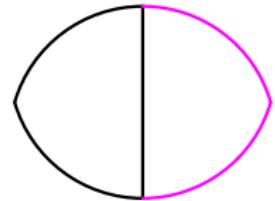
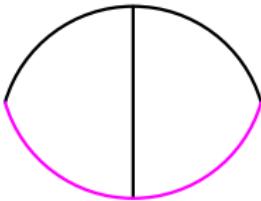
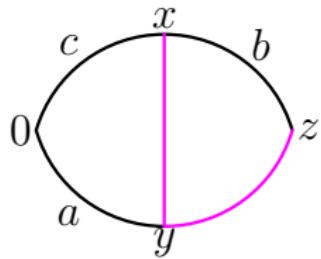
$$\begin{aligned}f(a_1, a_2, a_3, a_4, a_5, a_6) \\= 6\zeta_3 + 18\zeta_4 I_1 + 3\zeta_5 \left(I_1^2 + \frac{5}{2}I_2 \right) \\- 15(7\zeta_6 + 2\zeta_3^2)I_1^3 + 3 \left(\frac{25}{2}\zeta_6 - \zeta_3^2 \right) I_1 I_2 \\- 9 \left(\frac{439}{8}\zeta_7 + 20\zeta_4\zeta_3 \right) I_1^4 + 3 \left(\frac{211}{8}\zeta_7 - 6\zeta_4\zeta_3 \right) I_1^2 I_2 \\+ \frac{9}{8}\zeta_7 \left(\frac{35}{4}I_2^2 - 7I_4 \right) + \dots\end{aligned}$$

Gegenbauer polynomials



A. V. Kotikov, Phys.
Lett. **B 375** (1996) 240

3 non-integer indices



$$\lambda = d/2 - 1$$

$$A(a, b, c) = \frac{1}{\pi^d} \int \frac{d^d x \, d^d y}{(y^2)^a [(z-y)^2]^\lambda [(z-x)^2]^b (x^2)^c [(x-y)^2]^\lambda}$$

Gegenbauer polynomials

$$\frac{1}{[(y-z)^2]^\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)n!} y^{\mu_1 \dots \mu_n} z^{\mu_1 \dots \mu_n} \left[\frac{\theta(z^2 - y^2)}{(z^2)^{\lambda+n}} + \frac{\theta(y^2 - z^2)}{(y^2)^{\lambda+n}} \right]$$

Gegenbauer polynomials

$$\frac{1}{[(y-z)^2]^\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)n!} y^{\mu_1 \dots \mu_n} z^{\mu_1 \dots \mu_n} \left[\frac{\theta(z^2 - y^2)}{(z^2)^{\lambda+n}} + \frac{\theta(y^2 - z^2)}{(y^2)^{\lambda+n}} \right]$$

$$\begin{aligned} & \frac{1}{\pi^{d/2}} \int \frac{y^{\mu_1 \dots \mu_n}}{(y^2)^a [(x-y)^2]^\lambda} \theta(y^2 - x^2) d^d y \\ &= \frac{1}{\Gamma(\lambda)(a-1)(\lambda+n)} \frac{x^{\mu_1 \dots \mu_n}}{(x^2)^{a-1}} \\ & \frac{1}{\pi^{d/2}} \int \frac{y^{\mu_1 \dots \mu_n}}{(y^2)^a [(x-y)^2]^\lambda} \theta(x^2 - y^2) d^d y \\ &= \frac{1}{\Gamma(\lambda)(n+\lambda-a+1)(\lambda+n)} \frac{x^{\mu_1 \dots \mu_n}}{(x^2)^{a-1}} \end{aligned}$$

Gegenbauer polynomials

$$\frac{1}{[(y-z)^2]^\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)n!} y^{\mu_1 \dots \mu_n} z^{\mu_1 \dots \mu_n} \left[\frac{\theta(z^2 - y^2)}{(z^2)^{\lambda+n}} + \frac{\theta(y^2 - z^2)}{(y^2)^{\lambda+n}} \right]$$

$$\begin{aligned} & \frac{1}{\pi^{d/2}} \int \frac{y^{\mu_1 \dots \mu_n}}{(y^2)^a [(x-y)^2]^\lambda} \theta(y^2 - x^2) d^d y \\ &= \frac{1}{\Gamma(\lambda)(a-1)(\lambda+n)} \frac{x^{\mu_1 \dots \mu_n}}{(x^2)^{a-1}} \\ & \frac{1}{\pi^{d/2}} \int \frac{y^{\mu_1 \dots \mu_n}}{(y^2)^a [(x-y)^2]^\lambda} \theta(x^2 - y^2) d^d y \\ &= \frac{1}{\Gamma(\lambda)(n+\lambda-a+1)(\lambda+n)} \frac{x^{\mu_1 \dots \mu_n}}{(x^2)^{a-1}} \end{aligned}$$

Then $\int d^d x \Rightarrow$ combination of several ${}_3F_2(1)$

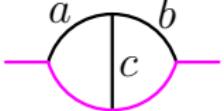
Solving IBP for 3 non-integer indices



D. J. Broadhurst, J. A. Gracey, D. Kreimer, Z. Phys. **C 75** (1997) 559 [hep-th/9607174]

D. J. Broadhurst, Nucl. Phys. Proc. Suppl. **116** (2003) 432
[hep-ph/0211194]

3 non-integer indices


$$= I(a, b, c, d) \quad d = a + b + c - \frac{D}{2}$$

IBP $G(a, b) = G(a)G(b)G(\bar{a} + \bar{b})$

$$aI(a+1, b, c, d+1) - (a+b-D+2)I(a, b, c, d)$$

$$= cG(1, d+1) \left(\frac{aG(a+1, c+1)}{d-b+1} - G(b, c+1) \right)$$

$$(c-D+2)I(a, b, c, d)$$

$$+ \frac{(a+c-D+1)(b+c-D+1)}{c-D/2+1} I(a, b, c-1, d-1)$$

$$= c(c+d-D+1)G(1, d) \left(\frac{G(a, c+1)}{d-b} + \frac{G(b, c+1)}{d-a} \right)$$

Ansatz

$$\begin{aligned} & \frac{I(a, b, c, d)}{cdG(1, d+1)} = \\ & \frac{G(a_1, b+1)}{D-3} S\left(\frac{D}{2} - a - 1, b - 1, \frac{D}{2} + a - d - 2, d - b\right) \\ & + (a \leftrightarrow b) \end{aligned}$$

$$S(a, b, c, d) = S(b, a, c, d) = -S(c, d, a, b)$$

$$aS(a, b, c, d) = 1 + \frac{(a+c)(a+d)}{a+b+c+d} S(a-1, b, c, d)$$

Solution

$$S(a, b, c, d) = \frac{\pi \cot \pi c}{H(a, b, c, d)} - \frac{1}{c} - \frac{b+c}{bc} F(a+c, -b, -c, b+d)$$

$$H(a, b, c, d)$$

$$= \frac{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+d)\Gamma(1+a+b+c+d)}{\Gamma(1+a+c)\Gamma(1+a+d)\Gamma(1+b+c)\Gamma(1+b+d)}$$

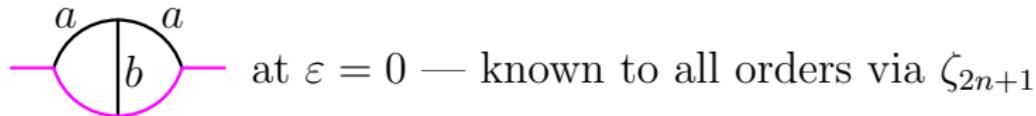
$$F(a, b, c, d) = {}_3F_2 \left(\begin{array}{c} 1, -a, -b \\ 1 + c, 1 + d \end{array} \middle| 1 \right) - 1$$

Solution

$$S(a, b, c, d) = \frac{\pi \cot \pi c}{H(a, b, c, d)} - \frac{1}{c} - \frac{b+c}{bc} F(a+c, -b, -c, b+d)$$

$$H(a, b, c, d) = \frac{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+d)\Gamma(1+a+b+c+d)}{\Gamma(1+a+c)\Gamma(1+a+d)\Gamma(1+b+c)\Gamma(1+b+d)}$$

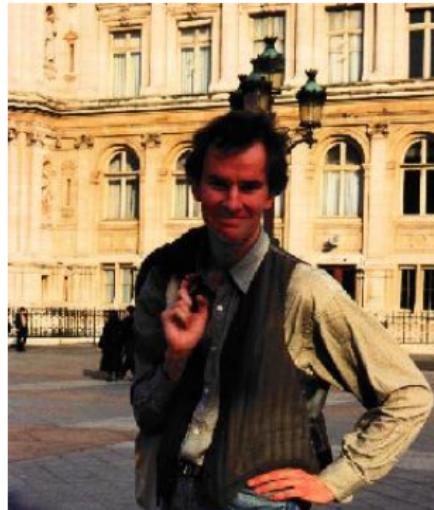
$$F(a, b, c, d) = {}_3F_2 \left(\begin{array}{c} 1, -a, -b \\ 1 + c, 1 + d \end{array} \middle| 1 \right) - 1$$



Expansion to ε^9 (from ${}_3F_2(1)$)

$$\begin{aligned}f(a_1, a_2, a_3, a_4, a_5, a_6) = & \cdots \\& + 3 \left(\frac{378}{5} \zeta_{53} - \frac{33523}{40} \zeta_8 + 10 \zeta_5 \zeta_3 \right) I_1^5 \\& - 3 \left(\frac{54}{5} \zeta_{53} - \frac{1009}{40} \zeta_8 + 42 \zeta_5 \zeta_3 \right) I_1^3 I_2 \\& - \frac{3}{2} \left(\frac{9}{5} \zeta_{53} - \frac{4023}{80} \zeta_8 + 7 \zeta_5 \zeta_3 \right) I_1 I_2^2 \\& + 3 \left(\frac{18}{5} \zeta_{53} - \frac{1083}{40} \zeta_8 + 8 \zeta_5 \zeta_3 \right) I_1 I_4 + \cdots\end{aligned}$$

Mellin–Barnes representation



I. Bierenbaum, S. Weinzierl Eur. Phys. J. C **32** (2003) 67
[hep-ph/0308311]

Mellin–Barnes representation

$$\begin{array}{c} a_1 \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad = \quad \begin{array}{c} a_1 \\ \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \text{where} \quad \begin{array}{c} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \bullet \end{array} \quad = \quad \begin{array}{c} a_2 \\ \diagup \quad \diagdown \\ a_3 \\ \diagup \quad \diagdown \\ a_4 \end{array}$$

$$I(a_1, a_2, a_3, a_4, a_5) = \frac{1}{\pi^{d/2}} \int \frac{d^d k_1}{[(k_1 - k)^2]^{a_1} (k_1^2)^{a_5}} V((k_1 - k)^2, k_1^2)$$

$$V((k_1 - k)^2, k_1^2) = \frac{1}{\pi^{d/2}} \int \frac{d^d k_2}{[(k_2 - k)^2]^{a_2} [(k_1 - k_2)^2]^{a_3} (k_2^2)^{a_4}}$$

Mellin–Barnes representation

$$\begin{array}{c} a_1 \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \quad | \\ a_5 \quad a_4 \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \quad | \\ a_1 \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \quad | \\ a_5 \end{array} = \begin{array}{c} a_1 \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \quad | \\ a_5 \end{array} \quad \text{where} \quad \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \quad | \\ a_3 \end{array} = \begin{array}{c} a_2 \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \quad | \\ a_3 \quad a_4 \end{array}$$

$$I(a_1, a_2, a_3, a_4, a_5) = \frac{1}{\pi^{d/2}} \int \frac{d^d k_1}{[(k_1 - k)^2]^{a_1} (k_1^2)^{a_5}} V((k_1 - k)^2, k_1^2)$$

$$V((k_1 - k)^2, k_1^2) = \frac{1}{\pi^{d/2}} \int \frac{d^d k_2}{[(k_2 - k)^2]^{a_2} [(k_1 - k_2)^2]^{a_3} (k_2^2)^{a_4}}$$

Mellin–Barnes ($k^2 = 1$)

$$V((k_1 - k)^2, k_1^2) = \frac{1}{(2\pi i)^2} \int dz_1 dz_2 [(k_1 - k)^2]^{z_1} (k_1^2)^{z_2} v(z_1, z_2)$$

$$I(a_1, a_2, a_3, a_4, a_5) = \frac{1}{(2\pi i)^2} \int dz_1 dz_2 G(a_1 - z_1, a_5 - z_2) v(z_1, z_2)$$

Mellin–Barnes representation

$$I(a_1, a_2, a_3, a_4, a_5) = \frac{1}{(2\pi i)^2 \Gamma(a_2)\Gamma(a_4)\Gamma(a_3)\Gamma(d-a_2-a_4-a_3)}$$
$$\int dz_1 dz_2 \frac{\Gamma(-z_1)\Gamma(\frac{d}{2}-a_4-a_3-z_1)\Gamma(\frac{d}{2}-a_1+z_1)}{\Gamma(a_1-z_1)}$$
$$\frac{\Gamma(-z_2)\Gamma(\frac{d}{2}-a_2-a_3-z_2)\Gamma(\frac{d}{2}-a_5+z_2)}{\Gamma(a_5-z_2)}$$
$$\frac{\Gamma(a_1+a_5-\frac{d}{2}-z_1-z_2)\Gamma(a_3+z_1+z_2)\Gamma(a_2+a_4+a_3-\frac{d}{2}+z_1+z_2)}{\Gamma(d-a_1-a_5+z_1+z_2)}$$

Residues

- ▶ Close z_1 contour to the right. 3 series of poles: $\Gamma(-z_1)$,
 $\Gamma(\frac{d}{2} - a_4 - a_3 - z_1)$, $\Gamma(a_1 + a_5 - \frac{d}{2} - z_1 - z_2)$:

$$I = I_1 + I_2 + I_3$$

Residues

- ▶ Close z_1 contour to the right. 3 series of poles: $\Gamma(-z_1)$, $\Gamma(\frac{d}{2} - a_4 - a_3 - z_1)$, $\Gamma(a_1 + a_5 - \frac{d}{2} - z_1 - z_2)$:

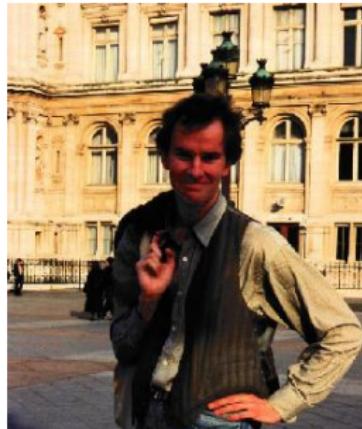
$$I = I_1 + I_2 + I_3$$

- ▶ Close z_1 contour to the right:

$$\begin{aligned} I &= I_{11} + I_{12} + I_{13} \\ &\quad + I_{21} + I_{22} + I_{23} \\ &\quad + I_{31} + I_{32} + I_{33} + I_{34} + I_{35} \end{aligned}$$

Each term is a double sum

Nested sums



S. Moch, P. Uwer, S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083]

S. Weinzierl, Comput. Phys. Commun. **145** (2002) 357
[math-ph/0201011] (`NestedSums`: C++, GiNaC)

S. Moch, P. Uwer, Comput. Phys. Commun. **174** (2006) 759 [math-ph/0508008] (`XSUMMER`: FORM)

Expanding hypergeometric functions in ε



T. Huber, D. Maître, Comput. Phys. Commun. **175** (2006)
122 [hep-ph/0507094] (HypExp: Mathematica, HPL)

Happy birthday, Dima!

