

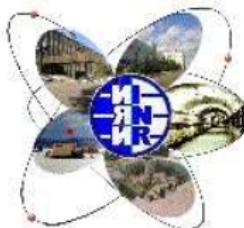
$R(s)$ and Z decay in $\mathcal{O}(\alpha_s^4)$: complete results

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in collaboration with

P. Baikov (MSU), J. Kühn (KIT) and J. Rittinger (KIT)

Advances of QFT, Dubna, 4-7.10.2011



Final results of a Large Karlsruhe based project:

- 10 years of work of the KIT-MSU-INR multiloop group:
 - + starting from the first publication Baikov, K. Ch. and Kühn, “Five-loop vacuum polarization in pQCD: $O(\alpha_s^4 N_f^2)$ results”, Phys.Rev.Lett. B559 (2003) 245.
 - + ...
 - + ...
 - + and finishing with recent works
Phys.Rev.Lett.101:012002,2008; arXiv:0801.1821
Phys.Rev.Lett.104:132004,2010; arXiv:1001.3606v1
Nucl.Phys.B837:186-220,2010; arXiv:1004.1153
Nucl.Phys.Proc.Suppl.205-206:237-241,2010;
P. Baikov, K. Ch., J. Kühn and J. Rittinger, in preparation

Aim of the project

Extending to NNNNLO (which means $\alpha_s^4!!!$) the classical NNNLO $\mathcal{O}(\alpha_s^3)$ results by

Gorishny, Kataev, Larin, (1991); Samuel, Surguladze (1991) on

the three “gold plated” (Bjorken, 1979) QCD observables:

$$R_Z = \Gamma(Z_0 \rightarrow \textit{hadrons}) / \sigma(Z_0 \rightarrow \mu^+ \mu^-)$$

$$R_\tau = \Gamma(\tau \rightarrow \textit{hadrons} + \nu_\tau) / \Gamma(\tau \rightarrow l + \bar{\nu}_l + \nu_\tau)$$

$$R(s) = \sigma_{tot}(e^+ e^- \rightarrow \textit{hadrons}) / \sigma(e^+ e^- \rightarrow \mu^+ \mu^-)$$

Why “gold plated”?

- As predictions are very clean theoretically: inclusive observables, expressible completely through two-point correlators.
- As R_Z , R_τ and, to a lesser extent, $R(s)$ are very precisely measured.
- At last, the ratio $R(s)$ is related in unique way with the vacuum polarization operator, which contribute to the few important physical quantities. For example, the running of the effective α^{QED} in the SM as well as the hadronic contribution to the muon anomalous magnetic moment.

Why NNNNLO could be of any importance at all?

- Because, with the NNNLO accuracy the theoretical errors were comparable with the experimental ones and, in despair, everybody was using the famous Kataev&Starshenko /1993/ estimation of the α_s^4 term which (incidentally?) has happened to be quite close to the true number!
- After our calculations the situation has become significantly better, especially for Γ_Z , where the the theoretical error was reduced by a factor of four! (more discussion will follow).

New Results to report

- (**new!**) result for the singlet contribution into the (massless) $\langle VV \rangle$ corelator \implies final complete $\mathcal{O}(\alpha_s^4)$ result for the R(s)
- (**new!**) as spin-off: the complete QED β -function in five loops
- (**new!**) final complete result for $Z \rightarrow$ hadrons at $\mathcal{O}(\alpha_s^4)$ (the latter including in full power-not-suppressed top-mass dependence)

Z Boson Decay Rate into Hadrons

$$\Gamma(Z \rightarrow \text{hadrons}) = \sum_{f_{QCD}} \int d\Phi \left| \mathcal{M}(Z \rightarrow f_{QCD}) \right|^2$$

$$= \int d\Phi \left| \begin{array}{c} Z \\ \text{---} \\ \text{---} \end{array} \right. \left. \begin{array}{c} f_{QCD} \\ \text{---} \\ \text{---} \end{array} \right|^2 + \dots \xrightarrow{\text{Opt.Th.}} \text{Im} \left[\begin{array}{c} Z \\ \text{---} \\ \text{---} \end{array} \right. \left. \begin{array}{c} Z \\ \text{---} \\ \text{---} \end{array} \right] + \dots$$

$$\text{QCD} \left(\begin{array}{c} Z \\ \mu \\ \nu \end{array} \right) \left(\begin{array}{c} Z \\ \nu \\ \mu \end{array} \right) = \Pi^{\mu\nu} = i \int e^{iqx} \langle 0 | T j_Z^\mu(x) j_Z^\nu(0) | 0 \rangle dx$$

$$= g^{\mu\nu} \Pi_1(-q^2) + q^\mu q^\nu \Pi_2(-q^2)$$

$$\Gamma(Z \rightarrow \text{hadrons}) = \Gamma_0 \cdot \frac{2\pi i}{s} \left(\Pi_1(s - i\varepsilon) - \Pi_1(s + i\varepsilon) \right)$$

$$= \Gamma_0 \cdot R(s) \quad \left(\Gamma_0 = \frac{G_F M_Z^3}{8\pi\sqrt{2}} \right)$$

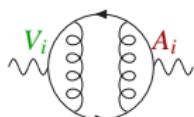
Vector and Axial Contributions to Z-decay

Z current to fermion i :

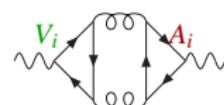
$$j_{Zi}^\mu = g_i^V j_{Vi}^\mu + g_i^A j_{Ai}^\mu = g_i^V \overbrace{\bar{\psi}_i \gamma^\mu \psi_i}^{=V_i} + g_i^A \overbrace{\bar{\psi}_i \gamma^\mu \gamma_5 \psi_i}^{=A_i}$$

$$\begin{aligned} \text{QCD } \overset{Z}{\mu} \text{---} \overset{Z}{\nu} &= \sum_i (g_i^V)^2 \text{---} \overset{V_i}{\text{---}} + \sum_{i,j} g_i^V g_j^V \text{---} \overset{V_i}{\text{---}} \text{---} \overset{V_j}{\text{---}} \\ &+ \sum_i (g_i^A)^2 \text{---} \overset{A_i}{\text{---}} + \sum_{i,j} g_i^A g_j^A \text{---} \overset{A_i}{\text{---}} \text{---} \overset{A_j}{\text{---}} \\ i, j &= \{t, b, c, s, u, d\} \end{aligned}$$

NO

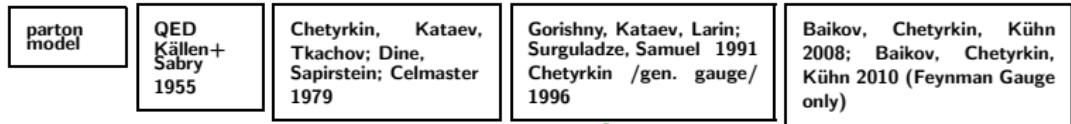


and

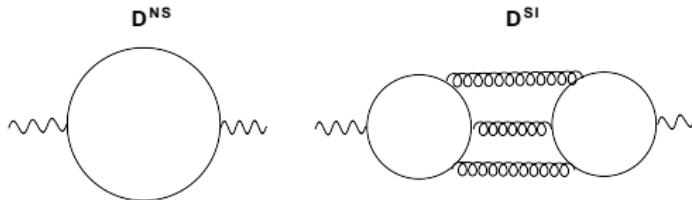


• status of theory (in the massless limit) •

$$R^{NS} = 3 \sum_i Q_i^2 \left(1 + \frac{\alpha_s}{\pi} + \# \left(\frac{\alpha_s}{\pi} \right)^2 + \# \left(\frac{\alpha_s}{\pi} \right)^3 + \# \left(\frac{\alpha_s}{\pi} \right)^4 + \dots \right)$$



$$R^{SI} = \left(\sum_i Q_i \right)^2 \left(\# \left(\frac{\alpha_s}{\pi} \right)^3 + \boxed{?? \left(\frac{\alpha_s}{\pi} \right)^4} + \dots \right)$$



Singlet contribution to the (vector) Adler function

(Last missing term!)

$$D^{SI}(Q^2) = d_R \left(\sum_{i=3}^{\infty} d_i^{SI} a_s^i(Q^2) \right)$$

$$d_3^{SI} = \frac{d^{abc} d^{abc}}{d_R} \left(\frac{11}{192} - \frac{1}{8} \zeta_3 \right), \quad d_4^{SI} = \frac{d^{abc} d^{abc}}{d_R} (C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T n_f d_{4,3}^{SI})$$

$$d_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad d_{4,3}^{SI} = \frac{-149}{576} + \frac{13}{32}\zeta_3 - \frac{5}{16}\zeta_5 + \frac{1}{8}\zeta_3^2$$

$$d_{4,2}^{SI} = \frac{3893}{4608} - \frac{169}{128}\zeta_3 + \frac{45}{64}\zeta_5 - \frac{11}{32}\zeta_3^2$$

Phenomenological implications for $\sigma_{tot}(e^+e^- \rightarrow \text{hadrons})$

Numerically:

$$\begin{aligned} R(s) &= 3 \sum_f Q_f^2 \left\{ 1 + a_s + a_s^2 (1.986 - 0.1153 n_f) \right. \\ &+ a_s^3 (-6.637 - 1.200 n_f - 0.00518 n_f^2) \left. \right\} \\ &- \left(\sum_f Q_f \right)^2 \left(1.2395 a_s^3 + \frac{(-17.8277 + 0.57489 n_f) a_s^4}{n_f} \right) \end{aligned}$$

for $n_f = 5$

$$\frac{11}{3} [1 + a_s + a_s^2 1.409 - 12.767 a_s^3 - 79.98 a_s^4] + \frac{1}{9} [-1.240 a_s^3 - 14.95 a_s^4]$$

Extra suppression factor $\frac{3}{99} \approx 0.03!$

QED β -function in five loops

By a proper change of color factors we arrive at the **full** Adler function

of QED in five loops \Rightarrow the QED β -function;
for a QED with one charged fermion we get ($A \equiv \frac{e^2}{16\pi^2}$)

$$\beta^{QED} = \frac{4}{3}A + 4A^2 - \frac{62}{9}A^3 - A^4 \left(\frac{5570}{243} + \frac{832}{9}\zeta_3 \right)$$

Gorishny, Kataev,
Larin, Surguladze, 1991

$$-A^5 \left(\frac{195067}{486} + \frac{800}{3}\zeta_3 + \frac{416}{3}\zeta_4 - \frac{6880}{3}\zeta_5 \right)$$

Numerically ($A = \frac{\alpha}{4\pi} \approx 5.81 \cdot 10^{-4}$)

$$\beta^{QED} = \frac{4}{3}A (1 + 3A - 5.1667A^2 - 100.534A^3 + 1129.51A^4)$$

No hope for a non-trivial fixed point solution $\beta(A^*) = 0!$

Tool-box for massless correlators at α_s^4 :

- IRR / Vladimirov, (78)/ + IR R^* -operation /Chetyrkin, Smirnov (1984)/ + resolved combinatorics /Chetyrkin, (1997)/
- reduction to Masters: “direct and automatic” construction of CF’s through $1/D$ expansion within the Baikov’s representation for Feynman integrals (Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Supp.116:378-381,2003)
- all master integrals (we need 4-loop massless propagators) are reliably known in analytical form /see next slide/
- computing: MPI-based (PARFORM) as well as thread-based (TFORM) versions of FORM
Vermaseren, Retey, Fliegner, Tentyukov, ... (2000 – . . .)

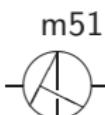
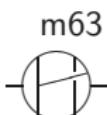
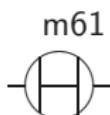
(non-trivial) 4-loop master integrals

/used in all calculations we are talking about/

$$\text{Diagram} = \frac{441\zeta_7}{8} + \mathcal{O}(\varepsilon)$$

$$\text{Diagram} = \frac{20\zeta_5}{\epsilon} - 80\zeta_5 + 68\zeta_3^2 + 50\zeta_6 + \mathcal{O}(\varepsilon)$$

Kazakov, D.I., The Method
Of Uniqueness, A New
Powerful Technique For
Multiloop Calculations.,
Phys.Lett., B133 (1983) 406

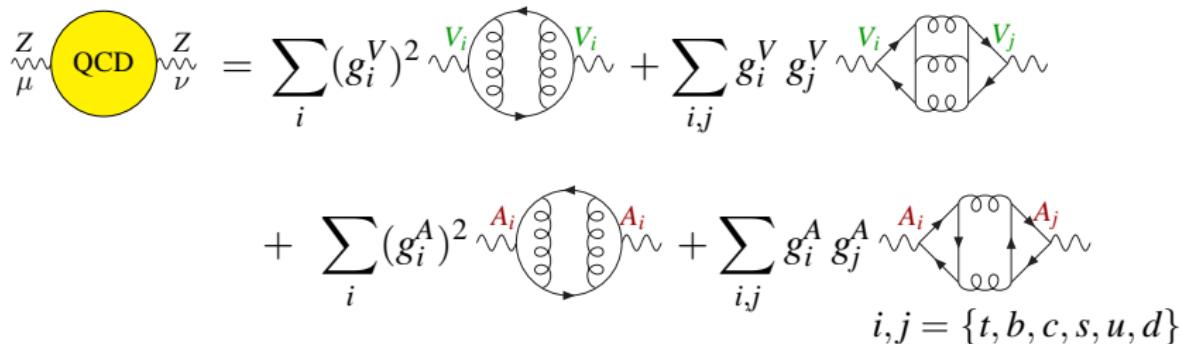


Baikov, K.Ch.
(2010); R. Lee,
A. Smirnov, V.
Smirnov, (2011)

Vector and Axial Contributions to the Z-decay

Z current to fermion i :

$$j_{Zi}^\mu = g_i^V j_{Vi}^\mu + g_i^A j_{Ai}^\mu = g_i^V \overbrace{\bar{\psi}_i \gamma^\mu \psi_i}^{=V_i} + g_i^A \overbrace{\bar{\psi}_i \gamma^\mu \gamma_5 \psi_i}^{=A_i}$$



- scale $\sqrt{s} = M_Z \Rightarrow m_l = 0 \quad (l = \{b, c, s, u, d\})$
- 2 scales $\sqrt{s} = M_Z$ and m_t (but $M_Z^2/(4m_t^2) \ll 1$):
Decoupling of top (factorization of top effects)

't Hooft-Veltman-Larin Treatment of γ_5

Problem: naive γ_5 ($[\gamma_5, \gamma_\alpha] = 0$) is not applicable for the singlet diagrams.

Solution

't Hooft-Veltman '72: treat $\epsilon^{\mu\mu_1\mu_2\mu_3}$ as 4-dimensional object

$$\gamma_5 = \frac{i}{4!} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4}$$
$$(i = \{t, b, c, s, u, d\})$$

Larin '93: introduce a special prefactor

$$\zeta_A = 1 - \frac{4}{3} a_s + \dots$$

and define the axial vector current as:

$$A_i = \bar{\psi}_i \gamma^\mu \gamma_5 \psi_i = \zeta_A \frac{i}{6} \epsilon^{\mu\mu_1\mu_2\mu_3} \bar{\psi}_i \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \psi_i$$

$(\zeta_A$ effectively restores antisymmetry of γ_5 in NS diagrams; it is fixed **uniquely** from the (non-anomalous) Ward identity , or, equivalently, by requiring scale inv. of the current $\zeta_A \frac{i}{6} \epsilon^{\mu\mu_1\mu_2\mu_3} [\bar{\psi}_i \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \psi_i]_{\overline{\text{MS}}}$)

Decoupling of Top

Decoupling for QCD fields and coupling constant ($\zeta_2, \zeta_3, \bar{\zeta}_3$ and ζ_g):
6 flavour \rightarrow 5 flavour

Vector current: naive decoupling (due to Ward Identity)

$$V_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} 0 + \mathcal{O}(1/m_t^2), \quad V_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} V_b^{(5)} + \mathcal{O}(1/m_t^2)$$

Axial vector current: **no** naive decoupling (due to potential anomaly if it would!)

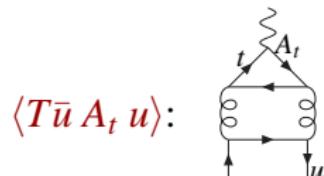
$$A_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} C_h A_S^{(5)} + \mathcal{O}(1/m_t^2), \quad A_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} A_b^{(5)} + C_\psi A_S^{(5)} + \mathcal{O}(1/m_t^2)$$

$$A_S^{(5)} = \sum_l A_l^{(5)}, \quad (A_t^{(6)} - A_b^{(6)}) \stackrel{m_t \rightarrow \infty}{=} (C_h - C_\psi) A_S^{(5)} - A_b + \mathcal{O}(1/m_t^2)$$

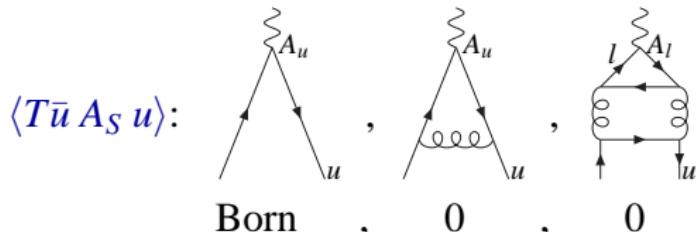
Decoupling of Top

Calculation of C_h with method of projectors (Gorishny, Larin '86):

$$\mathbf{A}_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} C_h(a^{(6)}(\mu), \mu/m_t) \mathbf{A}_S^{(5)}$$



$\langle T\bar{u} A_i u \rangle|_{p=0}$: TAD's



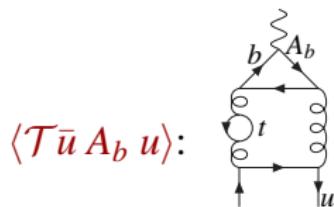
Calculation of massive tadpoles:

- matching to topologies with EXP (Seidensticker and Steinhauser)
- reduction with Laporta alg. via CRUSHER (Marquard, Seidel)
- all master integrals up to 4 loop are known (Schröder, Vuorinen '05)

Decoupling of Top

Calculation of C_ψ :

$$A_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} A_b^{(5)} + C_\psi(a^{(6)}(\mu), \mu/m_t) A_S^{(5)}$$



Decoupling of Top

$$A_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} C_h A_S^{(5)}, \quad A_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} A_b^{(5)} + C_\psi A_S^{(5)}.$$

$$\begin{aligned} C_h = & + \left(\frac{\alpha^{(6)}(\mu)}{\pi} \right)^2 \left[+ 0.125 - 0.5 \ln \left(\frac{\mu^2}{m_t^2} \right) \right] \\ & + \left(\frac{\alpha^{(6)}(\mu)}{\pi} \right)^3 \left[- 0.515 - 0.417 \ln \left(\frac{\mu^2}{m_t^2} \right) - 0.875 \ln^2 \left(\frac{\mu^2}{m_t^2} \right) \right] \\ & + \left(\frac{\alpha^{(6)}(\mu)}{\pi} \right)^4 \left[- 18.335 + 15.563 \ln \left(\frac{\mu^2}{m_t^2} \right) - 0.767 \ln^2 \left(\frac{\mu^2}{m_t^2} \right) - 1.531 \ln^3 \left(\frac{\mu^2}{m_t^2} \right) \right] \end{aligned}$$

$$\begin{aligned} C_\psi = & + \left(\frac{\alpha^{(6)}(\mu)}{\pi} \right)^3 \left[- 0.265 - 0.014 \ln \left(\frac{\mu^2}{m_t^2} \right) - 0.084 \ln^2 \left(\frac{\mu^2}{m_t^2} \right) \right] \\ & + \left(\frac{\alpha^{(6)}(\mu)}{\pi} \right)^4 \left[- 1.282 + 0.393 \ln \left(\frac{\mu^2}{m_t^2} \right) - 0.406 \ln^2 \left(\frac{\mu^2}{m_t^2} \right) - 0.132 \ln^3 \left(\frac{\mu^2}{m_t^2} \right) \right] \end{aligned}$$

- 2 loop Collins, Wilczek and Zee '78
- 3 loop log enhanced terms Chetyrkin, Kühn '93
- 3 loop Larin, Ritbergen, Vermaseren '93; Chetyrkin, Tarasov '94
- 4 loop is new

Axial Vector Correlator

$$\Gamma(Z \rightarrow \text{hadrons}) = \Gamma_0 \cdot R(s) = \Gamma_0 \cdot (R^V(s) + R^A(s))$$

Optical Theorem:

$$\begin{aligned}
 \text{QCD} &= \sum_i (g_i^V)^2 \text{ (left)} + \sum_{i,j} g_i^V g_j^V \text{ (right)} \\
 &\quad + \sum_i (g_i^A)^2 \text{ (left)} + \sum_{i,j} g_i^A g_j^A \text{ (right)}
 \end{aligned}$$

R^{NS} $R_{i,j}^{A,S}$

$$\text{Decoupling: } A_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} C_h A_S^{(5)}, \quad A_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} A_b^{(5)} + C_\psi A_S^{(5)}.$$

$$R^A \stackrel{\mathcal{O}(\alpha^4)}{=} (5 R^{NS} + C_h^2 - 2(C_h - C_\psi)(R^{NS} + R_{b,S}^{A,S}) + R_{b,b}^{A,S})$$

Master Formula for Γ_Z^h

$$\Gamma_Z^h = \Gamma_0 \times$$

$$\begin{aligned}
& \left(\sum_i (g_i^V)^2 + \sum_i (g_i^V)^2 \right) \left\{ 1 + \textcolor{blue}{a_s} + 1.40923 \textcolor{blue}{a_s^2} - 12.7671 \textcolor{blue}{a_s^3} - 79.9806 \textcolor{blue}{a_s^4} \right\} \\
& + \left(\sum_i g_i^V \right)^2 \left(-0.41318 \textcolor{blue}{a_s^3} - 4.9841 \textcolor{blue}{a_s^4} \right) \\
& + \left[\left(-3.08333 - \ln_{M_t} \right) \textcolor{blue}{a_s^2} \right. \\
& \quad \left. + \left(-15.9877 - 3.72222 \ln_{M_t} + 1.91667 \ln_{M_t}^2 \right) \textcolor{blue}{a_s^3} \right. \\
& \quad \left. + \left(49.162 + 17.6822 \ln_{M_t} + 14.7153 \ln_{M_t}^2 - 3.67361 \ln_{M_t}^3 \right) \textcolor{blue}{a_s^4} \right] \left\} \right.
\end{aligned}$$

$$\Gamma_0 = \frac{G_F M_Z^3}{8 \pi \sqrt{2}} , \quad \textcolor{blue}{a}_s = \frac{\alpha_s^{(5)}}{\pi} , \quad \ln_{M_t} = \ln \left(\frac{M_t^{\text{pole}}}{M_Z} \right)^2$$

Effective Master Formula for Γ_Z^h

Using as inputs $M_t = 172.0$, $s_W^2 = 0.232$ and $\alpha_s(M_Z) = .1190$ we arrive to an “effective” Master Formula illustrating the total effect of subleading singlet contributions in comparison to the non-singlet ones:

$$\Gamma_Z = \Gamma_0 \left(\sum_i (g_i^V)^2 + \sum_i (g_i^A)^2 \right) \left\{ \begin{array}{l} 1 + a_s + 1.409 a_s^2 - 12.77 a_s^3 - \boxed{79.99 a_s^4} \iff \text{NS} \\ -0.1054 a_s^3 - \boxed{1.272 a_s^4} \iff \text{V SI} \\ -0.6315 a_s^2 - 3.0341 a_s^3 + \boxed{11.452 a_s^4} \iff \text{A SI} \end{array} \right\}$$

Conclusions I

- The “10 years +” project of computing $R(s)$ and $\Gamma(Z \rightarrow \text{hadrons})$ at order α_s^4 is finished!
- The last missing ingredients — singlet contributions — to the (massless) VV and AA as well as (massive) $\langle A_t A_t \rangle$, $\langle A_t A_b \rangle$ and $\langle A_b A_b \rangle$ correlators at $\mathcal{O}(\alpha_s^4)$ are now available
- singlet $\mathcal{O}(\alpha_s^4)$ contributions are numerically tiny
- the net effects of $\mathcal{O}(\alpha_s^4)$ term in Γ_Z^h are: an increase of $\delta\alpha_s(M_Z) = \mathbf{0.0005}$

$$\mathcal{O}(\alpha_s^3) : \quad \alpha_s(M_Z)^{NNLO} = \mathbf{0.1185} \pm 0.0026^{\text{exp}} \pm 0.002^{\text{th}}$$

$$\mathcal{O}(\alpha_s^4) : \quad \alpha_s(M_Z)^{NNNLO} = \mathbf{0.1190} \pm 0.0026^{\text{exp}} \pm 0.0005^{\text{th}}$$

and *four-fold* decrease of the theory error!

/ K.Ch, Baikov and Kühn, PLR 101 (2008) 012002/

Conclusions II

- the **5-loop** QED β -function is computed \Leftarrow the first example of 5-loop RG function in a (normal) 4-D gauge theory: almost exactly thirty years after similar result in a ϕ^4 model /K.Ch, Gorishnii, Larin, and Tkachov, Phys.Lett. B132 (1983) 351/; D.I. Kazakov, Phys.Lett. B133 (1983) 406; Kleinert, Neu, Schulte-Frohlinde , K.Ch., and Larin, Phys.Lett. B272 (1991) 39/
- All our methods and tools are equally well applicable to evaluation of the **5-loop** β -functions and anomalous dimensions in general **non-Abelian** gauge theories. A couple of interesting examples could be:
 - ① QCD β -function (important for a better understanding of the τ -lepton decay rate within the so-called contour-improved method)
 - ② the anom. dim. of the Konishi operator in N=4 supersymmetrical YM theory at **5 loops** (important for better understanding of Ads/CFT correspondence, integrability, ABA and TBA . . .)

Conclusion III

The qualification above about “normal” theories is essential, in supersymmetric theories life is different, for example,

M. Grisaru, D. Kazakov and D. Zanon,
“Five Loop Divergences For The N=2 Supersymmetric Nonlinear Sigma Model”, Nucl.Phys. B287 (1987) 189

Personal Conclusion

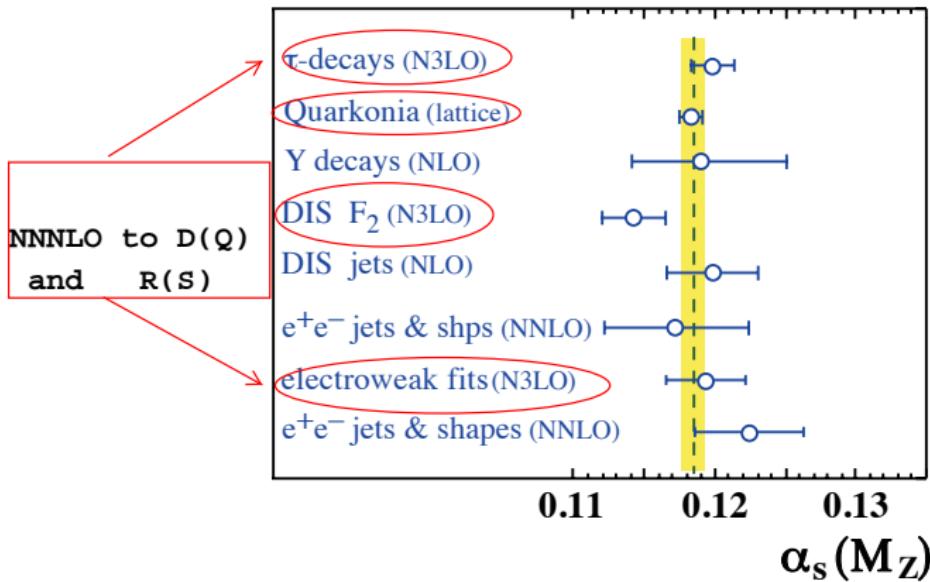
Dear Dima,

*We sincere congratulate you
with your 60-th Jubilee and
wish you all the best and many new
exciting results!*

and remember: There is life after 60!

Ira and Kostja

World Summary of α_s 2009:



$$\rightarrow \alpha_s(M_Z) = 0.1184 \pm 0.0007$$

(Bethke, arXiv:0908.1135)