BLTP, JINR

Generalized Skyrme model, Skyrme crystals and self-dual solitons

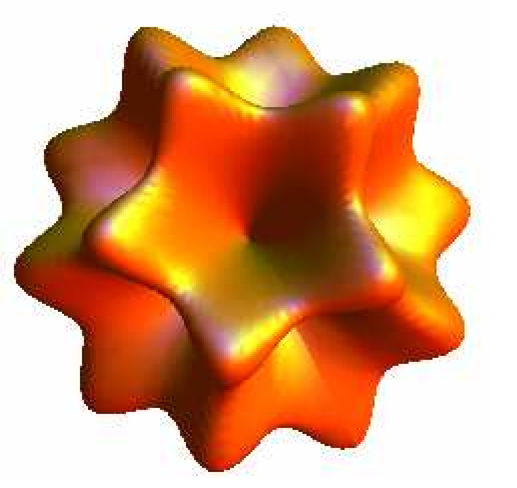
Ya Shnir

Thanks to my collaborators:
C.Adam, I.Perapechka,
A.Samoilenka and A. Wereszczynski

Dubna, 7.03 2018

Outline

- Skyrme models and Skyrmions
- Generalized Skyrme model
- Self-dual Skyrme models
- Skyrme crystals
- Summary



Skyrme model

- **QCD:** $$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}_i [i\gamma^\mu D_\mu - m\delta_{ij}]\psi_j$$

- **Low energy meson theory:**

$$L = \frac{f_\pi^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U\right) + \ldots$$

Tony Hilton Royle Skyrme

Skyrmes' motivations (1962):

- The idea of unifying bosons and fermions in a common framework
- Consideration of localised field configurations instead of point-like particles
- The desire to eliminate fermions from a fundamental formulation of theory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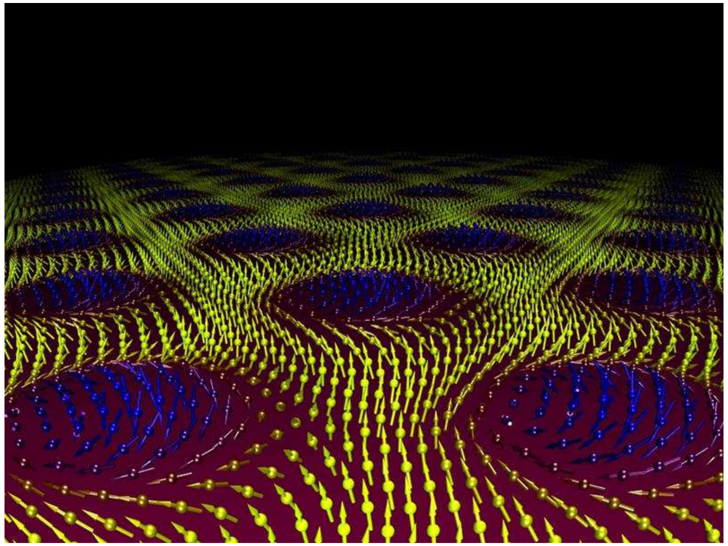
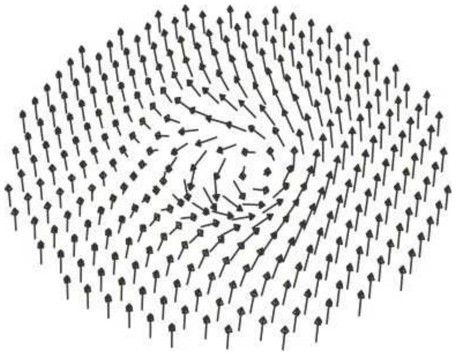
Nanosize magnetic whirlpools could be the future of data storage
(2 August 2016)
Could magnetic skyrmions hold the answer to better data storage?
The use of nanoscale magnetic whirlpools, known as magnetic skyrmions, to create novel and efficient ways to store data will be explored in a new £7M research programme led by Durham University.
Skyrmions, which are a new quantum mechanical state of matter, could be used to make our day-to-day gadgets, such as mobile phones and laptops, much smaller and cheaper whilst using less energy and generating less heat.
It is hoped better and more in-depth knowledge of skyrmions could address society's ever-increasing demands for processing and storing large amounts of data and improve current hard drive technology.
Revolutionise data storage
Scientists first predicted the existence of skyrmions in 1962 but they were only discovered experimentally in magnetic materials in 2009.
The UK team, funded by the Engineering and Physical Sciences Research Council (EPSRC), now aims to make a step change in our understanding of skyrmions with the goal of producing a new type of demonstrator device in partnership with industry.
Skyrmions, tiny swirling patterns in magnetic fields, can be created, manipulated and controlled in certain magnetic materials. Inside a skyrmion, magnetic moments point in different directions in a self-organised vortex.

Baby Skyrmions in action: Condensed Matter Systems

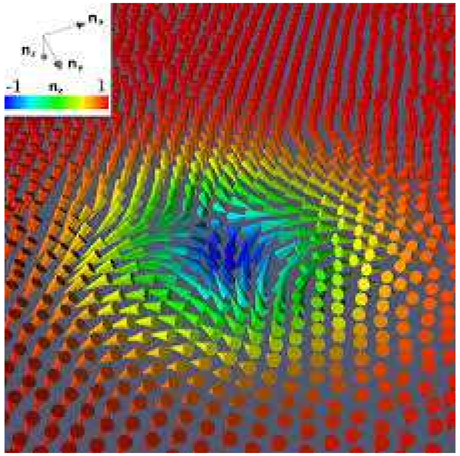




Rößler et al.
Nature 442 (2006) 797

Chiral Skyrmions in noncentrosymmetric magnets

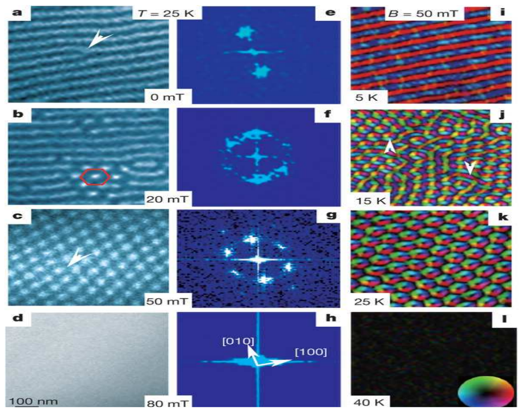
J. Garaud, J. Carlström, and E. Babaev,
Phys. Rev. Lett. 107, 197001 (2011);



Three-component superconductors

Yu, Onose et al. Nature 465, 901 (2010)

Experimental realization in a thin film of $Fe_{0.5}Co_{0.5}Si$



Skyrme model

(Skyrme, 1961)

- **The Skyrme field:** $U(\vec{r},t) \underset{r\to\infty}{\longrightarrow} \mathbb{I}$

  $U: \; S^3 \to S^3$

$f_\pi = 186\ MeV, \;\; m_\pi = 136\ MeV$

$$L = \boxed{\frac{f_\pi^2}{4}\mathrm{Tr}\;\left(\partial_\mu U \partial^\mu U^\dagger\right)} + \boxed{\frac{1}{32e^2}\mathrm{Tr}\;\left(\left[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\right]^2\right)} + \boxed{\frac{m_\pi^2 f_\pi^2}{8}\mathrm{Tr}\;(U - \mathbb{I})}$$

Sigma-model term

Skyrme term

Potential term

- **The topological charge:** $$\boxed{Q = \frac{1}{24\pi^2}\varepsilon_{ijk}\int d^3x \mathrm{Tr}\;\left[(U^\dagger\partial^i U)(U^\dagger\partial^j U)(U^\dagger\partial^k U)\right]}$$

- **The su(2) current:** $R_i = (\partial_i U)U^\dagger \Rightarrow Q = -\frac{1}{24\pi^2}\varepsilon_{ijk}\int d^3x \mathrm{Tr}(R_i R_j R_k)$

Rescaling: $x_\mu \to 2x_\mu/(ef_\pi); \quad m = 2m_\pi/(f_\pi e)$

$$E = \frac{f_\pi}{4e}\int d^3x \left\{-\frac{1}{2}\mathrm{Tr}\;(R_i R^i) - \frac{1}{16}\mathrm{Tr}\;([R_i, R_j])^2 + m^2 \mathrm{Tr}(\mathrm{U} - \mathbb{I})\right\}$$

Skyrme model

- **The Skyrme field:** $U = \phi_0 \mathbb{I} + i\sigma^a \cdot \pi^a$ $\qquad \phi^a = (\phi_0, \pi^a); \quad \phi^a \cdot \phi^a = 1$

$$L = \boxed{\partial_\mu \phi^a \partial^\mu \phi^a} - \boxed{\frac{1}{2}(\partial_\mu \phi^a \partial_\mu \phi^a)^2 + \frac{1}{2}(\partial_\mu \phi^a \partial_\nu \phi^a)(\partial^\mu \phi^b \partial^\nu \phi^b)} - \boxed{m^2(1 - \phi^a \phi^a_\infty)}$$

Sigma-model term

Skyrme term

Potential term

- **The topological charge:** $\boxed{Q = \frac{1}{12\pi^2} \int d^3x \; \varepsilon_{abcd} \varepsilon^{ijk} \phi^a \partial_i \phi^b \partial_j \phi^c \partial_k \phi^d}$

Topological bound: $\boxed{E \geq 12\pi^2 |Q|}$

Topological bound is not saturated in the Skyrme model,
the solitons interact

Skyrmions from instantons

M.Atiyah and N.Manton, Phys. Lett. B 222, 438 (1989)

SU(2) Yang-Mills → **Skyrme model**

$$L = \frac{1}{2g^2}\mathrm{Tr}\ F_{\mu\nu}^2 \quad \longrightarrow \quad L = \frac{f_\pi^2}{4}\mathrm{Tr}\,(\partial_\mu U \partial^\mu U) + \ldots$$

Instanton's holonomy:

$$U(\mathbf{x}) = \mathcal{P}\exp\left(i\int_{-\infty}^{\infty} dx_0 A_0(\mathbf{x}, x_0)\right) \in \ SU(2) \ \xrightarrow[\mathbf{x}\to\infty]{} \mathbb{I}$$

Q=1

- **YM Instanton**
- **Skyrmion**

$$A_0 = i\hat{r^a}\cdot\tau^a\left(\frac{1}{r^2+x_0^2+\lambda} - \frac{1}{r^2+x_0^2}\right) \quad \longrightarrow \quad \begin{cases} U(r) = \exp\left[i\tau^a\hat{r}^a F(r)\right] \\ F(r) = \pi\left(1 - \dfrac{r}{\sqrt{r^2+\lambda^2}}\right)\end{cases}$$

Pontryagin index → **Skyrmion winding number**

$$\frac{1}{16\pi^2}\int d^4x F_{\mu\nu}\widetilde{F}_{\mu\nu} \quad \longrightarrow \quad \frac{1}{24\pi^2}\varepsilon_{ijk}\int d^3x \mathrm{Tr}\ \left[(U^\dagger\partial^i U)(U^\dagger\partial^j U)(U^\dagger\partial^k U)\right]$$

Skyrme model

- **Spherically symmetric skyrmion:**

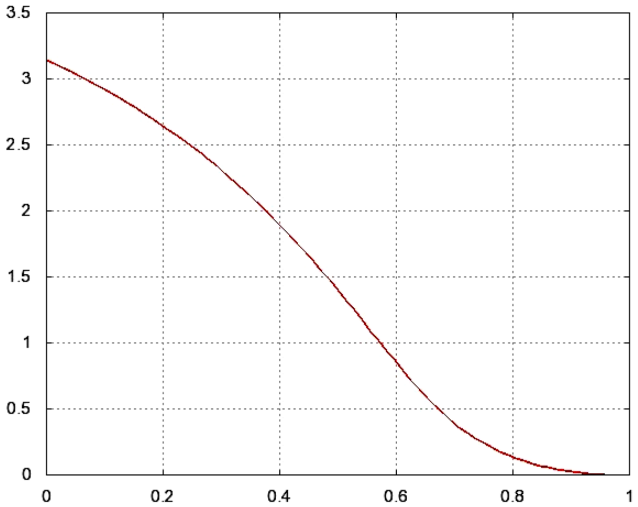
$$\boxed{U(r) = \exp\left[i\tau^a \hat{r}^a F(r)\right]}$$

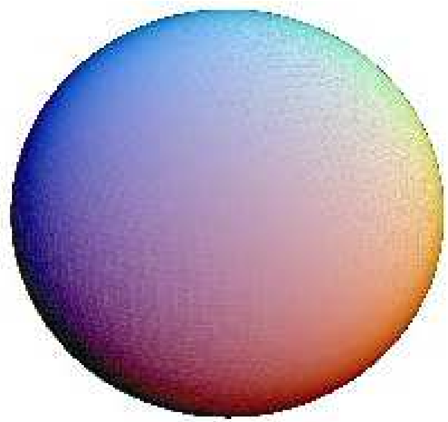
(Hedgehog ansatz)

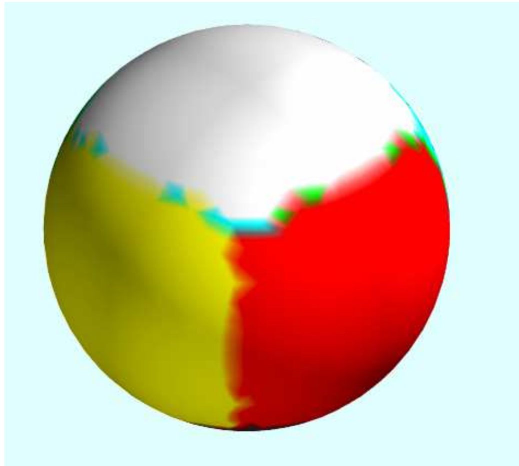
$$Q = \frac{1}{\pi}\left[F(r) - \frac{\sin 2F(r)}{2}\right]_0^\infty$$

The boundary conditions

$$F(0) = \pi,\ \ F(\infty) = 0 \Longrightarrow Q = 1$$







$$U(r) = \sigma + \pi^a \cdot \tau^a = \cos F(r) + i\hat{\mathbf{n}} \cdot \tau \sin F(r)$$

$$\boxed{\phi^a = (\sigma, \pi^1, \pi^2, \pi^3)}$$

$$L = -\int d^3x \left\{\tfrac{1}{2}(\partial_\mu\phi^a)^2 - \tfrac{1}{4}\left[(\partial_\mu\phi^a\partial_\nu\phi^a)^2 - (\partial_\mu\phi^a)^4\right] + m^2\ (1-\phi^3)\right\}$$

Skyrmions: asymptotic field

- **Field equations:**

$$(1-\partial_\nu\phi^b\partial^\nu\phi^b)\partial_\mu\partial^\mu\phi^a+(\partial_\mu\phi^b\partial_\nu\partial^\nu\phi^b-\partial^\nu\phi^b\partial_\mu\partial_\nu\phi^b)\partial^\mu\phi^a$$
$$+\partial^\mu\phi^b\partial^\nu\phi^b\partial_\mu\partial_\nu\phi^a+m^2\phi^a_\infty=0\,.$$

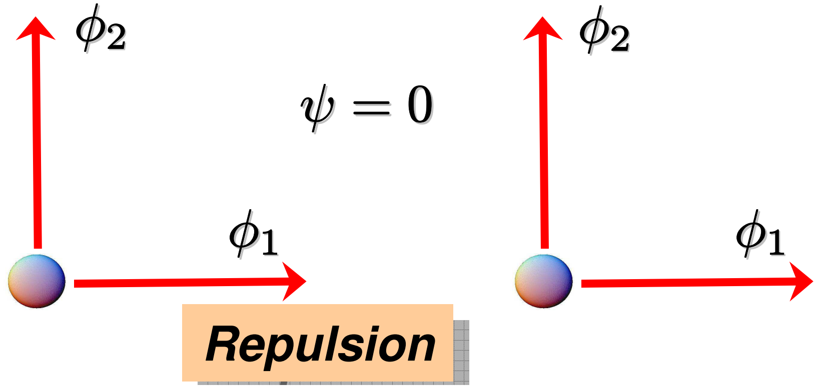
For a Hedgehog:

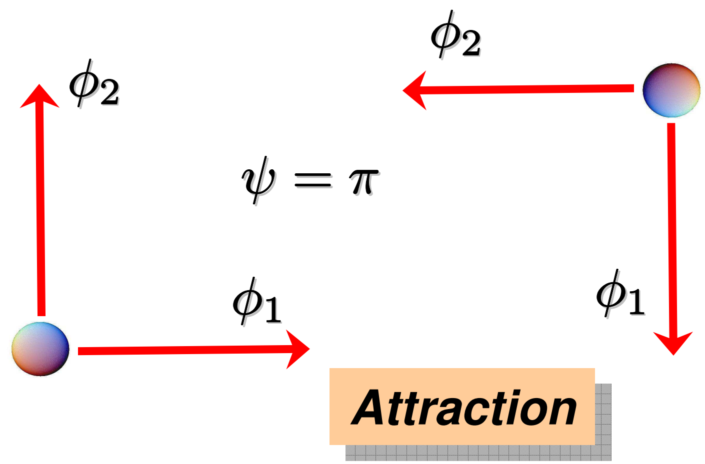
$$(r^2+2\sin^2F)F''+2rF'-\sin 2F\left(1-F'^2+\frac{\sin^2F}{r^2}\right)+m^2\sin F=0$$

m=0: $$F(r)\sim\frac{d}{4\pi r^2}+O\left(\frac{1}{r^8}\right),\quad \pi_i=\sin F(r)\hat{r}_i=\frac{dr_i}{4\pi r^3},\quad \text{as } r\to\infty$$

The field of a Skyrmion represents **a triplet of orthogonal scalar dipoles**

The energy of interaction of 2 dipoles:





Rotating Skyrmions

Symmetries of the Skyrme model:

$$U(\vec{r},t) \xrightarrow[r\to\infty]{} \mathbb{I}$$

- **Poincare group** $\mathbb{R}^{1,3} \times SO(3,1)$
- **SO(4) chiral invariance** $SO(4) \cong \dfrac{SU(2)\times SU(2)}{\mathbb{Z}_2} \Rightarrow SO(3)$

The Euler-Lagrange equations:

$$R_\mu = (\partial_\mu U)U^\dagger; \qquad L_\mu = (\partial_\mu U^\dagger)U$$

$$\partial^\mu \bar{R}_\mu \equiv \partial^\mu R_\mu + \frac{1}{4}[R_\nu,[R^\nu,R_\mu]] = 0 \qquad \partial^\mu \bar{L}_\mu \equiv \partial^\mu L_\mu + \frac{1}{4}[L_\nu,[L^\nu,L_\mu]] = 0$$

Note: the currents R_μ, L_μ satisfy the zero curvature equation (Maurer–Cartan identity)

$$\partial_\mu R_\nu - \partial_\nu R_\mu + [R_\mu, R_\nu] = 0 \qquad \partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] = 0$$

- **Collective coordinates: (Isorotations)** $U(r,t) = A(t)U(r)A^\dagger(t); \qquad A(t) \in SU(2)$
- **Spacial rotations:** $\vec{r} \to O(t)\vec{r}; \qquad O(t) \in SO(3)$

Rotating Skyrmions

The angular velocities $\Omega_k = -i\mathrm{Tr}\,(\tau_k \dot{O} O^\dagger);\qquad \omega_k = -i\mathrm{Tr}\,(\tau_k \dot{A} A^\dagger)$

Spacial rotations

Isorotations

Effective Lagrangian:

$$\boxed{L = \frac{1}{2}\omega_i U_{ij}\omega_j + \frac{1}{2}\Omega_i V_{ij}\Omega_j - \omega_i W_{ij}\Omega_j - M}$$

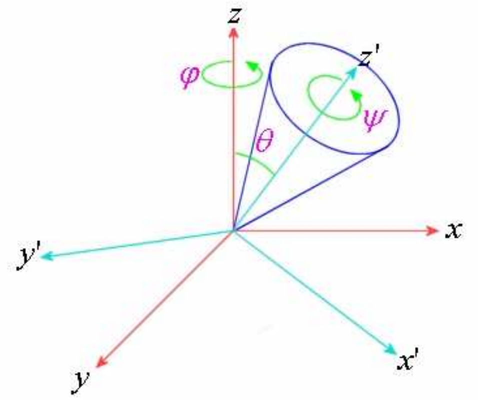
Moments of inertia

The body-fixed spin and isospin angular momenta

$$L_i = -W_{ij}\omega_j + V_{ij}\Omega_j;\qquad K_i = U_{ij}\omega_j - W_{ij}\Omega_j$$

$$J_i = -O_{ij}^\dagger L_j;\qquad I_i = -A_{ij}K_j$$

The space-fixed spin and isospin angular momenta



Integrals of motion

For the hedgehog ansatz any spatial rotation is equivalent to a group rotation because $O_{ij} = \frac{1}{2}\,\mathrm{Tr}(A\tau_i A^\dagger \tau_j)$

Spinning a hedgehog

Q=1

$$U(r) = \exp\left[i\tau^a \hat{r}^a F(r)\right]$$

Spacial rotations can be absorbed into the group rotations:

$$U(r) \to A(t)U(Or)A^\dagger(t) = \widetilde{A}(t)U(r)\widetilde{A}(t) \qquad \text{where} \quad \widetilde{A} = OA \in SU(2)$$

The angular velocities

$$\Omega_L^a = i\ \mathrm{Tr}(A^\dagger \dot{A}\tau^a) \qquad \Omega_R^a = i\ \mathrm{Tr}\dot{A}A^\dagger\tau^a)$$

Note: $L = M + \frac{I}{2}\Omega_L^2 = M + \frac{I}{2}\Omega_R^2 = M + I\ \mathrm{Tr}\ \dot{A}^\dagger\dot{A}$

Spherical top

$$I = \frac{8\pi}{3}\int_0^\infty r^2 dr \sin^2 f(r)\left[\frac{F_\pi^2}{4} + \frac{1}{e^2}\left(f'(r)^2 + \frac{\sin^2 f(r)}{r^2}\right)\right]$$

The moment of inertia of the Skyrmion

The canonical quantization procedure: Introduce two sets of angular momenta, J_L (canonically conjugate to Ω_L) and J_R (conjugate to Ω_R).

$$\boxed{J_{L,R} = I\,\Omega_{L,R}}$$

Rigid body?

$I = \text{const}$ $\Rightarrow$ $$H = M + \frac{J^2}{2I} = M + \frac{j(j+1)}{2I}$$

$$[J_i, J_j] = \varepsilon_{ijk} J_k$$

Hedgehog approximation – spin = isospin

- **J=1/2, S=1/2:** **Nucleon**

  $$m_n = 939\ MeV$$

- **J=3/2, S=3/2:** **Δ-resonance**

  $$m_\Delta = 1232\ MeV$$

1980s-2000s: Everybody knows that..

- **The Skyrme model can be considered as a low-energy effective theory of hadrons**
- **its solitonic solutions are identified with nucleons**
- **the topological charge is identified with baryon number**
- **it can thus be used to study the structure of nuclear matter at high densities**

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$$m_\Delta = 1232\ MeV$$

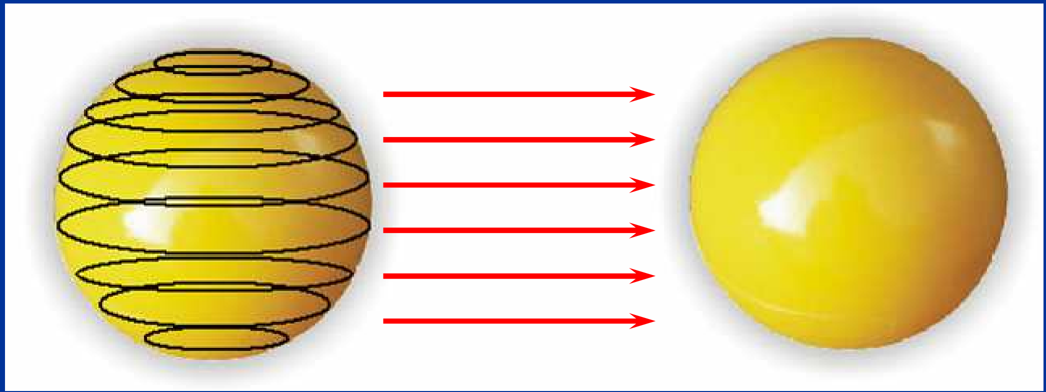
2010.. But… Skyrme model requires for ~20% accuracy

- $f_\pi(p) / f_\pi(\pi) \sim 1/2$
- Rigid body approximation
- $m_n / m_\pi \sim 2$
- “Nuclei” do not at all look like nuclei
- Binding energy of nucleons if too high

The usual Skyrme model with pion fields only does not work

Rational map Skyrmions

The Skyrme field is effectively a map U: $S_3 \to SU(2) \sim S_3$



(N.S. Manton, C.Houghton & P.Sutcliffe, 1998)

Stereographic Projection $z = \tan(\theta/2)e^{i\varphi}$

$$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}\left(\frac{z+\bar{z}}{1+z\bar{z}}, i\frac{z^*-z}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+\bar{z}}\right)$$

$$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Domain space

The idea of the rational map ansatz:

- Separate the radial and the angular dependence of the Skyrme field **as**

$$U = \exp\{if(r)\hat{\mathbf{n}}_Z \cdot \sigma\}$$

- *I*dentify spheres S_2 with concentric spheres in compactified R_3
- *I*dentify target space $S_2$with spheres of latitude on S_3

$$\hat{n}_Z = \left(\frac{Z+\bar{Z}}{1+Z\bar{Z}}, i\frac{\bar{Z}-Z}{1+Z\bar{Z}}, \frac{1-Z\bar{Z}}{1+Z\bar{Z}}\right)$$

$$Z = P(z)/Q(z)$$

Target space

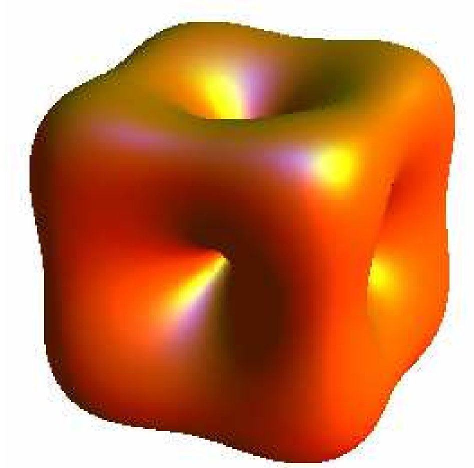
Rational map approximation

- **Static energy:** $$E = 4\pi \int \left(r^2 f'^2 + 2Q(f'^2 + 1)\sin^2 f + W \frac{\sin^4 f}{r} \right) dr$$

$$4\pi Q = \int \left(\frac{1+|z|^2}{1+|Z|^2} \left| \frac{dZ}{dz} \right| \right)^2 \frac{dz d\bar{z}}{(1+|z|^2)^2}$$

$$W = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|Z|^2} \left| \frac{dZ}{dz} \right| \right)^4 \frac{dz d\bar{z}}{(1+|z|^2)^2}$$

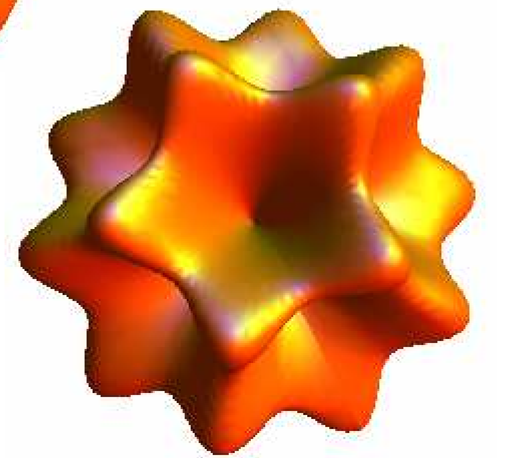
The holomorphic maps of degree Q:



Q= 4: $$Z(z) = \frac{z^4 + 2i\sqrt{3}z^2 + 1}{z^4 - 2i\sqrt{3}z^2 + 1}$$

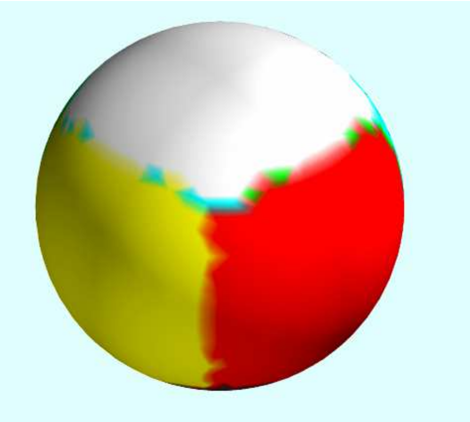
(Octahedral Skyrmions)

Q= 7: $$Z(z) = \frac{z^7 - 7z^5 - 7z^2 - 1}{z^7 + 7z^5 - 7z^2 + 1}$$

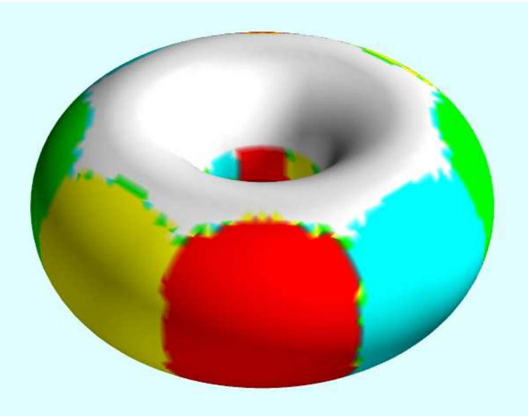


(Icosahedral Skyrmions)

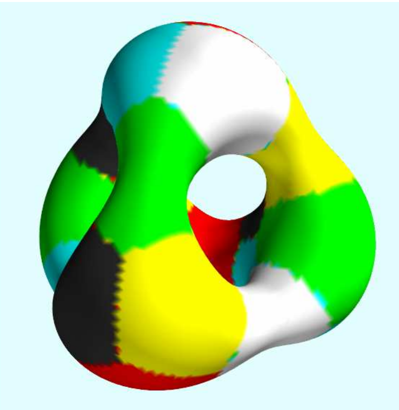
Skyrmions



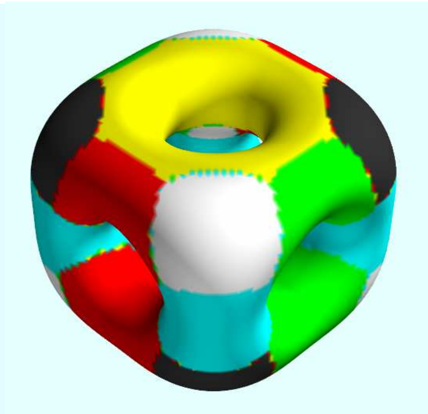
Q=1



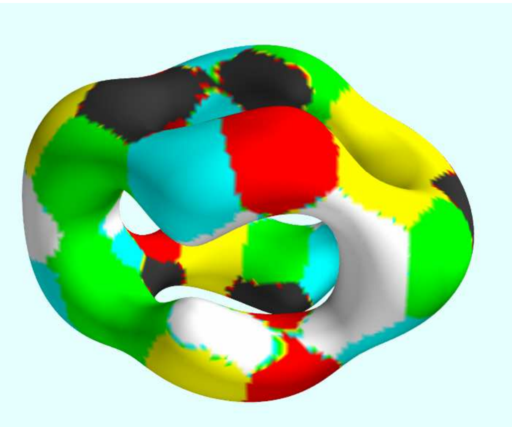
Q=2



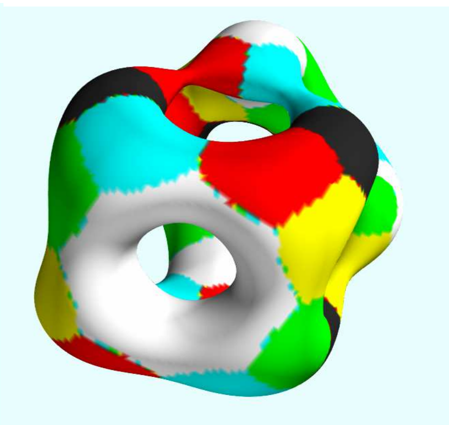
Q=3



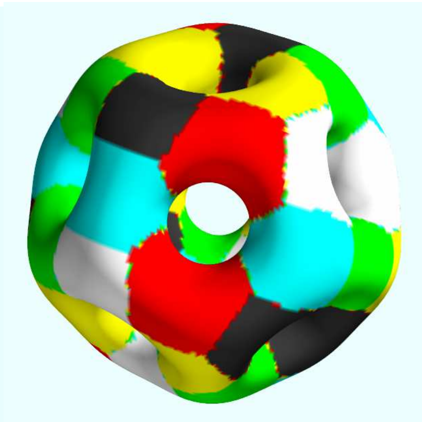
Q=4



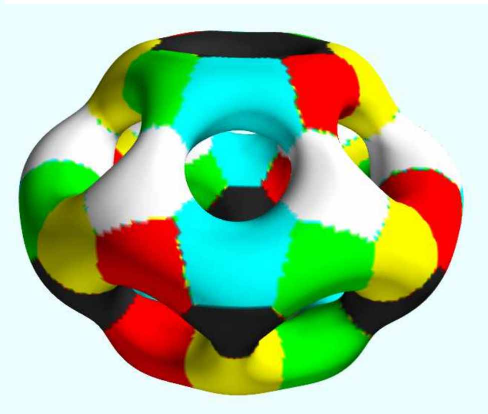
Q=5



Q=6



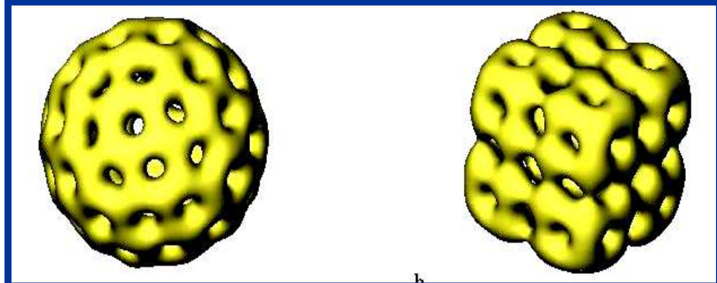
Q=7



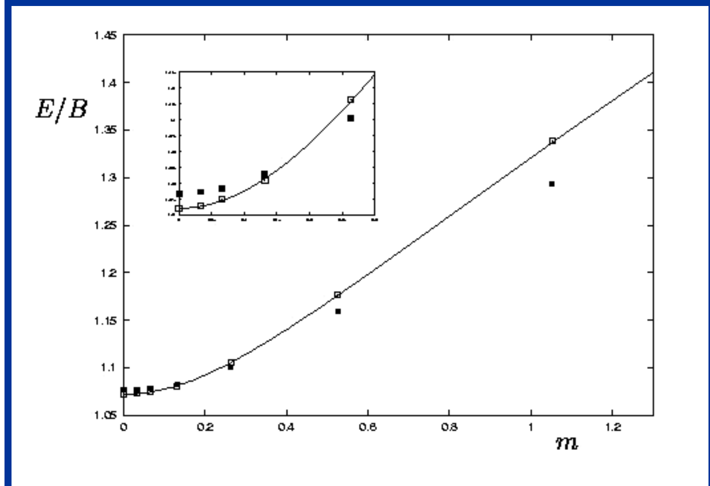
Q=8

Crystalline structure of nucleons

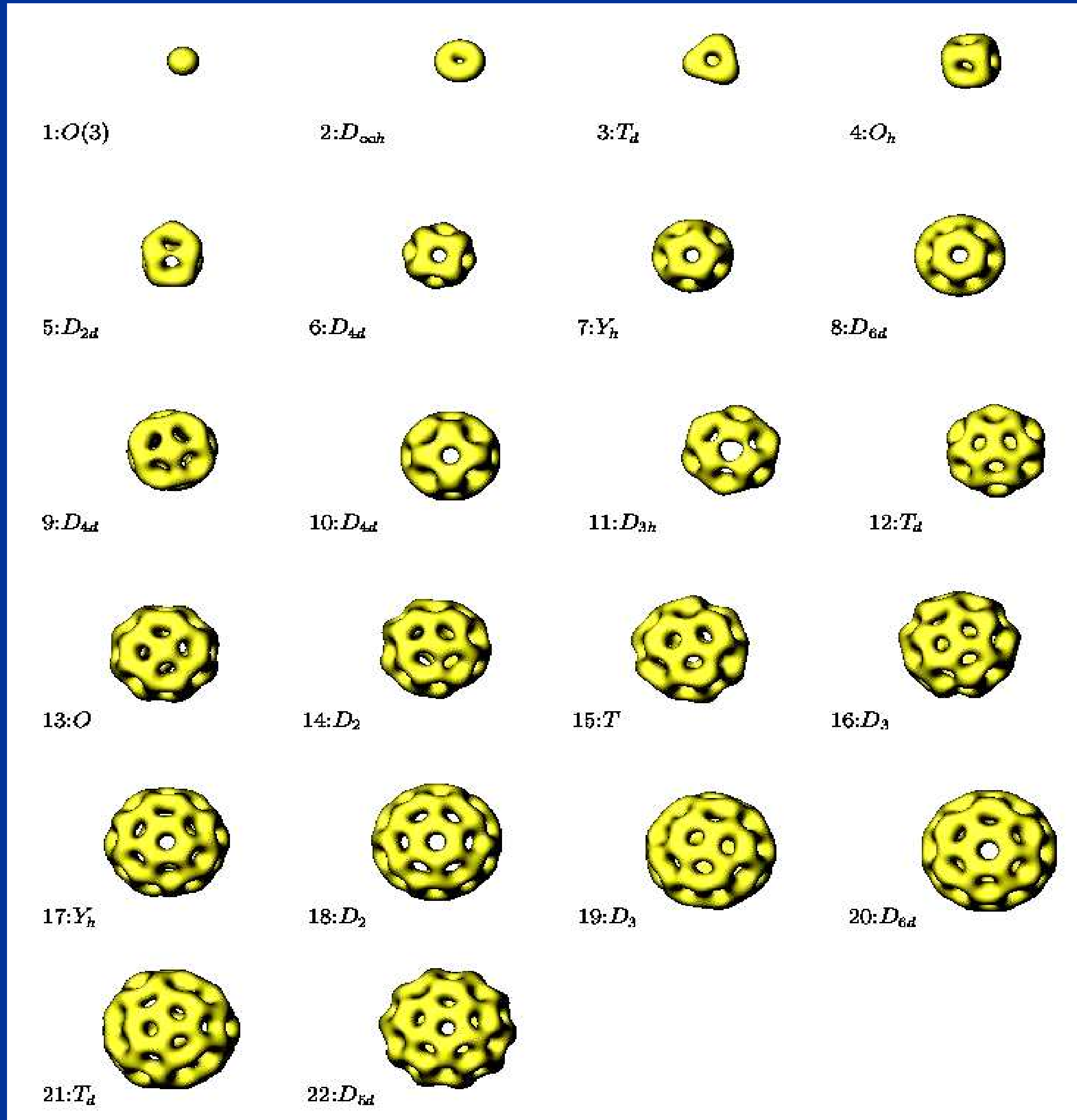
R. A. Battye, N. S. Manton,
C.Houghton
and P. Sutcliffe (1996,2004)



Shell vs. Crystal

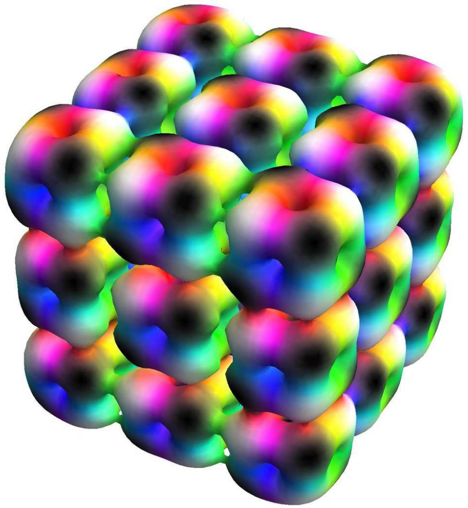


Shell wins for m $\geq$ 0.16



Skyrme crystals: $L_2 + L_4 + L_0$

(Klebanov, Kugler, Shtrikman, Manton..)

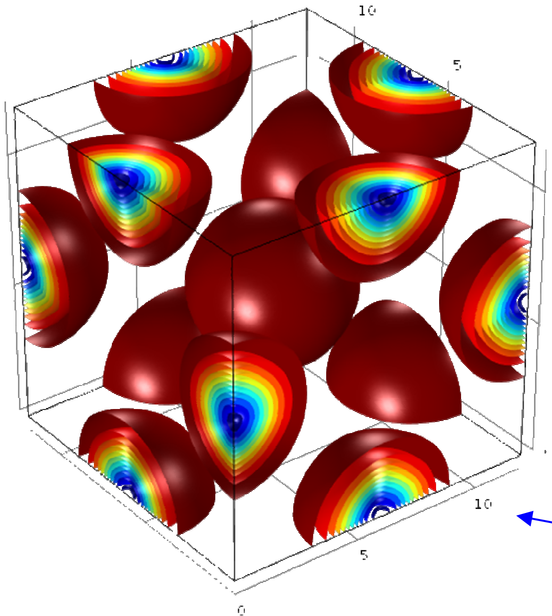


simple cubic lattice

$$(x_1, x_2, x_3) \longrightarrow (\sigma, \pi^1, \pi^2, \pi^3); \quad x_i = 2n_i L$$

$$U(x_1 + L, x_2, x_3) = \tau^2 U(x_1, x_2, x_3)\tau^2 \quad \text{etc}$$

Atractive channel of interaction between 6 nearest neighbors



face-centred cubuc lattice

$$(x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3); \quad (\sigma, \pi^1, \pi^2, \pi^3) \mapsto (\sigma, -\pi^1, \pi^2, \pi^3)$$

$$(x_1, x_2, x_3) \mapsto (x_2, x_3, x_1); \quad (\sigma, \pi^1, \pi^2, \pi^3) \mapsto (\sigma, \pi^2, \pi^3, \pi^1),$$

$$(x_1, x_2, x_3) \mapsto (x_1, x_3, -x_2); \quad (\sigma, \pi^1, \pi^2, \pi^3) \mapsto (\sigma, \pi^1, \pi^3, -\pi^2),$$

Atractive channel of interaction between 12 nearest neighbors

Low density phase

$$(x_1, x_2, x_3) \mapsto (x_1 + L, x_2 + L, x_3); (\sigma, \pi^1, \pi^2, \pi^3) \mapsto (\sigma, -\pi^1, -\pi^2, \pi^3)$$

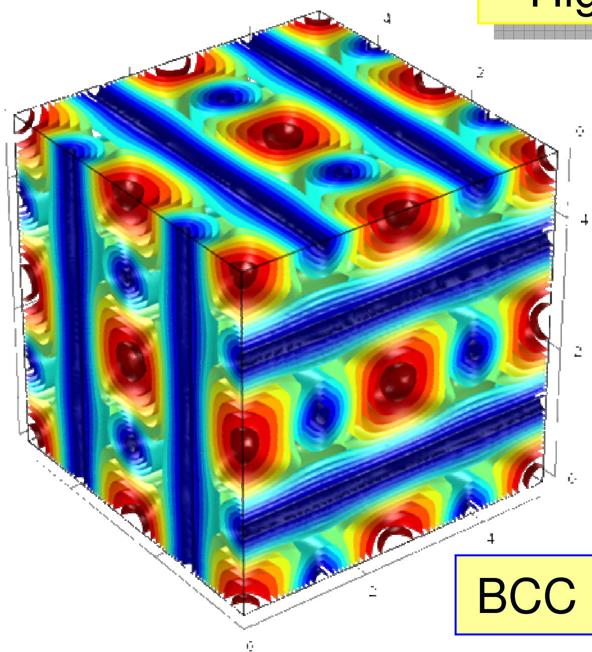
Skyrme crystals: $L_2 + L_4 + L_0$

$$\sigma = \mp \cos\xi_1 \cos\xi_2 \cos\xi_3\,;$$

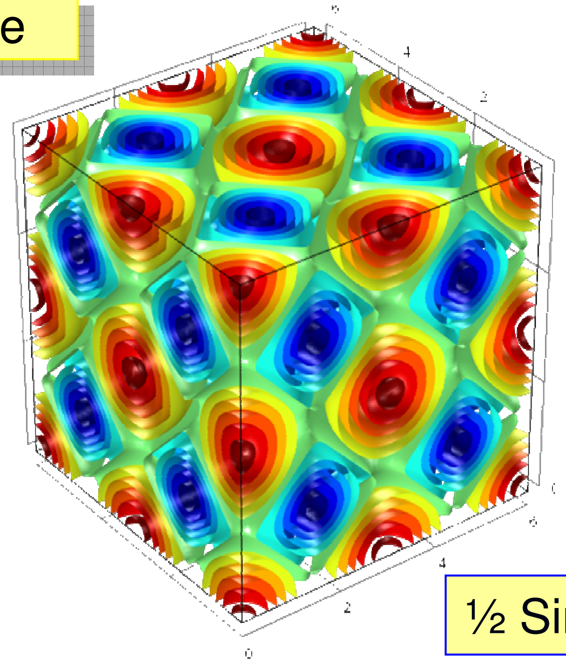
$$\pi^1 = \pm \sin\xi_1 \sqrt{1 - \frac{1}{2}\sin^2\xi_2 - \frac{1}{2}\sin^2\xi_3 + \frac{1}{3}\sin^2\xi_2\ \sin^2\xi_3} \qquad \xi_i = \pi x_i / L$$

$$(x_1, x_2, x_3) \mapsto (x_1 + L, x_2, x_3)\,; \quad (\sigma, \pi^1, \pi^2, \pi^3) \mapsto (-\sigma, -\pi^1, \pi^2, \pi^3)$$

High density phase

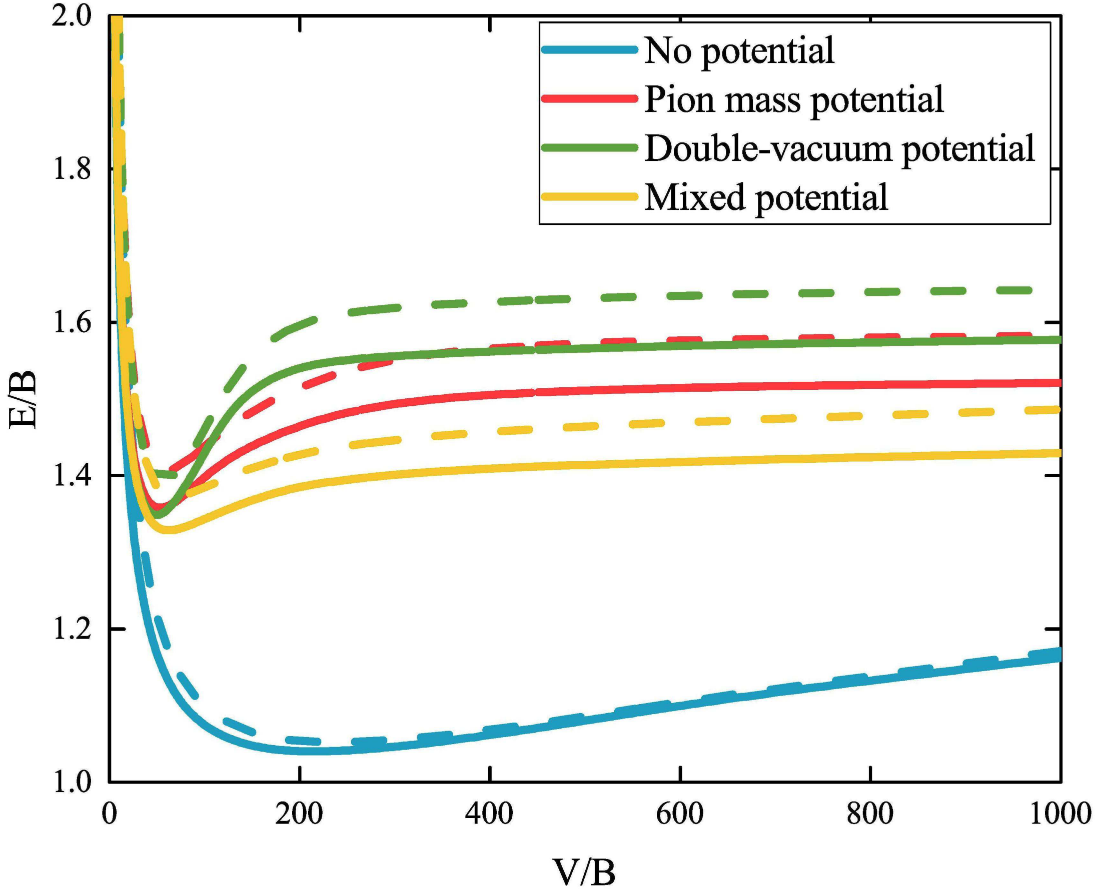


BCC



½ Simple cubic

Skyrme crystals: $L_2 + L_4 + L_0$



Generalized Skyrme model

(C.Adam, J.Sánchez-Guillén and A.Wereszczynski..)

- **Poincare invariance & standard Hamiltonian (quadratic in time derivatives):**

$$L = aL_2 + bL_4 + cL_6 + L_0$$

$$E \geq \pm Q$$

$$L_2 = \frac{f_\pi^2}{4}\mathrm{Tr}\ \left(U^\dagger\partial_\mu U U^\dagger \partial^\mu U\right) \quad \mathrm{L}_4 = \frac{1}{32e^2}\mathrm{Tr}\ \left(\left[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\right]^2\right) \quad L_0 = \mu^2 V$$

$$L_6 = -\frac{\lambda^2}{24\pi^2}\mathrm{Tr}\left(\varepsilon_{\mu\nu\rho\sigma}U^\dagger\partial_\mu U U^\dagger\partial_\nu U U^\dagger\partial_\rho U\right)^2 = B_\mu B^\mu$$

- **Submodel:** $L_6 + L_0$

Self-duality equation: $\mathrm{Tr}\left(\varepsilon_{\mu\nu\rho}U^\dagger\partial_\mu U U^\dagger\partial_\nu U U^\dagger\partial_\rho U\right) = \pm C\sqrt{L_0}$

$$\pi^2 Q \pm \mu\sqrt{V} = 0$$

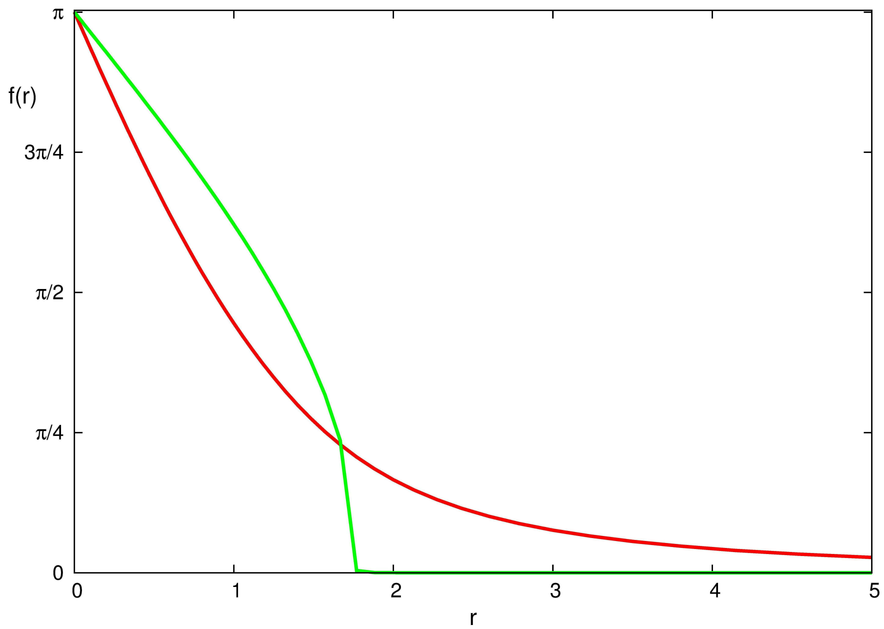
BPS Skyrmions - Compactons

Hedgehog parametrization:

$$U(r) = \exp\left[i\tau^a \hat{r}^a F(r)\right]$$

$$z = \frac{2\mu r^3}{3}$$

Field equation: $$\cos^3\frac{F}{2} = \pm\frac{3}{4}(z - z_0)$$



$$F(z) = \begin{cases} 2\arccos\sqrt[3]{\frac{3z}{4}}\,; & z \in [0, \frac{4}{3}] \\ 0\,, & z \geq \frac{4}{3}, \end{cases}$$

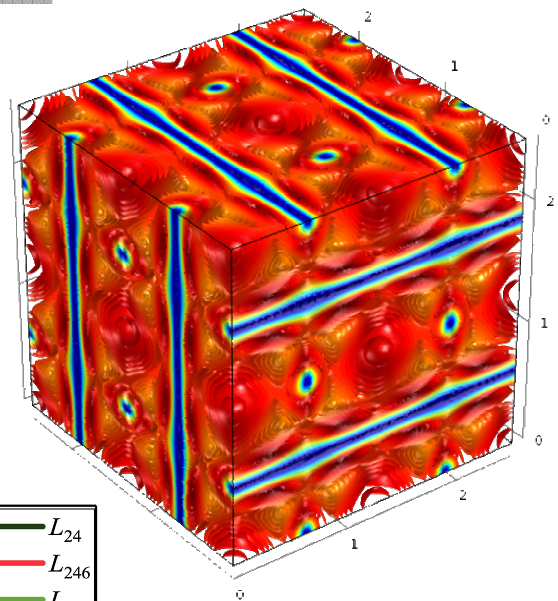
More compactons (non-BPS):

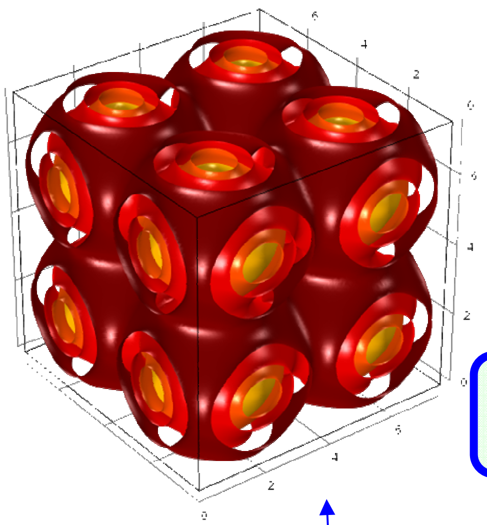
$$L = L_0 + L_4 + L_6$$

Skyrme crystals: submodels of $L_2 + L_4 + L_6 + L_0$

High density phase

$$L_2 + L_6$$

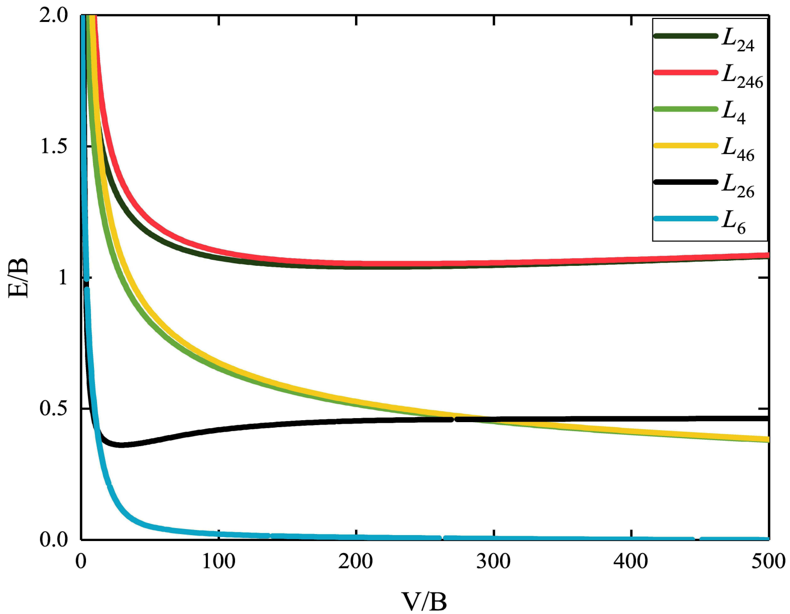




$$aL_2 + bL_4 + L_6 + L_0$$

$$a, b \to 0$$

Low density phase



Yet another self-dual Skyrme model

(L.A. Ferreira and Ya Shnir)

$$U = \phi_0 \mathbb{I} + i\sigma^a \cdot \phi^a \quad \Longrightarrow \quad Z_1 = \phi_1 + i\phi_2; \quad Z_2 = \phi_0 + i\phi_3; \quad Z_a^* Z_a = 1$$

$$\boxed{A_\mu = \frac{i}{2}\left(Z_a^* \partial_\mu Z_a - Z_a \partial_\mu Z_a^*\right); \qquad H_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

$$\mathcal{L} = \frac{m^2}{2} A_\mu^2 - \frac{1}{4e^2} H_{\mu\nu}^2; \qquad \mathcal{E} = \frac{1}{2}\int d^3x \left(m^2 A_n^2 + \frac{1}{e^2} B_n^2\right)$$

$$Q = \frac{1}{12\pi^2}\int d^3x\ \varepsilon_{abcd}\varepsilon^{ijk}\phi^a \partial_i \phi^b \partial_j \phi^c \partial_k \phi^d = \frac{1}{4\pi^2}\int d^3x\ A_n B_n$$

$$\boxed{\mathcal{E} = \frac{1}{2}\int d^3x \left(m A_n \pm \frac{1}{e} B_n\right)^2 \mp \frac{m}{e}\int d^3x A_n B_n}$$

- **Energy bound:** $\boxed{E \geq 4\pi^2 \frac{m}{e}|Q|}$
- **Self-duality eqs:** $\boxed{m\vec{A} = \frac{1}{e}\vec{B} = \frac{1}{e}\nabla \times \vec{A}}$

Yet another self-dual Skyrme model

The model is invariant w.r.t group of **conformal transformations in 3d**

Two commuting U(1):

$$\begin{cases} \partial_\varphi = x_2\partial_1 - x_1\partial_2 \\ \partial_\xi = \dfrac{x_3}{a}\left(x_1\partial_1 + x_2\partial_2\right) + \dfrac{1}{2a}\left(a^2 + x_3^2 - x_1^2 - x_2^2\right)\partial_3 \end{cases}$$

Toroidal coordinates:

$$z = \frac{4a^2\rho^2}{(r^2+a^2)^2}; \qquad \tan\xi = \frac{2ax_3}{r^2-a^2}; \qquad \tan\varphi = \frac{x_2}{x_1}$$

Ansatz:

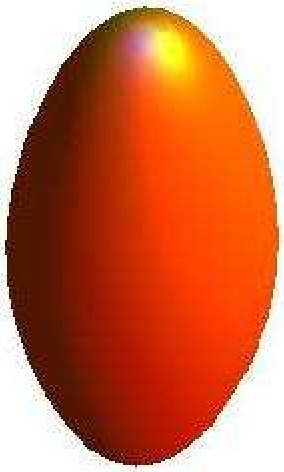
$$Z_1 = \sqrt{F(z)}e^{In\varphi}; \qquad Z_2 = \sqrt{1-F(z)}e^{Im\xi}$$

$$\boxed{m_0 f(\vec r)\vec A = \pm\frac{1}{e_0 f(\vec r)}\vec B} \Rightarrow \begin{cases} F(z) = \dfrac{m^2 z}{m^2 z + n^2(1-z)} \\ f^2 = \dfrac{2mn(1-\sqrt{1-z}\cos\xi)}{a[m^2z + n^2(1-z)]} \end{cases}$$

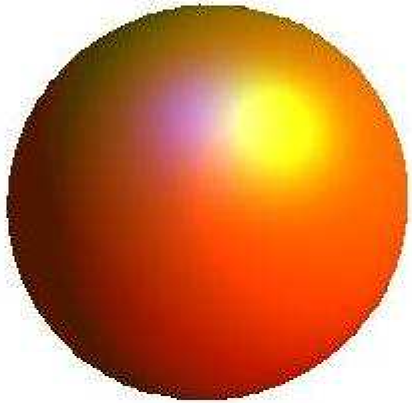
Yet another self-dual Skyrme model

Energy density and topological charge:

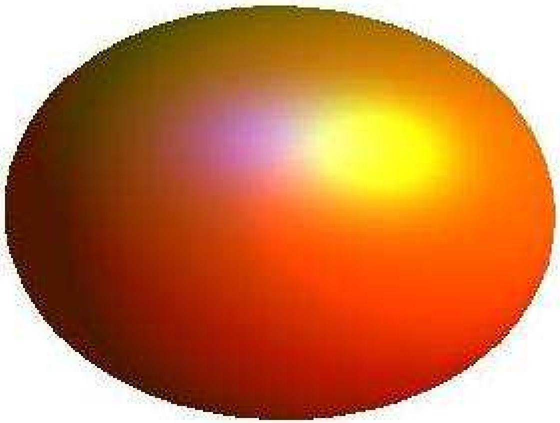
$$\mathcal{E} = \frac{16mn}{a^3} \frac{(r/a)^2+1}{[4(\rho/a)^2(m^2-n^2)+n^2((r/a)^2+1)^2]^2}; \qquad Q = mn$$



n=1, m=4



n=2, m=2



n=4, m=1

Summary

- Usual Skyrme model supports variety of different solitons which do not saturate the topological bound

- Construction of Skyrme crystal allows us to approach the self-duality bound

- Self-dual reduction of Skyrme models is possible

- New exact self-dual Skyrme model is constructed

Thank you!