Formation of equation of state at early stage of ultrarelativistic heavy-ion collisions

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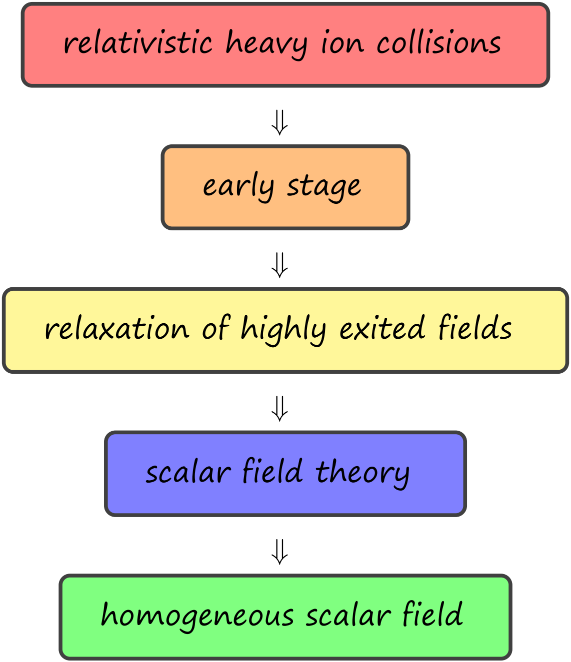
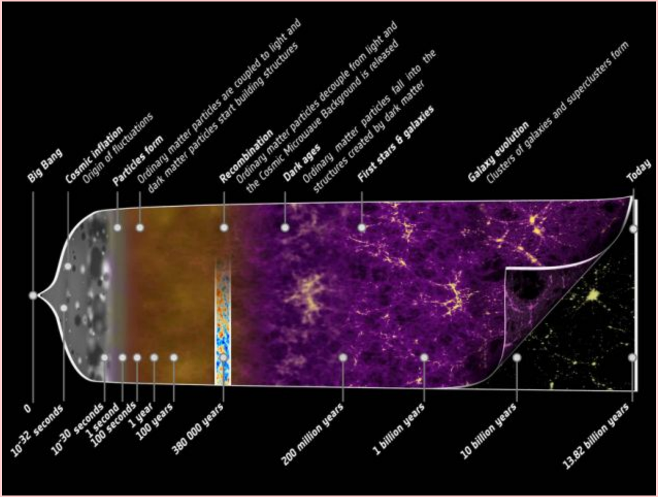
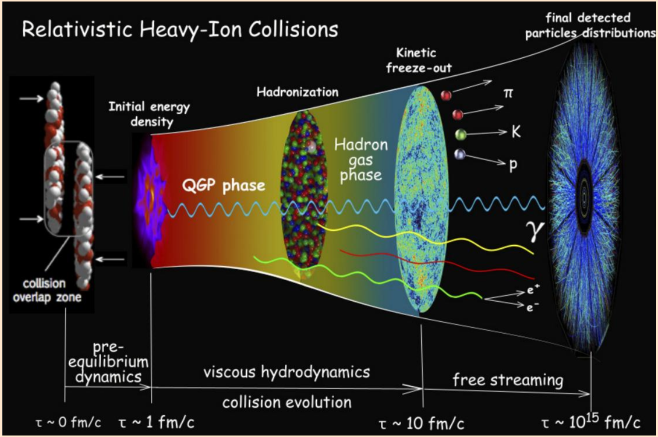

relativistic heavy ion collisions
early stage
relaxation of highly exited fields
scalar field theory
homogeneous scalar field

Big Bang



Big Bang
Cosmic inflation
Origin of fluctuations
Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwave Background is released
Dark ages
Ordinary matter particles fall into the structures created by dark matter
First stars & galaxies
Galaxy evolution
Clusters of galaxies and superclusters form
Today
0
10^{-32} seconds
10^{-30} seconds
1 second
100 seconds
1 year
100 years
380 000 years
200 million years
1 billion years
10 billion years
13.82 billion years

Little Bang



arxiv:1304.3634. Ulrich Heinz "Towards the Little Bang Standard Model"

Keldysh and S-matrix technique

S-matrix approach – The transition amplitude from ``in'' to ``out'' state

$$M = \langle out|in\rangle, \; \sigma \sim |M|^2$$

-Adiabatic switching of the interactions.

$$|in\rangle = \hat{U}(0,-\infty)|in\rangle_0, \quad |out\rangle = \hat{U}^\dagger(\infty,0)|out\rangle_0$$

$$M = \langle out|in\rangle = {}_0\langle out|\hat{U}(\infty,-\infty)|in\rangle_0$$

– But S-matrix for a process 2-> 100 ???

Keldysh-Schwinger approach used to analyze the time evolution of the density matrix $\hat{\rho}(t) = \sum_i P_i|\psi_i(t)\rangle\langle\psi_i(t)|$

$$\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0)$$

Any observable is given by

$$\langle A(t)\rangle = tr(\hat{A}\hat{\rho}) = tr(\hat{U}^\dagger(t,t_0)\hat{A}\hat{U}(t,t_0)\hat{\rho}(t_0))$$

Quantum evolution: general case

– For evolution starting at $t = t_0$ and ending at $t = t_1$

$$\langle F(\hat{\varphi})\rangle_{t_1} = tr(F(\hat{\varphi})\hat{\rho}(t_1)) = \int d\xi F(\xi)\langle\xi|U(t_1, t_0)\hat{\rho}(t_0)\hat{U}(t_0, t_1)|\xi\rangle$$

$$= \int d\xi \int d\xi_1 \int d\xi_2 F(\xi)\langle\xi|U(t_1, t_0)|\xi_1\rangle\langle\xi_1|\hat{\rho}(t_0)|\xi_2\rangle\langle\xi_2|\hat{U}(t_0, t_1)|\xi\rangle,$$

with the matrix elements of the evolution operator

$$\langle\xi|\hat{U}(t_1, t_0)|\xi_1\rangle = \int_{\eta_F(t_0)=\xi_1}^{\eta_F(t_1)=\xi} \mathcal{D}\eta_F(t) e^{iS[\eta_F]}$$

$$< \xi_2|\hat{U}(t_0, t_1)|\xi > = \int_{\eta_B(t_0)=\xi_2}^{\eta_B(t_1)=\xi} \mathcal{D}\eta_B(t) e^{-iS[\eta_B]}$$

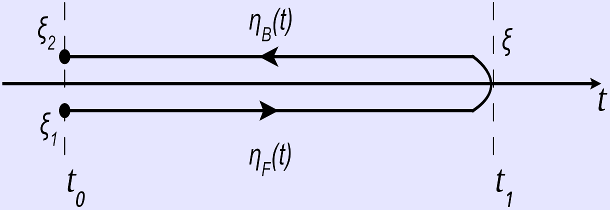
Quantum evolution: general case

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int d\xi \int d\xi_1 \int d\xi_2 \langle \xi_1|\hat{\rho}(t_0)|\xi_2\rangle \times$$

$$\times\, F(\xi) \int_{\eta_F(t_0)=\xi_1}^{\eta_F(t_1)=\xi} \mathcal{D}\eta_F \int_{\eta_B(t_0)=\xi_2}^{\eta_B(t_1)=\xi} \mathcal{D}\eta_B \; e^{iS_K[\eta]},$$

$$S_K[\eta] = S[\eta_F] - S[\eta_B]$$

- The fields η_F and η_B live on the Keldysh contour:



Quantum evolution: general case

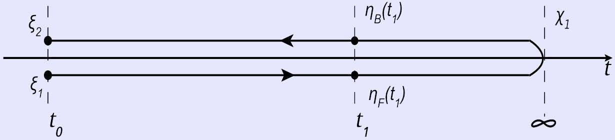
- It is convenient to extend time $t = \infty$

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int d\chi_1 \int d\xi_1 \int d\xi_2 \langle \xi_1|\hat{\rho}(t_0)|\xi_2\rangle$$

$$\times \int_{\eta_F(t_0)=\xi_1}^{\eta_F(\infty)=\chi_1} \mathcal{D}\eta_F \int_{\eta_B(t_0)=\xi_2}^{\eta_B(\infty)=\chi_1} \mathcal{D}\eta_B \; F\left(\frac{\eta_F(t_1)+\eta_B(t_1)}{2}\right) \; e^{iS_K[\eta]}$$

  Only observable F depends on observation time t_1

- The fields η_F and η_B now live on the extended Keldysh contour:



Quantum evolution: general case

Keldysh rotation:

$$\begin{aligned}
\phi_c &= \frac{\eta_F + \eta_B}{2}, \qquad \phi_q = \eta_F - \eta_B \\
\phi_c(t_0) &= \frac{\xi_1 + \xi_2}{2}, \qquad \phi_c(\infty) = \chi_1 \\
\phi_q(t_0) &= \xi_1 - \xi_2, \qquad \phi_q(\infty) = 0
\end{aligned}$$

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int d\chi_1 \int d\xi_1 \int d\xi_2 \langle \xi_1 | \hat{\rho}(t_0) | \xi_2 \rangle$$

$$\times \int_{\phi_c(t_0)=\frac{\xi_1+\xi_2}{2}}^{\phi_c(\infty)=\chi_1} \mathcal{D}\phi_c \int_{\phi_q(t_0)=\xi_1-\xi_2}^{\phi_q(\infty)=0} \mathcal{D}\phi_q \; F(\phi_c(t_1)) \; e^{iS_K[\phi_c,\phi_q]}$$

Quantum evolution: general case

Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\lambda}{4!}\varphi^4 - J\varphi$$

Keldysh action:

$$S_K[\phi_c, \phi_q] = \int_{t_0}^{\infty} dt[\dot{\phi}_c \dot{\phi}_q - \frac{\lambda}{4!}\phi_c \phi_q^3 - \frac{\lambda}{6}\phi_c^3 \phi_q - J\phi_q] =$$

$$\dot{\phi}_c(t_0)(\xi_1 - \xi_2) - \int_{t_0}^{\infty} dt\ (\phi_q[\ddot{\phi}_c + \frac{\lambda}{6}\phi_c^3 + J] - \frac{\lambda}{4!}\phi_c \phi_q^3)$$

$$\frac{1}{\hbar} S_K[\phi_c, \phi_q] = \int_{t_0}^{\infty} dt[\dot{\phi}_c \dot{\phi}_q - \hbar^2 \frac{\lambda}{4!}\phi_c \phi_q^3 - \frac{\lambda}{6}\phi_c^3 \phi_q - J\phi_q]$$

$$\phi_q \to \hbar \phi_q$$

Quantum evolution: general case

Expansion in ϕ_q is a quasiclassical expansion (not in λ)

$$< F(\hat{\varphi}) >_{t_1} = \int d\chi_1 \int d\xi_1 \int d\xi_2 < \xi_1|\hat{\rho}(t_0)|\xi_2 >$$

$$\times \int\limits_{\phi_c(t_0)=\frac{\xi_1+\xi_2}{2}}^{\phi_c(\infty)=\chi_1} \mathcal{D}\phi_c \; F[\phi_c(t_1)] e^{i\dot{\phi}_c(t_0)(\xi_1-\xi_2)}$$

$$\times \int\limits_{\phi_q(t_0)=\xi_1-\xi_2}^{\phi_q(\infty)=0} \mathcal{D}\phi_q \; e^{-i\int\limits_{t_0}^{\infty} dt(\phi_q A[\phi_c] - \frac{\lambda}{4!}\phi_c\phi_q^3)} =$$

$$\int\limits_{\phi_c(t_0)=\frac{\xi_1+\xi_2}{2}}^{\phi_c(\infty)=\chi_1} \mathcal{D}\phi_c \; F[\phi_c(t_1)] \int \frac{d\tilde{p}}{2\pi} e^{i\tilde{p}(\xi_1-\xi_2)} \delta(\tilde{p} - \dot{\phi}_c(t_0)) \; \delta(A[\phi_c])$$

$$e^{-i\frac{\lambda}{4!}\phi_c\phi_q^3} \approx 1 - \frac{\lambda}{4!}\phi_c\phi_q^3$$

Quantum evolution at the leading order in ϕ_q

In the leading order in ϕ_q we have $F[\phi_c(t_1)] = F[\phi_c^0(t_1)]$ where ϕ_c^0 is solution of EoM. That

$$\langle F(\hat{\varphi})\rangle_{t_1}^{LO} = \int \frac{d\tilde{p}}{2\pi} \int d\xi_1 \int d\xi_2 \langle \xi_1|\hat{\rho}(t_0)|\xi_2\rangle e^{i\tilde{p}(\xi_1-\xi_2)} F[\phi_c^0(t_1)]$$

Defining $(\xi_1+\xi_2)/2 = \alpha$ and $\xi_1 - \xi_2 = \beta$ one gets:

$$\begin{aligned}
\langle F(\hat{\varphi})\rangle_{t_1}^{LO} &= \int \frac{d\tilde{p}}{2\pi} \int d\alpha\, f_W(\alpha,\tilde{p},t_0)\ F[\phi_c^0(t_1)] \\
f_W(\alpha,\tilde{p},t_0) &= \int d\beta \langle \alpha + \frac{\beta}{2}|\hat{\rho}(t_0)|\alpha - \frac{\beta}{2}\rangle e^{i\tilde{p}\beta} \\
\phi_c^0(t_0) &= \alpha, \qquad \dot{\phi}_c^0(t_0) = \tilde{p} \\
\ddot{\phi}_c^0 &+ \frac{\lambda}{6}(\phi_c^0)^3 = 0
\end{aligned}$$

Quantum evolution at the leading order in ϕ_q

For spatially inhomogeneous fields

$$\langle F(\hat{\varphi})\rangle_{t_1}^{LO} = \int D\tilde{p}(x) \int D\alpha(x) f_W[\alpha(x), \tilde{p}(x), t_0] F[\phi_c^0(t_1, x)],$$

where $\mathcal{D}\phi(x)$ means the integration over 4-dimensional functions and symbol $D\phi(x)$ – over 3-dimensional ones and initial conditions

$$\Box\phi_c^0 + \frac{\lambda}{6}(\phi_c^0)^3 = 0, \quad \phi_c^0(t_0, x) = \alpha(x), \quad \dot{\phi}_c^0(t_0, x) = \tilde{p}(x).$$

Numerical solution

Observables for consideration

$$T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \left(\frac{1}{2} \partial_\sigma \varphi \partial^\sigma \varphi - \frac{\lambda}{24} \varphi^4 \right).$$

Dynamical interrelation between energy and pressure and possibility of reaching the "hydrodynamic" regime $\varepsilon = 3p$ $[trT^{\mu\nu} = 0]$

In the case of homogeneous field at the classical level

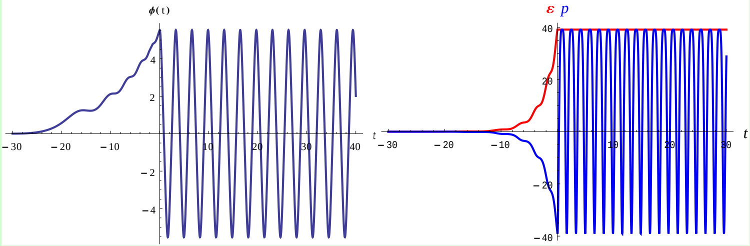
$$\varepsilon_0 = \frac{\dot{\varphi}^2}{2} + \frac{\lambda \varphi^4}{24}, \quad p_0 = \frac{\dot{\varphi}^2}{2} - \frac{\lambda \varphi^4}{24}$$

$$\varphi(t) = \phi_{max}\, cn \left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}} \phi_{max}\, t + C \right)$$

Jacobi Elliptical functions

$$u = \int_0^\phi \frac{dx}{\sqrt{1 - k^2 sin^2(x)}}, \quad cos(\phi) = cn(k^2, u), \quad T = 4K(k^2)$$

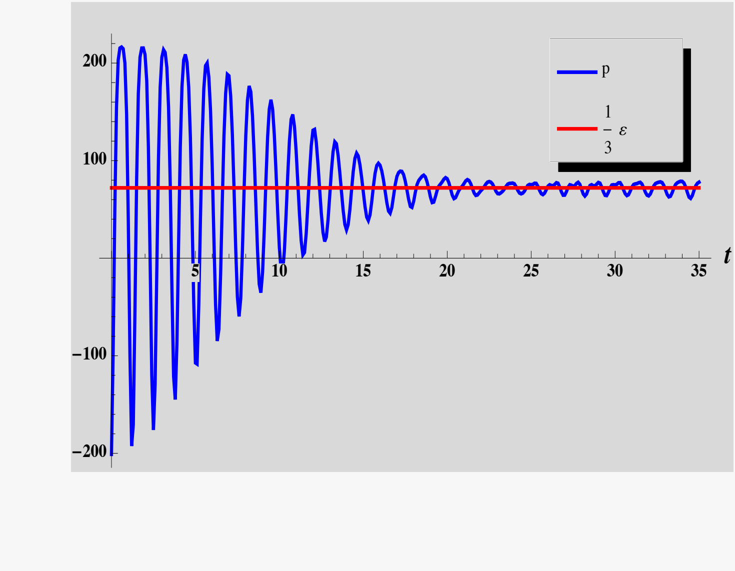
Classical evolution: field, energy and pressure



- Kevin Dusling (Brookhaven), Thomas Epelbaum, Francois Gelis (Saclay, SPhT), Raju Venugopalan (Brookhaven) [Nucl.Phys. A850 (2011) 69-109]
- only numerical
- nonregular derivation (only for $T^{\mu\nu}$ observables)
- without NLO

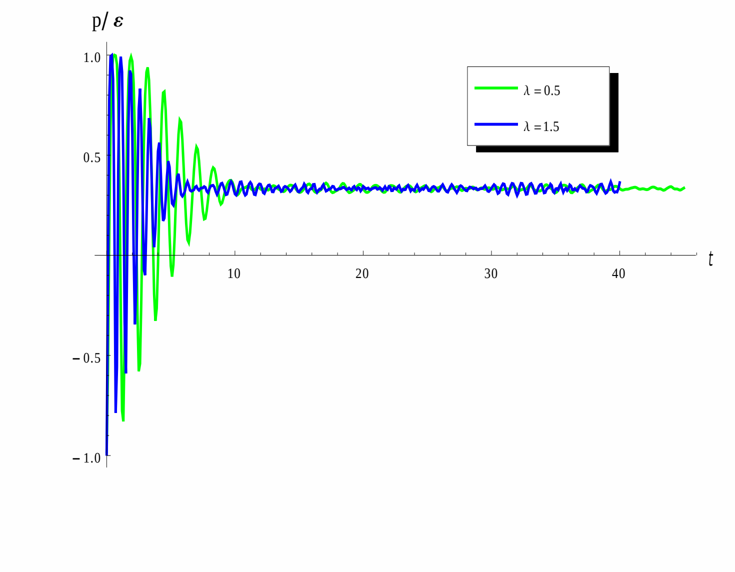
Quantum evolution at LO: pressure relaxation

Averaging over initial conditions with Gaussian Wigner function gives



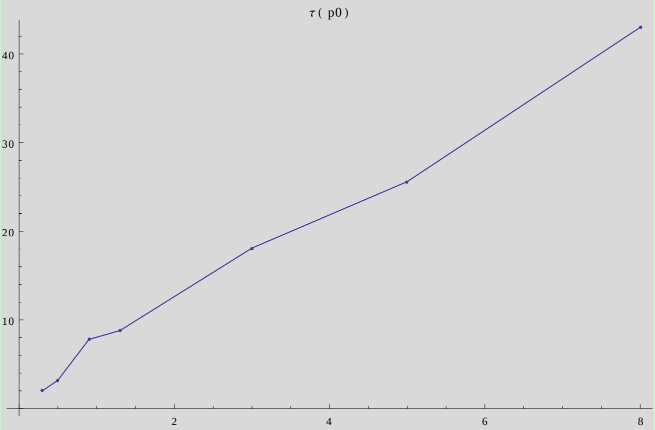
Quantum evolution at LO: pressure relaxation

Pressure relaxation as a function of coupling:



Quantum evolution at LO: pressure relaxation

- Let us define relaxation time τ as a time at which $\left|\frac{\varepsilon-3p}{\varepsilon}\right| = 0.1$



- $\tau(p_0)$ is approximately linear in p_0 in agreement with analytical expression

Quantum evolution at LO: analytical solution

Ansatz for the Wigner function:

$$f_W(\alpha, p, 0) = \frac{1}{\alpha_0 p_0 \pi} e^{-\frac{(\alpha - A)^2}{\alpha_0^2} - \frac{p^2}{p_0^2}}$$

Change variables

$$\varphi(t, \alpha, \tilde{p}) = \phi_{max}(\alpha, \tilde{p})\ cn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}(\alpha, \tilde{p})\ t + C(\alpha, \tilde{p})\right)$$

$$\alpha = \phi(0) = \phi_{max} cn(1/2, C),$$

$$p = \dot{\phi}(0) = -\sqrt{\frac{\lambda}{6}}\phi_{max}^2 sn(1/2, C) dn(1/2, C)$$

$$\int \frac{d\tilde{p}}{2\pi} \int d\alpha \to \int |J|\ d\phi_{max}\ dC,$$

$$|J| = \sqrt{\frac{\lambda}{6}}\phi_{max}^2.$$

Quantum evolution at LO: analytical solution

In the saddle point approximation

$$f_W(\phi_{max}, C, 0) \approx \frac{1}{\alpha_0 p_0 \pi} e^{-\frac{(\phi_{max}-A)^2}{\alpha_0^2} - \frac{C^2 A^4 \lambda}{6 p_0^2}} \qquad (1)$$

valid for $\alpha_0 \ll A$ and $p_0 \ll A^2\sqrt{\lambda/6}$, introducing a Fourier transform

$$cn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max} t + C\right) = \sum_{k=-\infty}^{\infty} u_k e^{\frac{2\pi i k}{T}\left(\sqrt{\frac{\lambda}{6}}\phi_{max} t + C\right)}, \qquad (2)$$

$$u_m = \frac{1}{T}\int_0^T cn\left(\frac{1}{2}; t\right) e^{-imt\frac{2\pi}{T}} dt,$$

one can receive for mean field

$$< \varphi_c >_{LO} = 2A \sum_{k=0}^{\infty} u_k e^{-\frac{6\pi^2 p_0^2}{\lambda A^4 T^2} k^2} e^{-\frac{\alpha_0^2 \pi^2 \lambda}{6 T^2} k^2 t^2} \cos\left(\frac{2A\pi k}{T}\sqrt{\frac{\lambda}{6}} t\right)$$

Quantum evolution at LO: analytical solution

In the same approximation the pressure reads

$$\frac{p_{LO}}{\varepsilon_{LO}} = -8\left(\frac{2\pi}{T}\right)^2 \sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty} k\, l\, u_k u_l\, e^{-\frac{6\pi^2 p_0^2}{\lambda A^4 T^2}(k+l)^2}$$

$$\times \quad e^{-\frac{\alpha_0^2\pi^2\lambda}{6T^2}(k+l)^2 t^2} \cos\left(\frac{2\pi A(k+l)}{T}\sqrt{\frac{\lambda}{6}}t\right) - 1$$

$$\varepsilon_{LO} = \frac{\lambda}{24}A^4 \qquad (3)$$

Let us consider the large time limit $t \to \infty$ ($q = 0$). Consider

$$I(q) = -\left(\frac{2\pi}{T}\right)^2 \sum_{k=-\infty}^{\infty} k(q-k)u_k u_{q-k} = \qquad (4)$$

$$\frac{1}{T}\int_0^T \left(\frac{d\ cn\left(\frac{1}{2};t\right)}{dt}\right)^2 e^{-\frac{2\pi i}{T}qt},$$

Quantum evolution at LO: analytical solution

The corresponding sum can be calculated analytically, $I(0) = 1/3$, so that

$$p_{LO}(t \to \infty) = \varepsilon_{LO}(4I(0) - 1) = \frac{\varepsilon_{LO}}{3}.$$

The next step is $q = 2, I(q) \approx 0.12$

$$p_{LO}(t \to \infty) = \varepsilon_{LO}\left[\frac{1}{3} + 8I(2) e^{-\frac{24\pi^2 p_0^2}{\lambda A^4 T^2}} e^{-\frac{2\alpha_0^2 \pi^2 \lambda}{3T^2} t^2} \cos\left(\frac{4\pi A}{T}\sqrt{\frac{\lambda}{6}} t\right) + \dots\right]$$

"Thermalization time" t_{th} can be estimated as

$$t_{th} \sim \sqrt{\frac{3}{2}} \frac{T}{\pi\alpha_0\sqrt{\lambda}}. \quad (5)$$

Quantum evolution at LO: analytical solution

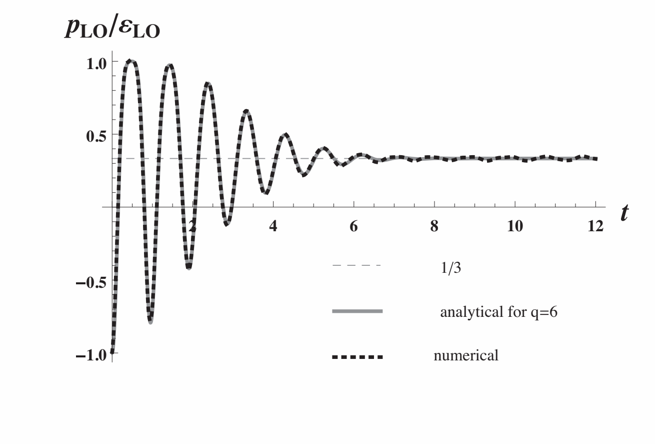


Figure : Pressure relaxation: comparison of numerical result an analytical expression with terms up to $q = 6$ taken into account. The parameter values are $p_0 = 1.5\sqrt{2}$, $\alpha_0 = 1/p_0$, $A = 10$, $\lambda = 0.9$ ($I(4) \approx -0.04$, $I(6) \approx -0.006$)

Quantum evolution at NLO

$$e^{-i\frac{\lambda}{4!}\int_{t_0}^{\infty} dt'\phi_c\phi_q^3} \approx 1 - \frac{i\lambda}{4!}\int_{t_0}^{\infty} dt'\phi_c(t')\phi_q^3(t') + O(\phi_q^6) \quad (6)$$

Using procedure described above and relations

$$\frac{\delta}{\delta J(t)} e^{iS_K[\phi_c,\phi_q]} = i\phi_q(t)e^{iS_K[\phi_c,\phi_q]} \quad (7)$$

and

$$\frac{\delta\phi(t_1)}{\delta J(t')} = 0 \;\; if \;\; t' \geq t_1, \quad (8)$$

which follows from causality, where ϕ is the solution of EoM with nonzero J

$$\ddot{\phi} + \frac{\lambda}{6}\phi^3 = J \quad (9)$$

Quantum evolution at NLO

One can obtain for NLO correction

$$\langle F(\hat{\varphi})\rangle_{t_1}^{NLO} = \int \frac{d\tilde{p}}{2\pi} \int d\alpha \; f_W(\alpha, \tilde{p}, t_0) \times \left(F[\phi_c^0(t_1)] + \frac{\lambda}{4!} \int_{t_0}^{t_1} dt' \phi(t') \frac{\delta^3 F[\phi(t_1)]}{\delta J^3(t')}\bigg|_{J=0} \right).$$

Where

$$\frac{\delta^3 F[\phi(t_1)]}{\delta J^3(t')} = \frac{dF}{d\phi}\Phi_3(t_1, t') + 3\frac{d^2F}{d\phi^2}\Phi_2(t_1, t')\Phi_1(t_1, t') + \frac{d^3F}{d\phi^3}\Phi_1(t_1, t')^3,$$

with

$$\frac{\delta\phi(t_1)}{\delta J(t')} = \Phi_1(t_1, t'), \; \frac{\delta^2\phi(t_1)}{\delta J^2(t')} = \Phi_2(t_1, t'), \; \frac{\delta^3\phi(t_1)}{\delta J^3(t')} = \Phi_3(t_1, t').$$

Quantum evolution at NLO

Variation of equation of motion gives

$$\frac{\delta^3}{\delta J^3(t')}(\ddot{\phi} + \frac{\lambda}{6}\phi^3 + J)_{t_1} = 0$$
$$\hat{L}_t = \partial_t^2 + \frac{\lambda}{2}\phi^2(t)$$

evolution equations for field variations:

$$\begin{aligned}
\hat{L}_{t_1}\Phi_1(t_1,t') &= \delta(t_1 - t'),\\
\hat{L}_{t_1}\Phi_2(t_1,t') &= -\lambda\phi(t_1)\Phi_1^2(t_1,t'),\\
\hat{L}_{t_1}\Phi_3(t_1,t') &= -\lambda\Phi_1^3(t_1,t') - 3\lambda\phi(t_1)\Phi_1(t_1,t')\Phi_2(t_1,t').
\end{aligned}$$

Quantum evolution at NLO

2-point correlation functions in the Keldysh formalism:

$$< \phi_c(t_1)\phi_c(t_2) >= \int \frac{d\tilde{p}}{2\pi} \int d\alpha f_W(\alpha, \tilde{p}, t_0)\phi(t_1)\phi(t_2),$$

$$< \phi_c(t_1)\phi_q(t_2) >= -i \int \frac{d\tilde{p}}{2\pi} \int d\alpha f_W(\alpha, \tilde{p}, t_0) \frac{\delta\phi(t_1)}{\delta J(t_2)}$$

$$= -i \int \frac{d\tilde{p}}{2\pi} \int d\alpha f_W(\alpha, \tilde{p}, t_0)\Phi_1(t_1, t_2),$$

$$< \phi_q(t_1)\phi_q(t_2) >= 0 \text{ by construction.}$$

In order to find $\Phi_1(t, t')$ one can note that

$$\partial_t \; [\ddot{\phi}_c^0(t) + \frac{\lambda}{6}(\phi_c^0(t))^3] = 0 \text{gives}$$

$$[\partial_t^2 + \frac{\lambda}{2}(\phi_c^0(t))^2]\dot{\phi}_c^0(t) = \hat{L}_t\dot{\phi}_c^0(t) = 0.$$

It means that $\dot{\phi}_c^0(t) \equiv f_1(t)$ is the first particular solution of equation on $\Phi_1(t, t')$ (or Green function $G(t,t')$)

Quantum evolution at NLO

$$\Phi_1(t,t') = G(t,t') = \theta(t-t')[f_1(t')f_2(t) - f_2(t')f_1(t)].$$

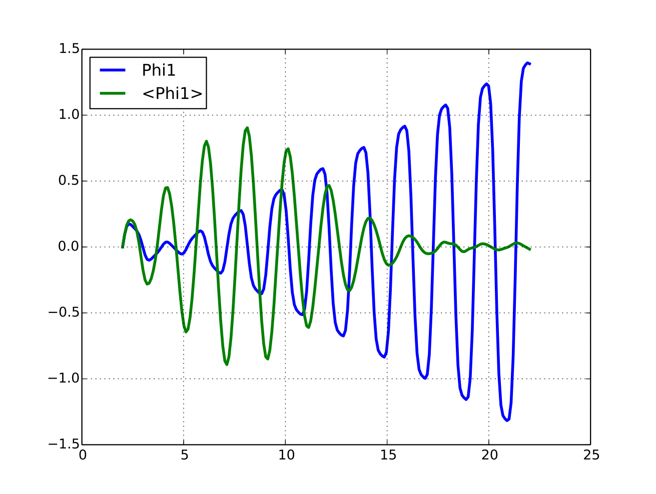
Or in other terms

$$\Phi_1(t,t') = \frac{6}{\lambda\phi_{max}^4}[\dot{\phi}_c^0(t')\phi_c^0(t) - \dot{\phi}_c^0(t)\phi_c^0(t') + \dot{\phi}_c^0(t)\dot{\phi}_c^0(t')(t-t')]\theta(t-t')$$

$$\phi_c^0(t) = \phi_{max} cn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\ t + C\right)$$

$$\dot{\phi}_c^0(t) = -\sqrt{\frac{\lambda}{6}}\phi_{max}^2 sn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\ t + C\right) \cdot dn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\ t + C\right)$$

Quantum evolution at NLO



Let formulate once again the main results obtained:

1. The systematic procedure of computing quantum corrections in the framework of Keldysh formalism is described.
2. Analytical expressions for pressure relaxation in the scalar field model are presented.
3. Explicit equations for the next-to-leading order corrections are written down.