Гидростатическое равновесие звезд в отсутствие локальной электронейтральности

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Семинар «Адронная материя при экстремальных условиях»
БЛТФ ОИЯИ, Дубна, 14 марта 2018

I. Introduction

ELECTROMAGNETIC FORCE
vs.
GRAVITATIONAL FORCE:

$$\lambda_{\rm G} = \frac{e^2}{G m_{\rm u}^2} \approx 1.25 \times 10^{36}$$

Complete screening → gravity should play the dominant role in determining the stellar structure

Hydrostatic equilibrium equations are usually supplemented by:

LOCAL ELECTRONEUTRALITY CONSTRAINT

I. Introduction

LOCAL ELECTRONEUTRALITY CONSTRAINT (LEC)

Similar to

$$A_\varepsilon : \varepsilon \frac{du}{dx} = -u + x, \quad 0 \leqslant x \leqslant 1; \quad u(0) = 1.$$

$$A_0 : 0 = -u + x$$

= связь, приводящая к потере начального условия

An example from:

Васильева Аделаида Борисовна
Бутузов Валентин Федорович

АСИМПТОТИЧЕСКИЕ МЕТОДЫ
В ТЕОРИИ СИНГУЛЯРНЫХ ВОЗМУЩЕНИЙ

How fundamental is LEC?

- **ABSOLUTELY FUNDAMENTAL FOR ISOLATED SYSTEMS (thermodynamics)**
- **NOT FUNDAMENTAL FOR SYSTEMS IN EXTERNAL FIELDS (multi-component Gibbs' condition)**

I. Introduction

chemical potential:

$\mu = \partial E / \partial N$

Systems in external fields obey

Gibbs' CONDITION

$$\mu + V(x) = \text{const}$$

Основные уравнения ГР
A_ε типа следуют из
этих уравненийй

For *n*-component fluid

$$\mu_a + V_a(x) = \text{const} \quad (a = 1, 2, \dots, n)$$

O. Klein (1949)
Kodama, Yamada (1972)
Olson and Bailyn (1975)
} **in GRT**

Multi-component Gibbs' condition is INCOMPARTIBLE WITH LEC

I. Introduction

As early as 1924, Rosseland in a paper recommended for publication by Sir Arthur Eddington showed that in thermodynamic equilibrium in stars,

within A_ε problem,

$$\rho_{\text{charge}} \sim \rho_{\text{total}} / \lambda_G \sim 10^{-36} \rho_{\text{total}}$$

The approximate character of LEC (= A_0)

But for a particular, regular solution only

Similar results:

S. B. Pikel'ner (1961)

E. Olson and M. Bailyn (1978)

L. Neslusan (2001)

M. Rotondo et al. (2011)

R. Belvedere et al. (2012)

I. Introduction

Up to date astrophysicists are discussing particular, regular solutions of the A_ε problem.

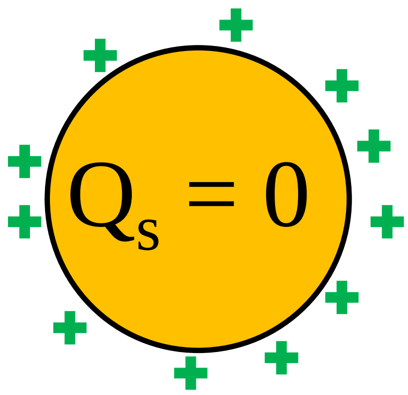
The problem is to construct the general solution to the A_ε problem

SOME OF THE PROPERTIES OF THE GENERAL SOLUTION CAN BE FORESEEN ON THE BASIS OF THE FOLLOWING SIMPLE ARGUMENTS:

(TWO HINTS)

I. Introduction

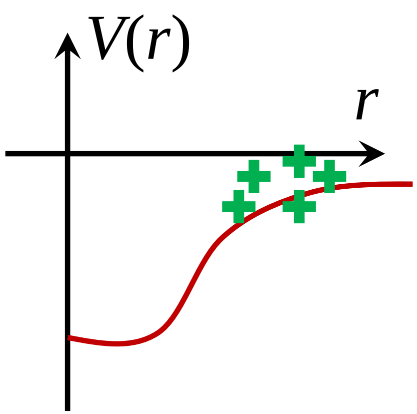
Ionosphere



$$\frac{e^2 \Delta N_p^{\,2}}{R_s} <> \frac{G M_s \Delta N_p m_p}{R_s}$$

$$\Rightarrow \Delta N_p \le \frac{G M_s m_p}{e^2} \sim 10^{21} \frac{M_s}{M_\odot}.$$

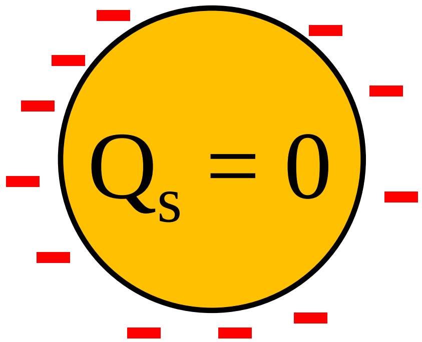
$$0 < Q_s < 150\,\mathrm{C}.$$



Expected to follow from the general solution

I. Introduction

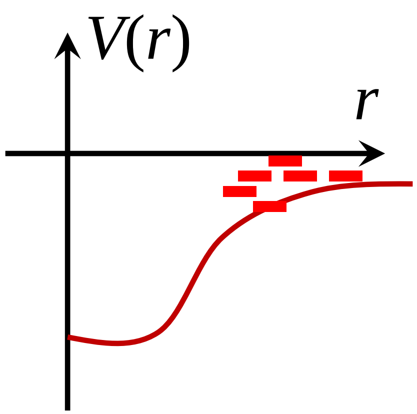
Electrosphere



$$\frac{e^2 \Delta N_e^{\ 2}}{R_s} <> \frac{G M_s \Delta N_e m_e}{R_s}$$

$$\Rightarrow \Delta N_e \leq \frac{G M_s m_e}{e^2} \sim 5 \times 10^{17} \frac{M_s}{M_\odot}.$$

$$-0.1\,\mathrm{C} < Q_s < 150\,\mathrm{C}.$$



EXPECTED TO FOLLOW FROM THE GENERAL SOLUTION

I. Introduction

The general solution is not regular in the gravitational constant G at G=0, as indicated by

- **the Poincare theorem on analyticity and**
- **Dyson's argument**

↑

EXPECTED TO FOLLOW FROM THE GENERAL SOLUTION

I. Introduction

Content:

I. Introduction

II. Regular solution to the unconstrained hydrostatic equilibrium equations
- A. Two-fluid model
- B. Two-fluid model with equal polytropic indices
- C. A power-series expansion near the center
- D. A power-series expansion in G
- E. Global stellar parameters

III. Electrosphere
- A. A polytropic model
- B. Electrically neutral stars

a non-singular problem, briefly

IV. Exactly solvable model
- A. General properties of solutions
  1. Poincaré theorem on analyticity
  2. Dyson's argument
- B. Two-fluid model with unit polytropic indices
  1. Closer inspection of analyticity in G
  2. Electro- and ionospheres
  3. Charge-mass-radius relation

I. Introduction

Content:

V. General solution to the unconstrained hydrostatic equilibrium equations

A. Irregular component in the WKB approximation
B. Correction to the irregular component
C. General solution

VI. Conclusions

II. Regular solution to the unconstrained hydrostatic equilibrium equations

A. Two-fluid model

Multi-component Gibbs' condition:

$$
\begin{aligned}
\mu_{\rm i} + m_{\rm i}\varphi_{\rm G} + Ze\varphi_{\rm E} &= \text{const}, \\
\mu_{\rm e} + m_{\rm e}\varphi_{\rm G} - e\varphi_{\rm E} &= \text{const},
\end{aligned}
$$

i = ions, $m_i = Am_u$, $m_u = 931$ MeV

e = electrons , $m_e = 0.51$ MeV

φ_E = electro-static potential

φ_G = gravitational potential

chemical potential:

$\mu = \partial E/\partial N$

$$
\begin{aligned}
\triangle\varphi_{\rm G} &= 4\pi G\rho_{\rm m}, \\
\triangle\varphi_{\rm E} &= -4\pi\rho_{\rm e},
\end{aligned}
$$

$$
\begin{aligned}
\rho_{\rm m} &= m_{\rm i}n_{\rm i} + m_{\rm e}n_{\rm e}, \\
\rho_{\rm e} &= Zen_{\rm i} - en_{\rm e}.
\end{aligned}
$$

$$
\begin{aligned}
\triangle_r(Z\mu_{\rm e} + \mu_{\rm i}) &= -4\pi G(m_{\rm i}+Zm_{\rm e})\rho_{\rm m}, \\
\triangle_r(m_{\rm i}\mu_{\rm e} - m_{\rm e}\mu_{\rm i}) &= -4\pi e(m_{\rm i}+Zm_{\rm e})\rho_{\rm e}.
\end{aligned}
$$

II. Regular solution to the unconstrained hydrostatic equilibrium equations

В ТЕОРИИ ПОЛИТРОП:

- **Pressure** $$P_{\rm k} = K_{\rm k} n_{\rm k}^{1+1/\eta_{\rm k}}, \quad {\rm k} = ({\rm i}, {\rm e})$$
- **Density** $$n_{\rm k} \equiv n_{\rm k0} \theta_{\rm k}^{\eta_{\rm k}}$$
- **Chemical potential** $$\mu_k = \mu_{k0} \theta_k,$$ $$\mu_{k0} = K_k (1+\eta_k) n_{k0}^{1/\eta_k}.$$
- **Dimensionless coordinate x:**

$$r = r_0 x, \quad r_0^2 = \frac{Z \mu_{\rm e0}}{4\pi G m_{\rm i} (m_{\rm i} + Z m_{\rm e}) n_{\rm i0}}$$

II. Regular solution to the unconstrained hydrostatic equilibrium equations

ОСНОВНАЯ СИСТЕМА УРАВНЕНИЙ

$$\begin{aligned} \Delta_x(\theta_\mathrm{e} + \Lambda_\mathrm{i}\theta_\mathrm{i}) &= -(\theta_\mathrm{i}^{\eta_\mathrm{i}} + \Lambda_\mathrm{m}\Lambda_\mathrm{e}\theta_\mathrm{e}^{\eta_\mathrm{e}}) \\ \Delta_x(\theta_\mathrm{e} - \Lambda_\mathrm{m}\Lambda_\mathrm{i}\theta_\mathrm{i}) &= -\Lambda_\mathrm{G}(\theta_\mathrm{i}^{\eta_\mathrm{i}} - \Lambda_\mathrm{e}\theta_\mathrm{e}^{\eta_\mathrm{e}}) \end{aligned}$$

с начальными условиями

$$\theta_\mathrm{k}(0) = 1, \qquad \theta'_\mathrm{k}(0) = 0.$$

и параметрами

$$\Lambda_\mathrm{e} = \frac{n_\mathrm{e0}}{Z n_\mathrm{i0}} \approx 1 \quad (= \mathbf{LEC}) \qquad \Lambda_\mathrm{m} = \frac{Z m_\mathrm{e}}{m_\mathrm{i}}$$

$$\Lambda_\mathrm{i} = \frac{\mu_\mathrm{i0}}{Z\mu_\mathrm{e0}} \approx \frac{(1+\eta_\mathrm{i})P_\mathrm{i0}}{(1+\eta_\mathrm{e})P_\mathrm{e0}} \ll 1 \qquad \Lambda_\mathrm{G} = \frac{Z^2 e^2}{G m_\mathrm{i}^2} = \left(\frac{Z}{A}\right)^2 \lambda_\mathrm{G}.$$

II. Regular solution to the unconstrained hydrostatic equilibrium equations

ОСНОВНАЯ СИСТЕМА УРАВНЕНИЙ

$$\triangle_x(\theta_{\rm e} + \Lambda_{\rm i}\theta_{\rm i}) = -(\theta_{\rm i}^{\eta_{\rm i}} + \Lambda_{\rm m}\Lambda_{\rm e}\theta_{\rm e}^{\eta_{\rm e}})$$

$$\triangle_x(\theta_{\rm e} - \Lambda_{\rm m}\Lambda_{\rm i}\theta_{\rm i}) = -\Lambda_{\rm G}(\theta_{\rm i}^{\eta_{\rm i}} - \Lambda_{\rm e}\theta_{\rm e}^{\eta_{\rm e}})$$

$$\theta_{\rm k}(0) = 1, \qquad \theta'_{\rm k}(0) = 0.$$

$$0 \le x \le x_b.$$

1. Задача Коши для системы ОДУ 4-го порядка.
2. Сингулярно возмущенная система, $\varepsilon = 1/\Lambda_G << 1$.
3. Тихоновская система ОДУ:

$$A_\varepsilon \begin{cases} \varepsilon \dfrac{dz}{dx} = F(z, y, x, \varepsilon), \quad \dfrac{dy}{dx} = f(z, y, x, \varepsilon), \\ 0 \leqslant x \leqslant x_0, \\ z(0) = z^0, \quad y(0) = y^0. \end{cases}$$

II. Regular solution to the unconstrained hydrostatic equilibrium equations

B. Two-fluid model with equal polytropic indices

A particular solution for $\theta_i(x) = \theta_e(x) = \theta(\tilde{x})$ and $\eta_i = \eta_e = \eta$ with

$$\tilde{x} = x\sqrt{\frac{1+\Lambda_m\Lambda_e}{1+\Lambda_i}},$$

R. Emden

can be found from the Lane-Emden equation

$$\triangle_{\tilde{x}}\theta(\tilde{x}) = -\theta^{\eta}(\tilde{x})$$

Free parameters Λ_i and Λ_e become functions of Λ_G

$$\Lambda_i = \frac{K_i}{K_e\Lambda_e^{1/\eta}Z^{1+1/\eta}}. \qquad \Lambda_e = \Lambda_e^{reg} = \frac{\Lambda_G(1+\Lambda_i) - (1-\Lambda_m\Lambda_i)}{\Lambda_G(1+\Lambda_i) + \Lambda_m(1-\Lambda_m\Lambda_i)}.$$

NB: Λ_e measures deviations from LEC

The limit $\Lambda_G \rightarrow \infty$ is smooth, the solution is regular.

II. Regular solution to the unconstrained hydrostatic equilibrium equations

The limit $\Lambda_G \to \infty$ is smooth, the solution is regular.

Similar to $A_\varepsilon : \varepsilon \dfrac{du}{dx} = -u + x, \quad 0 \leqslant x \leqslant 1; \quad u(0) = 1.$

$A_0 : 0 = -u + x$ **= связь, приводящая к потере начального условия**

В задаче A_{reg} условие регулярности приводит к $\Lambda_e = \Lambda_e^{reg}(\Lambda_G)$ и, соответственно, также к

потере начального условия.

При $\Lambda_e = \Lambda_e^{reg}(\Lambda_G)$ пограничная функция = 0, т.е. регулярное решение является точным

$$x(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \left(x_k(t) + \Pi_k(\tau) \right).$$

$x_k(t)$: **Taylor**

$\Pi_k(\tau) \to$ **0**: **Laurent**

$\tau = t / \varepsilon,$

$\varepsilon = 1 / \Lambda_G \ll 1$

II. Regular solution to the unconstrained hydrostatic equilibrium equations

C. A power-series expansion near the center

$$\theta_{\mathrm{k}} = 1 + \sum_{p=1}^{\infty} \beta_{\mathrm{k}p} x^{2p}, \quad \mathrm{k} = (\mathrm{i}, \mathrm{e}).$$

The zero-order term of the expansion gives

$$\begin{aligned} 6(\beta_{\mathrm{e}1} + \Lambda_{\mathrm{i}}\beta_{\mathrm{i}1}) &= -(1 + \Lambda_{\mathrm{m}}\Lambda_{\mathrm{e}}), \\ 6(\beta_{\mathrm{e}1} - \Lambda_{\mathrm{m}}\Lambda_{\mathrm{i}}\beta_{\mathrm{i}1}) &= -\Lambda_{\mathrm{G}}(1 - \Lambda_{\mathrm{e}}). \end{aligned}$$

The condition of boundness to have a regular solution:

$$1 - \Lambda_e = \frac{\alpha_e}{\Lambda_G}, \quad \alpha_e = O(1).$$

β_{e1} and β_{i1} are then fixed in terms of α_e, but α_e is not fixed (!)

II. Regular solution to the unconstrained hydrostatic equilibrium equations

The second order terms require

$$20(\beta_{e2} + \Lambda_i \beta_{i2}) = -(\beta_{i1}\eta_i + \Lambda_m \Lambda_e \beta_{e1} \eta_e),$$
$$20(\beta_{e2} - \Lambda_m \Lambda_i \beta_{i2}) = -\Lambda_G(\beta_{i1}\eta_i - \Lambda_e \beta_{e1} \eta_e).$$

A new constraint discovered

$$\beta_{i1}\eta_{i1} = \beta_{e1}\eta_{e1} + \frac{\gamma_1}{\Lambda_G}$$

which allows to fix the lowest order parametrs

$$\frac{\beta_{e1}}{\eta_i} = \frac{\beta_{i1}}{\eta_e} = -\frac{1+\Lambda_m}{6(\eta_i + \Lambda_i \eta_e)},$$
$$\alpha_e = (\eta_i - \Lambda_m \Lambda_i \eta_e)\frac{1+\Lambda_m}{\eta_i + \Lambda_i \eta_e}.$$

Finally, we construct $\Lambda_e = \Lambda_e^{reg}(\Lambda_G)$

II. Regular solution to the unconstrained hydrostatic equilibrium equations

D. A power-series expansion in G

$$\theta_{\mathrm{k}} = \theta_{\mathrm{k}0} + \theta_{\mathrm{k}1}\Lambda_{\mathrm{G}}^{-1} + O(\Lambda_{\mathrm{G}}^{-2}),$$
$$\Lambda_{\mathrm{e}} = \Lambda_{\mathrm{e}0} + \Lambda_{\mathrm{e}1}\Lambda_{\mathrm{G}}^{-1} + O(\Lambda_{\mathrm{G}}^{-2}).$$

The initial conditions $\theta_{\mathrm{e}0}(0) = \theta_{\mathrm{i}0}(0) = 1$ give LEC

$$\theta_{\mathrm{i}0}^{\eta_{\mathrm{i}}}(x) \equiv \theta_{\mathrm{e}0}^{\eta_{\mathrm{e}}}(x), \quad \Lambda_{\mathrm{e}0} = 1.$$

1st order:
$$\triangle_x(\theta_{\mathrm{e}0} + \Lambda_{\mathrm{i}}\theta_{\mathrm{i}0}) = -(1+\Lambda_{\mathrm{m}})\theta_{\mathrm{e}0}^{\eta_{\mathrm{e}}} = -(1+\Lambda_{\mathrm{m}})\theta_{\mathrm{i}0}^{\eta_{\mathrm{i}}},$$

2nd order:
$$\triangle_x(\theta_{\mathrm{e}1} + \Lambda_{\mathrm{i}}\theta_{\mathrm{i}1}) = -\theta_{\mathrm{e}0}^{\eta_{\mathrm{e}}}\left(\eta_{\mathrm{i}}\frac{\theta_{\mathrm{i}1}}{\theta_{\mathrm{i}0}} + \Lambda_{\mathrm{m}}\eta_{\mathrm{e}}\frac{\theta_{\mathrm{e}1}}{\theta_{\mathrm{e}0}} + \Lambda_{\mathrm{m}}\Lambda_{\mathrm{e}1}\right),$$
$$\triangle_x(\theta_{\mathrm{e}0} - \Lambda_{\mathrm{m}}\Lambda_{\mathrm{i}}\theta_{\mathrm{i}0}) = -\theta_{\mathrm{e}0}^{\eta_{\mathrm{e}}}\left(\eta_{\mathrm{i}}\frac{\theta_{\mathrm{i}1}}{\theta_{\mathrm{i}0}} - \eta_{\mathrm{e}}\frac{\theta_{\mathrm{e}1}}{\theta_{\mathrm{e}0}} - \Lambda_{\mathrm{e}1}\right),$$

The region near the surface should be treated separately

II. Regular solution to the unconstrained hydrostatic equilibrium equations

E. Global stellar parameters

The mass of the star takes the form

ρ_m = mass density

$$M_s = \int_0^{R_s} 4\pi r^2 (n_i m_i + n_e m_e) dr$$

$$= -4\pi r_0^3 n_{i0} m_i x^2 \frac{d}{dx} (\theta_e + \Lambda_i \theta_i) \Big|_{x=x_b}$$

The total stellar charge (at the boundary of the l component) is equal to

ρ_e = charge density

$$Q_s = \int_0^{R_s} 4\pi r^2 (Zen_i - en_e) dr$$

$$= -4\pi r_0^3 n_{i0} \frac{Ze}{\Lambda_G} x^2 \frac{d}{dx} (\theta_e - \Lambda_m \Lambda_i \theta_i) \Big|_{x=x_b}$$

II. Regular solution to the unconstrained hydrostatic equilibrium equations

E. Global stellar parameters

The stellar charge can be estimated as

$$Q_s \sim \frac{ZeM_s}{Am_u\Lambda_G} \simeq 10^{21} e \frac{A}{Z} \frac{M_s}{M_\odot}$$

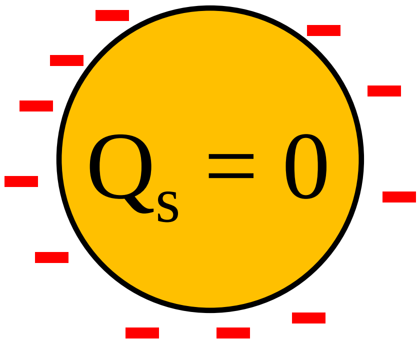
The total uncompensated electric charge of a star of one solar mass **Q = 100 C**, in agreement with

Pikel'ner (1961)
Bally and Harrison (1978)
Neslusan (2001)

One mole of ^{12}C contains $6N_A$ protons with $Q = 6\ 10^5$ C.
Earth: $Q = -(4 - 5.7)\ 10^5$ C.
The solar charge $|Q| < (0.4 - 1)\ 10^{18}$ C, Iorio (2012).

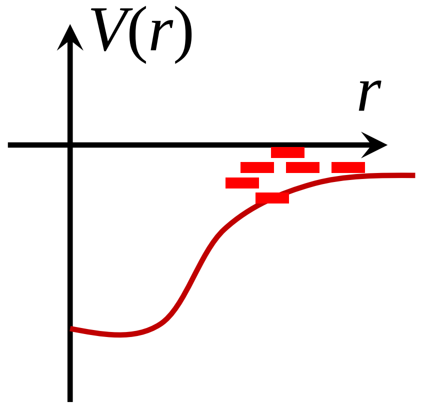
III. Electrosphere

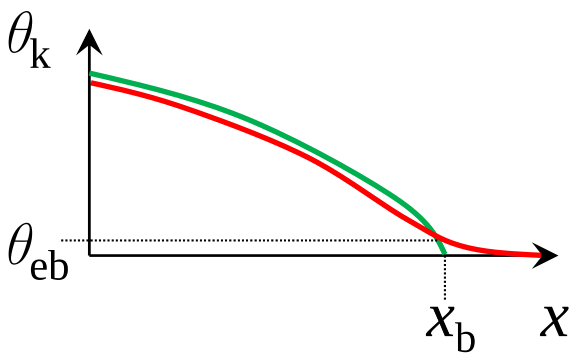
Electron envelopes @: the boundaries of



- solids W ~ 3 eV
- strange stars W ~ 30 MeV
- phases,

in nuclear matter W ~ 30 MeV





III. Electrosphere

A. A polytropic model

Gibbs' condition:

$$\mu_{\rm e} + m_{\rm e}\varphi_{\rm G} - e\varphi_{\rm E} \;=\; {\rm const},$$

Applying Laplacian, we get the Thomas-Fermi equation:

$$\Delta_r \mu_{\rm e} = 4\pi e^2 n_{\rm e}\left(1 - \frac{G m_{\rm e}^2}{e^2}\right)$$

with the boundary conditions:

- μ_e is continuous
- μ_e' also is continuous due to the balance of pressure & EM + gravity:

$$-\frac{1}{n_{\rm e}}\frac{dP_{\rm e}}{dr} = \frac{eQ_{\rm s}}{R_{\rm s}^2} + \frac{G m_{\rm e} M_{\rm s}}{R_{\rm s}^2}$$

III. Electrosphere

A. A polytropic model

Dimensionless units:

$$\lambda_{\rm m} = \frac{m_{\rm e}}{m_{\rm u}}$$

$$r = R_{\rm s} + r_{\rm a} y, \quad r_{\rm a}^2 = \frac{\mu_{\rm a0}}{4\pi e^2 \left(1 - \lambda_{\rm m}^2/\lambda_{\rm G}\right) n_{\rm a0}}$$

The main equation:

$$\frac{d^2\theta_{\rm a}}{dy^2} = \theta_{\rm a}^{\eta_{\rm e}}(y).$$

with the boundary conditions $\theta_{\rm a}(0) = 1$

$$\left.\frac{d\theta_{\rm a}}{dy}\right|_{y=0} = -(q_{\rm s} + \lambda_{\rm m})\sqrt{\frac{P_{\rm un}}{(1+\eta_{\rm e})P_{\rm a0}(\lambda_{\rm G} - \lambda_{\rm m}^2)}}$$

where $P_{\rm un} = \dfrac{GM_{\rm s}^2}{4\pi R_{\rm s}^4}.$

THE PROBLEM IS THUS FORMULATED.

III. Electrosphere

A. A polytropic model

The main equation:

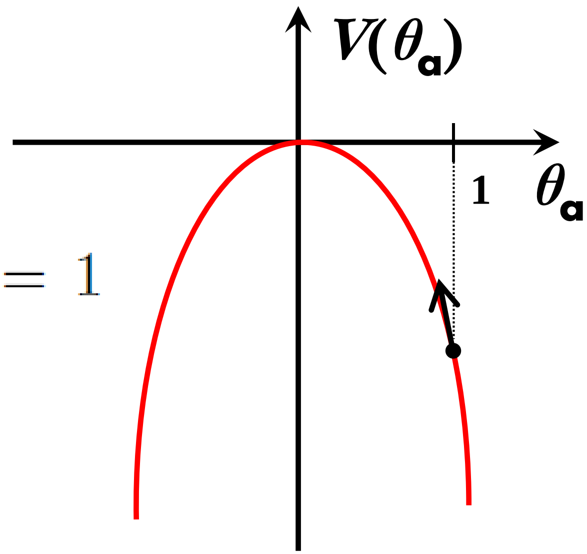
$$\frac{d^2\theta_a}{dy^2} = \theta_a^{\eta_e}(y).$$

looks like the 2nd Newton law, y is time, the potential and the energy:

$$V(\theta_a) = -\frac{|\theta_a|^{1+\eta_e}}{1+\eta_e}$$

$$W = \frac{\theta_a'^2}{2} + V(\theta_a)$$

$$W_0 = \frac{\theta_a'^2(0)}{2} - \frac{1}{1+\eta_e}$$



$\theta_a(0) = 1$

3 TYPES OF SOLUTIONS:

1. $W_0 > 0$ charged stars
2. $W_0 = 0$ neutral stars
3. $W_0 < 0$ *(compressed states)*

III. Electrosphere

A. A polytropic model

Implicit solution:

$$y = y_a - \frac{\theta_a}{\sqrt{2(W_0 - V(\theta_a))}} \times {}_2F_1\left(\frac{1}{2}, 1; \frac{2+\eta_e}{1+\eta_e}; \frac{-V(\theta_a)}{W_0 - V(\theta_a)}\right)$$

Charge of the envelope:

$$Q_e = -4\pi R_s^2 \int_0^{y_a} e n_e r_a dy = 4\pi R_s^2 e n_{a0} r_a \left(\theta_a'(0) + \sqrt{\theta_a'^2(0) - \frac{2}{1+\eta_e}}\right).$$

$Q_e = -\kappa Q_s$, where $0 \leq \kappa \leq \kappa_{max}$. $\kappa_{max} = \frac{1 + \lambda_m / q_s}{1 - \lambda_m^2 / \lambda_G} > 1$

$Q_{tot} = Q_s + Q_e$ $Q_{tot}^{max} \sim -0.05\ \mathrm{C}$

III. Electrosphere

A. A polytropic model

Estimate of thickness of the electron envelope:

$\eta = 3/2$:

$$\frac{r_a}{R_s} \sim \left[\left(\frac{a_B}{R_s} \right)^3 \frac{m_u}{M_s} \lambda_G \right]^{1/5}$$

$$\approx 2.2 \times 10^{-16} \left(\frac{R_\odot}{R_s} \right)^{3/5} \left(\frac{M_\odot}{M_s} \right)^{1/5}$$

$y_a r_a \sim 10^6\, a_B$ for $R_s = R_\odot$, $M_s = M_\odot$.

$\eta = 3$:

$$\frac{r_a}{R_s} \sim \sqrt{\frac{\lambda_G}{\alpha^{3/2}} \frac{m_u}{M_s}} \approx 1.3 \times 10^{-9} \left(\frac{M_\odot}{M_s} \right)^{1/2}$$

$y_a r_a \sim 1$ millimeter for $R_s = 10$ km, $M_s = M_\odot$.

IV. Exactly solvable model

A. General properties of solutions

- **Poincare theorem on analyticity:**

Solutions to ODE systems, when they exist, are analytic functions of the initial coordinates and parameters in the region of analyticity of the ODEs.

Since analytic functions are determined by their singularities, one can talk about THE SINGULARITIES instead of THE REGION OF ANALYTICITY.

- **Dyson's argument (1952)**

provides an effective qualitative criterion for non-analyticity of observables in terms of the system parameters.

IV. Exactly solvable model

A. General properties of solutions

Poincare theorem on analyticity

Дж. Чью, *Аналитическая теория S-матрицы* (Мир, Москва 1968), стр. 11-12:

Формально указанная связь является отражением интуитивно понятной теоремы Пуанкаре, которая, грубо говоря, гласит следующее: если коэффициенты дифференциального уравнения аналитически зависят от некоторой величины, то и решения уравнения будут аналитическими функциями этой величины. Иными словами, Пуанкаре утверждает, что в теориях, основанных на дифференциальных уравнениях, сохраняется любая аналитичность, которую мы вводили в коэффициенты.

IV. Exactly solvable model

A. General properties of solutions

Poincare theorem on analyticity

The normal form, introducing $\pi_k = \theta'_k$, and define the vector

$$\Phi = (\pi_e, \theta_e, \pi_i, \theta_i)$$

and the vector function $\boldsymbol{F}(\mathbf{x}, \boldsymbol{\Phi}, \boldsymbol{\Lambda})$. The main ODE system:

$$\Phi' = \mathfrak{F}(x, \Phi, \boldsymbol{\Lambda})$$

where $\boldsymbol{\Lambda} = (\Lambda_e, \Lambda_i, \Lambda_m, \Lambda_G)$, with the initial conditions:

$$\Phi(x = 0) = (0, 1, 0, 1).$$

NB: $\boldsymbol{F}(\mathrm{x}, \boldsymbol{\Phi}, \boldsymbol{\Lambda})$ is analytic in $\boldsymbol{\Phi}, \boldsymbol{\Lambda} \in \mathbb{C}^8 \backslash \infty$; linear in $\boldsymbol{\Lambda}$;
and singular for $\boldsymbol{\Lambda} = \infty$ (simple pole).

The solutions inherit analyticity and thereby singularities of ODEs. We expect $\Lambda = \infty$ to be a singular point of the general solution.

IV. Exactly solvable model

A. General properties of solutions

Poincare theorem on analyticity

How it works?

Example:

$$A_\varepsilon : \varepsilon \frac{du}{dx} = -u + x, \quad 0 \leqslant x \leqslant 1; \quad u(0) = 1.$$

$$\frac{du}{dx} = \frac{-u+x}{\varepsilon},$$ r.h.s. has a simple pole at $\varepsilon = 0$.

Solution:

$$u_\varepsilon(x) = (1+\varepsilon)\exp(-x/\varepsilon) + x - \varepsilon.$$

The simple pole of ODE turns to an essential singularity of the solution

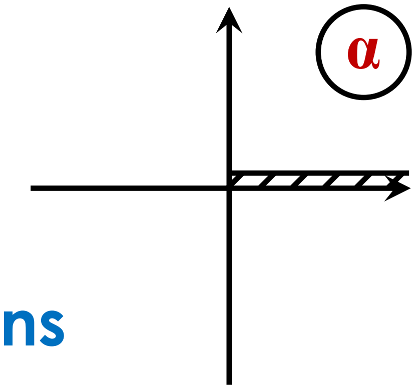
IV. Exactly solvable model

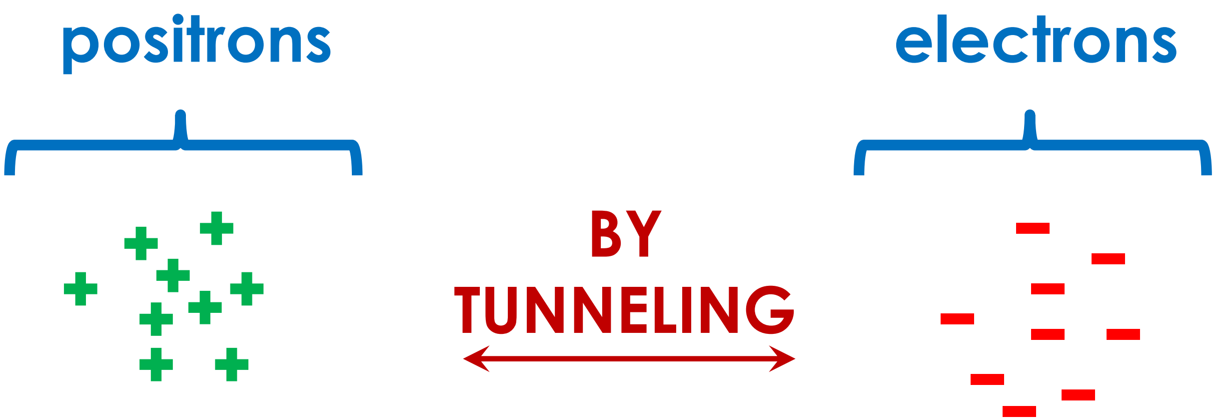
A. General properties of solutions

$\alpha > 0$: $0 < H_{QED}$

$\alpha < 0$: $-\infty < H_{QED}$

Dyson's argument in QED





If $\alpha < 0$: $E_{tot} < E_{vac} = 0$ for $N_{e+e-} \gtrsim 1/\alpha^{3/2}$

$Gm^2 \longrightarrow \alpha$: "Chandrasekhar limit"

VACUUM IS UNSTABLE & α =0 A BRANCH POINT

IV. Exactly solvable model

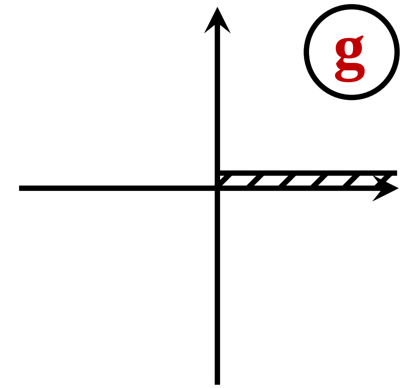
A. General properties of solutions

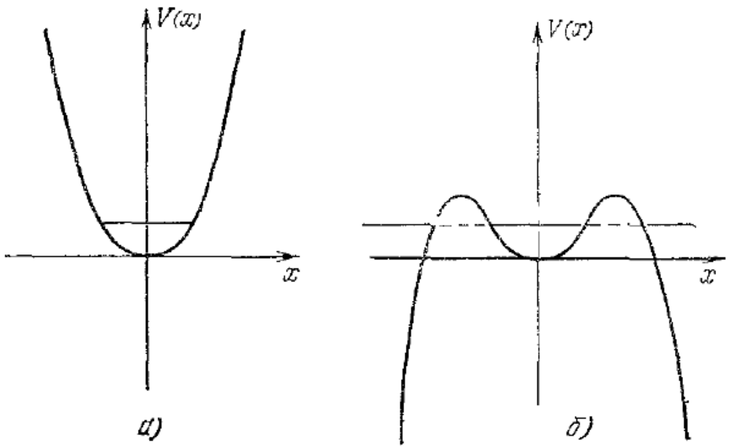
$g > 0$: $0 < H_4$

$g < 0$: $-\infty < H_4$

Dyson's argument

in Quantum mechanics





BOUND ($g > 0$) AND UNBOUND ($g < 0$) HAMILTONIANS ARE QUALITATIVELY DIFFERENT

⬍

$g = 0$ is a branch point of E_0

$$V(x) = m\omega^2 x^2/2 + gx^4$$

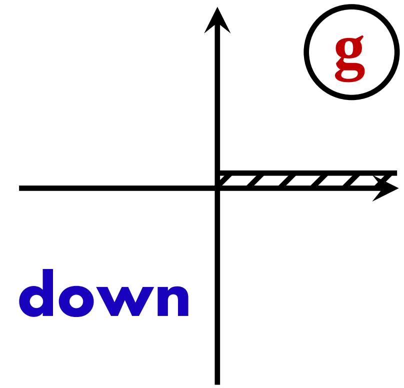
$$E_0 = E_0(g)$$

А. И. Вайнштейн, Препринт ИЯФ СО АН СССР. Новосибирск, (1964)
А. В. Турбинер, УФН 144 35 (1984)

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument

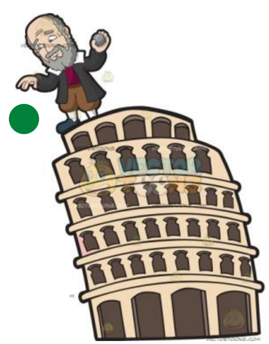


in Classical theory

1. A school level example: Stone thrown down

$$\frac{mv^2}{2} = mgh,$$

$$v = \sqrt{2gh}.$$



g > 0

FINITE (g > 0) AND INFINITE (g < 0) MOTIONS ARE QUALITATIVELY DIFFERENT

⬍

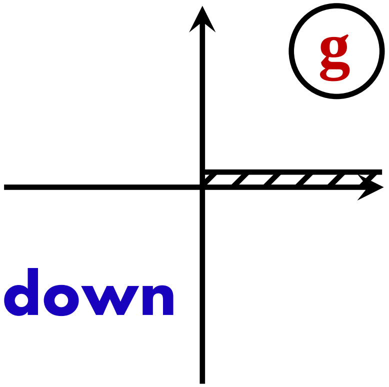
g = 0 is a root branch point

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument

in Classical theory



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$$\frac{mv^2}{2} = mgh,$$

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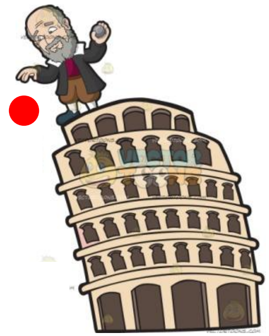


g < 0

FINITE (g > 0) AND INFINITE (g < 0) MOTIONS ARE QUALITATIVELY DIFFERENT

↕

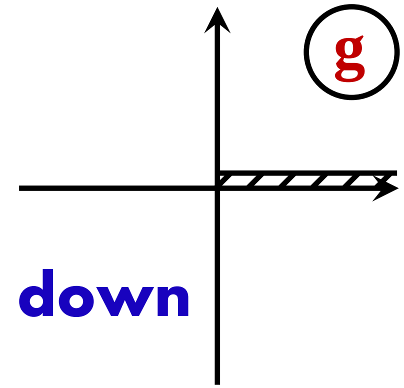
g = 0 is a root branch point

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument

in Classical theory



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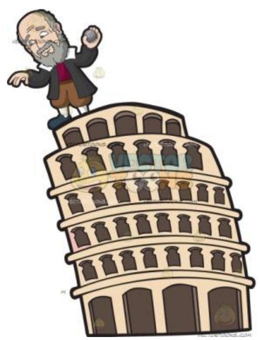
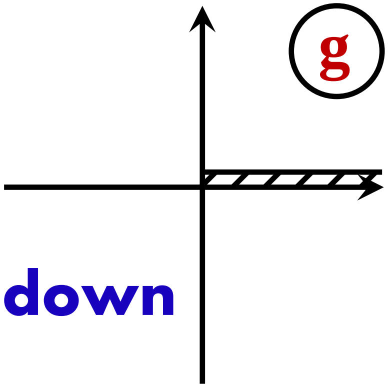


FINITE (g > 0) AND INFINITE (g < 0) MOTIONS ARE QUALITATIVELY DIFFERENT

g = 0 is a root branch point

Dyson's argument

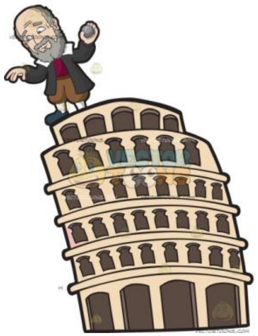
in Classical theory



1. A school level example: **Stone thrown down**

$$\frac{mv^2}{2} = mgh,$$

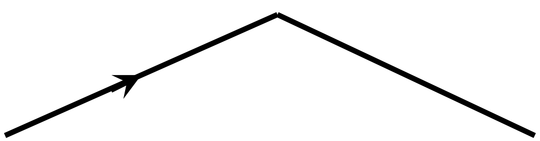
$$v = \sqrt{2gh}.$$



FINITE (g > 0) AND INFINITE (g < 0) MOTIONS ARE QUALITATIVELY DIFFERENT

⇕

g = 0 is a root branch point



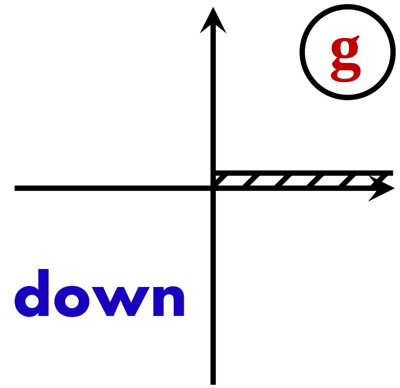
Aristotle

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument

in Classical theory



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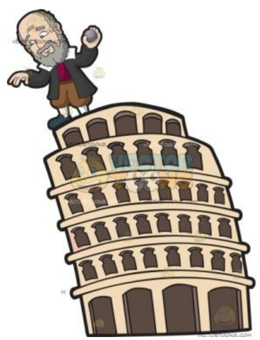
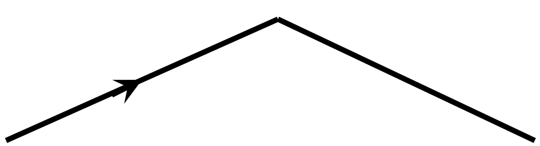


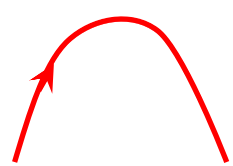
FINITE (g > 0) AND INFINITE (g < 0)
MOTIONS ARE
QUALITATIVELY DIFFERENT

⇕

g = 0 is a root branch point



Aristotle



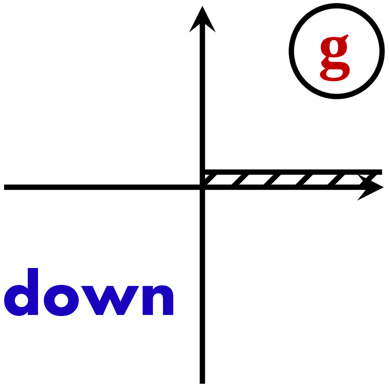
Parabola

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument

in Classical theory



 # 1. A school level example: Stone thrown down

$$\frac{mv^2}{2} = mgh,$$

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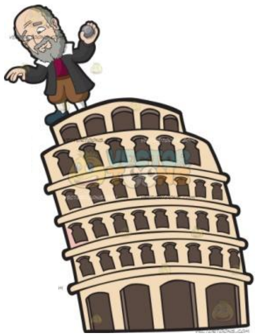
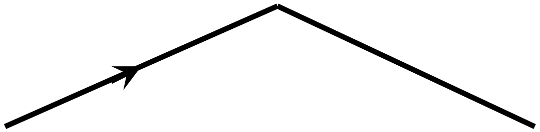


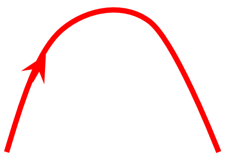
FINITE (g > 0) AND INFINITE (g < 0) MOTIONS ARE QUALITATIVELY DIFFERENT

⬍

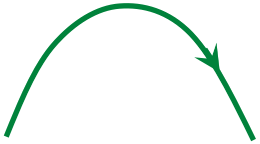
g = 0 is a root branch point



Aristotle



Parabola



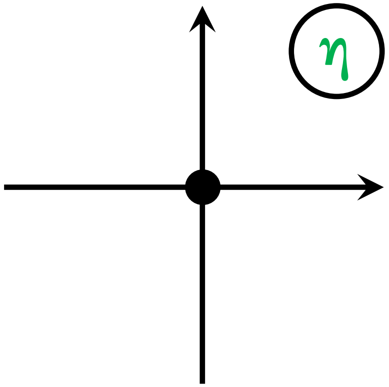
Ellipse, due to Kepler laws

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument

in Classical theory



2. Viscosity η in Navier–Stokes equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}\right)=-\nabla p+\boxed{\eta\Delta \mathbf{v}+\frac{\eta}{3}\nabla(\nabla \mathbf{v})}$$

Example: water flow in a tube:

$$v=\frac{\Delta p}{4\eta l}(R^2-r^2).$$

$\eta > 0$ IS OKEY

$\eta < 0$ is NOT PHYSICAL

TWO STRONGLY DIFFERENT CASES

⬍

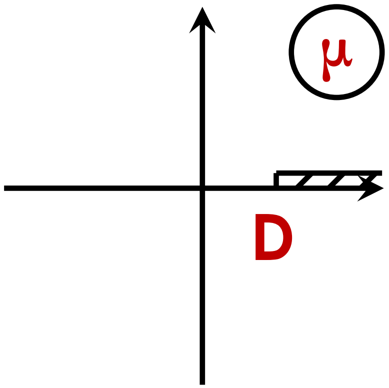
$\eta = 0$ is a singular point of v

IV. Exactly solvable model

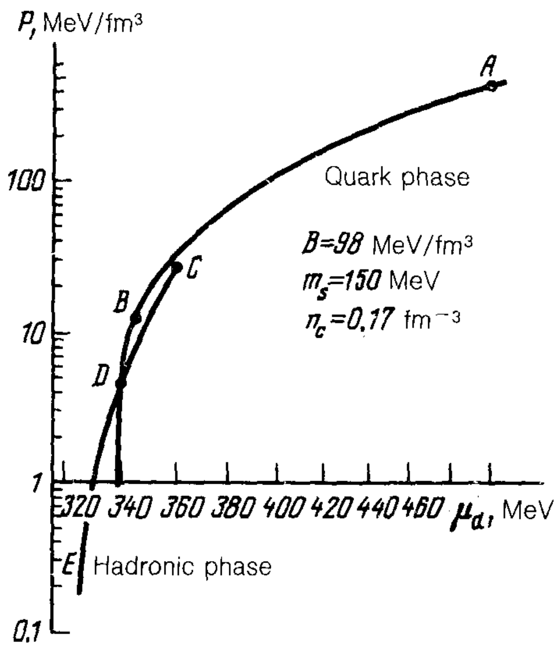
A. General properties of solutions

Dyson's argument

in Classical theory



3. 1st order phase transition



D – new phase occurs

TWO PHASES ARE

QUALITATIVELY DIFFERENT

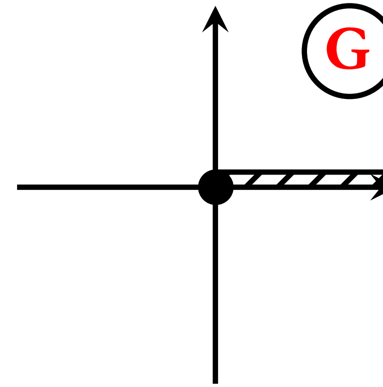
⇕

μ_D is a singular point of EoS

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument



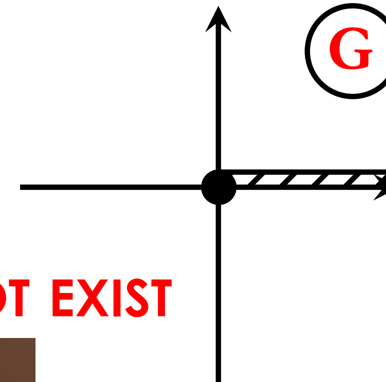
in our case:

G > 0 STARS EXIST

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument



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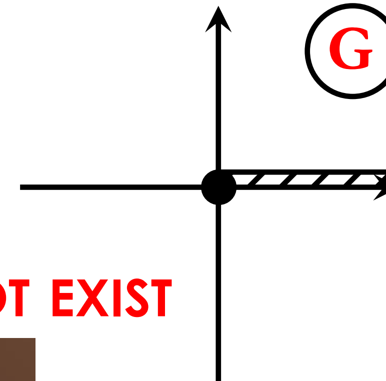
$G > 0$ STARS EXIST

$G < 0$ STARS DO NOT EXIST

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument



in our case:

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TWO CASES LOOK
QUITE DIFFERENT

$G = 0$ is a singular point of solutions of hydrostatic equilibrium equations

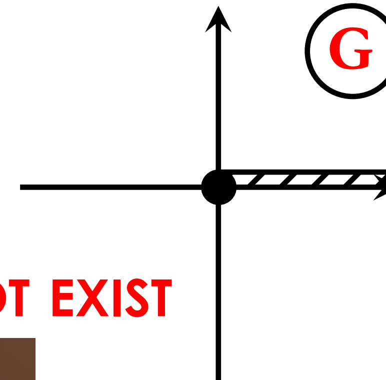
IV. Exactly solvable model

A. General properties of solutions

G <0: $0 < H_G$

G >0: $-\infty < H_G$

Dyson's argument



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TWO CASES LOOK
QUITE DIFFERENT

↕

G = 0 is a singular point of solutions of hydrostatic equilibrium equations

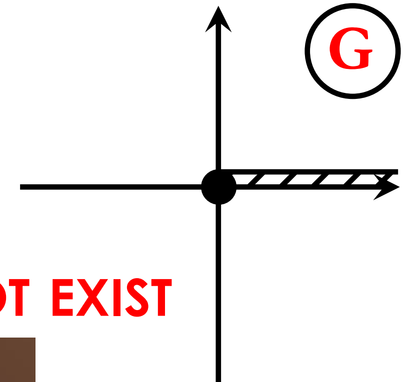
IV. Exactly solvable model

A. General properties of solutions

G <0: 0 < H_G

G >0: -∞ < H_G

Dyson's argument



in our case:

G > 0 STARS EXIST

G < 0 STARS DO NOT EXIST

TWO CASES LOOK QUITE DIFFERENT

⬍

G = 0 is a singular point of solutions of hydrostatic equilibrium equations

MOREOVER,
RADIUS AND MASS OF STAR:

$$r_0^2 = \frac{Z\mu_{e0}}{4\pi G m_i (m_i + Z m_e) n_{i0}}$$

$$M_s = \int_0^{R_s} 4\pi r^2 (n_i m_i + n_e m_e) dr$$

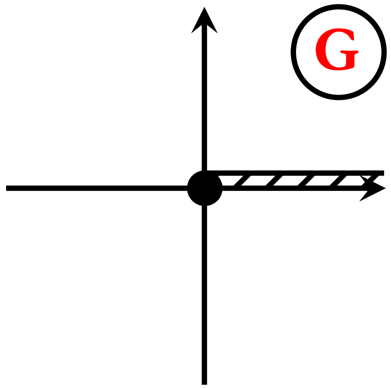
$$= -4\pi r_0^3 n_{i0} m_i x^2 \frac{d}{dx} (\theta_e + \Lambda_i \theta_i) \Big|_{x=x_b}$$

IV. Exactly solvable model

A. General properties of solutions

Dyson's argument

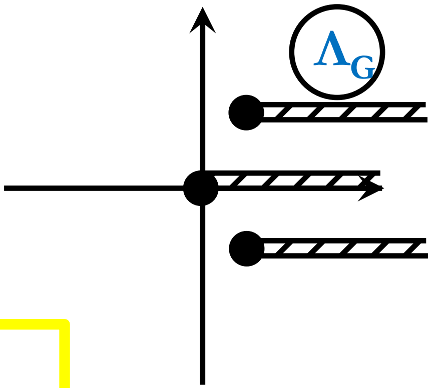
in our case:



The general solution should be sought in the class of functions that depend on Λ_G at $\Lambda_G = \infty$ in an irregular manner.

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices



The above statements can be precisely formulated using the model $\eta_i = \eta_e = 1$. In terms of

$$\varphi_k \equiv x\theta_k(x)$$

the main system of HE equations become

$$\begin{aligned} \varphi_e'' + \Lambda_i \varphi_i'' &= -(\varphi_i + \Lambda_m \Lambda_e \varphi_e) \\ \varphi_e'' - \Lambda_m \Lambda_i \varphi_i'' &= -\Lambda_G (\varphi_i - \Lambda_e \varphi_e) \end{aligned}$$

We are looking for solutions in the form $\varphi_k = \alpha_k e^{\beta x}$ and obtain

$$\begin{aligned} (\alpha_e + \Lambda_i \alpha_i)\beta^2 &= -(\alpha_i + \Lambda_m \Lambda_e \alpha_e), \\ (\alpha_e - \Lambda_m \Lambda_i \alpha_i)\beta^2 &= -\Lambda_G (\alpha_i - \Lambda_e \alpha_e). \end{aligned}$$

The eigenvalues are as follows:

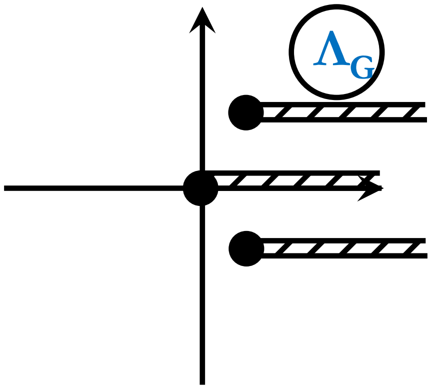
$$\beta_\pm^2 = \frac{1}{2\Lambda_i(1+\Lambda_m)} \Big\{ \Lambda_G(1+\Lambda_i\Lambda_e) - (1+\Lambda_m^2\Lambda_i\Lambda_e) \\ \pm \sqrt{\Lambda_G^2(1+\Lambda_i\Lambda_e)^2 - 2\Lambda_G\left[(1-\Lambda_m\Lambda_i\Lambda_e)^2 - \Lambda_i\Lambda_e(1+\Lambda_m)^2\right] + (1+\Lambda_m^2\Lambda_i\Lambda_e)^2} \Big\}$$

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

$$\beta_+^2 = \Lambda_G \frac{1+\Lambda_i\Lambda_e}{\Lambda_i(1+\Lambda_m)} + O(1),$$

$$\beta_-^2 = -\frac{\Lambda_e(1+\Lambda_m)}{1+\Lambda_i\Lambda_e} + O(\Lambda_G^{-1}).$$



The general solution: $\beta_1 = |\beta_+|$ & $\beta_2 = |\beta_-|$

$$\varphi_k(x) = \alpha_{k1}\sinh(\beta_1 x) + \alpha_{k2}\sin(\beta_2 x),$$

with the constraint

$$\alpha_{k1}\beta_1 + \alpha_{k2}\beta_2 = 1.$$

to fulfill $\varphi_k(0) = \varphi_k(0)'' = 0$, $\varphi_k(0)' = 1$.

NB: Для одноатомного идеального газа показатель адиабаты $\gamma = 5/3$, у нас $\gamma = 1 + 1/\eta = 6/3$.

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

1. Closer inspection of analyticity in G

The normal form: introducing $\pi_{\mathrm{k}} = \varphi'_{\mathrm{k}}$, and define the vector

$$\Phi \equiv \left(\pi_{\mathrm{i}}, \varphi_{\mathrm{i}}, \pi_{\mathrm{e}}, \varphi_{\mathrm{e}}\right)$$

The main ODE system:

$$\Phi'(x) = A\Phi(x)$$

where $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\Lambda})$ is linear in $\boldsymbol{\Lambda} \in \mathbb{C}^4 \backslash \infty$; with simple poles for $\boldsymbol{\Lambda} = \infty$, $\boldsymbol{\Lambda} = \left(\Lambda_{\mathrm{e}}, \Lambda_{\mathrm{i}}, \Lambda_{\mathrm{m}}, \Lambda_{\mathrm{G}}\right)$, with the initial conditions:

$$\Phi(x = 0) = (0, 1, 0, 1).$$

The solution:

$$\Phi(x) = \exp(Ax)\Phi(0)$$

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

1. Closer inspection of analyticity in G

$$\Phi'(x) = A\Phi(x)$$

$$\Phi(x) = D(x)\Phi(0)$$

$$D(x) = \exp(Ax)$$

- D(x) is analytic in $\Lambda \in \mathbb{C}^4 \backslash \infty$.
- A simple pole of A at $\Lambda = \infty$ is transformed into a fixed (essential) singularity of D(x).
- Φ(x) are analytic in $\Lambda \in \mathbb{C}^4 \backslash \infty$ and Φ(0), in agreement with the Poincare theorem on analyticity.

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

1. *Closer inspection of analyticity in G*

Let ξ_n ($\equiv \pm\beta_\pm \in \mathbb{C}^1$) denote the eigenvalues of **A** (n=1,2,3,4) and let $|n\rangle$ and $\langle m|$ be their right and left eigenvectors:

$$\mathbf{A}|\mathbf{n}\rangle = \xi_n|\mathbf{n}\rangle \text{ and } \langle\mathbf{n}|\mathbf{A} = \langle\mathbf{n}|\xi_n.$$

The evolution operator admits the representation

$$D(x) = \sum_n |n\rangle\langle n| \exp(\xi_n x).$$

ξ_n **are fixed from**

$$\det\|A - \xi\| = 0$$

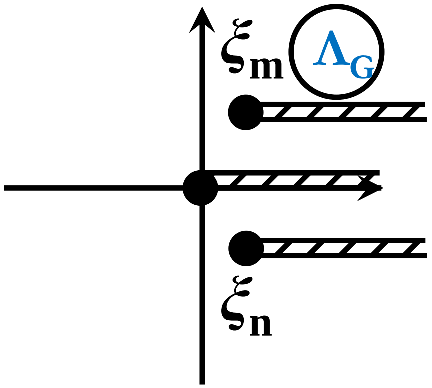
and acquire *ADDITIONAL SINGULARITIES,*

which cancel out from D(X).

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

1. *Closer inspection of analyticity in G*



$$D(x) = \sum_n |n\rangle\langle n| \exp(\xi_n x).$$

Frobenius representation:

$$|n\rangle\langle n| = \sum_{m \neq n} \frac{A - \xi_m}{\xi_n - \xi_m}.$$

D(x) is explicitly symmetric under the permutations

$$D(x) = \sum_n \exp(\xi_n x) \prod_{m \neq n} \frac{A - \xi_m}{\xi_n - \xi_m}.$$

$$\det\|A - \xi\| = 0 \;\Rightarrow\; \det\|A - \xi_n\| \equiv 0$$

после обхода доп. сингулярности $\xi_n \leftrightarrow \xi_m$

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

1. *Closer inspection of analyticity in G*

$\xi_n \equiv \pm\beta_\pm \in \mathbb{C}^1$ and the projection operators $|\mathbf{n}\rangle\langle\mathbf{n}|$ have 4 singular points

$\varphi_k(x)$ has one singular point $\Lambda = \infty$.

If we want a regular, non-trivial solution, Λ_e must be a function of Λ_G:

$$\Lambda_e \to \Lambda_e = f(\Lambda_G)$$

NB: Формальная математическая причина почему в регулярном решении параметры становятся функциями

- *С параметрами - точных регулярных решений нет (П ≠ 0)*
- *С функциями - точные регулярные решения есть (П ≡ 0)*

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

2. *Electro- and ionospheres*

The general solution:

$$\varphi_{\mathrm{k}}(x) = \alpha_{\mathrm{k}1}\sinh(\beta_1 x) + \alpha_{\mathrm{k}2}\sin(\beta_2 x),$$

with the constraint

$$\alpha_{\mathrm{k}1}\beta_1 + \alpha_{\mathrm{k}2}\beta_2 = 1.$$

Regular (*Π ≡ 0*) solution: $\alpha_{1\mathrm{k}} = 0$,

$$\begin{aligned}(1+\Lambda_{\mathrm{i}})\beta_2^2 &= 1+\Lambda_{\mathrm{m}}\Lambda_{\mathrm{e}},\\ (1-\Lambda_{\mathrm{m}}\Lambda_{\mathrm{i}})\beta_2^2 &= \Lambda_{\mathrm{G}}(1-\Lambda_{\mathrm{e}}).\end{aligned}$$

IS POSSIBLE FOR $\Lambda_{\mathrm{e}} = \Lambda_{\mathrm{e}}^{\mathrm{reg}} = \dfrac{\Lambda_{\mathrm{G}}(1+\Lambda_{\mathrm{i}}) - (1-\Lambda_{\mathrm{m}}\Lambda_{\mathrm{i}})}{\Lambda_{\mathrm{G}}(1+\Lambda_{\mathrm{i}}) + \Lambda_{\mathrm{m}}(1-\Lambda_{\mathrm{m}}\Lambda_{\mathrm{i}})}.$

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

2. *Electro- and ionospheres*

The general solution:

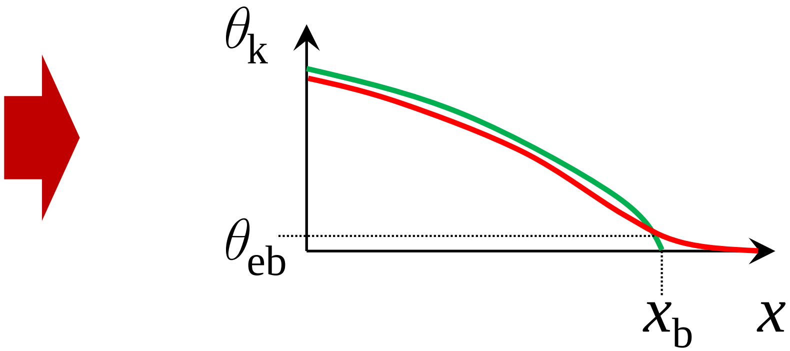
$$\theta_{\rm k}(x) = \alpha_{\rm k1}\frac{\sinh(\beta_1 x)}{x} + \alpha_{\rm k2}\frac{\sin(\beta_2 x)}{x},$$

$$\alpha_{\rm k2} = \frac{1-\alpha_{\rm k1}\beta_1}{\beta_2}.$$

A baryon boundary at $x_b = x_0 + \triangle x$ **where** $x_0 = \pi/\beta_{20}$

An electrosphere:

$$\theta_{\rm i}(x_b) = 0,$$
$$\theta_{\rm e}(x_b) = \theta_{\rm eb},$$



IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

2. *Electro- and ionospheres*

An electrosphere of A FINITE EXTENT:

$$\theta_{\rm eb} \leq \lambda_{\rm G}^{-1/(1+\eta_{\rm e})} \left[\frac{(1+\eta_{\rm e})\,(q_{\rm s}+\lambda_{\rm m})^2}{2\Lambda_{\rm e}(1+\Lambda_{\rm m})(1-\lambda_{\rm m}^2/\lambda_{\rm G})} \left(\frac{M_{\rm s}}{4\pi r_0^3 n_{\rm i0} m_{\rm i}} \right)^2 \left(\frac{r_0}{R_{\rm s}} \right)^4 \right]^{1/(1+\eta_{\rm e})}$$

For $\Delta x << x_0$

$$\Delta x = -\frac{1}{\beta_1} \mathcal{W} \left(\frac{\Delta\Lambda_{\rm e} \exp\left(\pi\beta_1/\beta_2\right)}{2(1+\Lambda_{\rm i})} \right)$$

where $\mathcal{W}(x)$ is the Lambert W-funciton which gives solution of the equation

$$\mathcal{W}(x)\exp(\mathcal{W}(x)) = x.$$

We note that $\Delta x < 0$ and $\Delta\Lambda_{\rm e} \sim \exp\left(-O(\sqrt{\Lambda_{\rm G}})\right) > 0.$

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

2. *Electro- and ionospheres*

An ionosphere of A FINITE EXTENT:

$$\theta_{\rm ib} \le \lambda_{\rm G}^{-1/(1+\eta_{\rm i})} \left[\frac{(1+\eta_{\rm i})(-q_{\rm s}+\lambda_{\rm m}/\Lambda_{\rm m})^2}{2\Lambda_{\rm i}(1+\Lambda_{\rm m})(1-1/\Lambda_{\rm G})} \left(\frac{M_{\rm s}}{4\pi r_0^3 n_{\rm i0} m_{\rm i}} \right)^2 \left(\frac{r_0}{R_{\rm s}} \right)^4 \right]^{1/(1+\eta_{\rm i})}$$

The same as electrosphere, with the replacements $\Lambda_{\rm i} \to 1/\Lambda_{\rm i}$,

$$\theta_{\rm eb} \to \theta_{\rm ib} \quad \& \quad \Delta\Lambda_{\rm e} \to -\Delta\Lambda_{\rm e}$$

We note that $\Delta x < 0$ and $\Delta\bar{\Lambda}_{\rm e} < 0$

We got a two-parameter set of the equations.

CONCENTRATIONS AT THE CENTER OF THE STAR ARE ARBITRARY,

however, in an exponentially small vicinity of the regular solution.

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

3. *Charge-mass-radius relation*

Mass:

$$\frac{M_\mathrm{s}}{4\pi r_0^3 n_{\mathrm{i}0} m_\mathrm{i}} = \frac{\pi(1+\Lambda_\mathrm{i})}{\beta_2},$$

Radius:

$$\frac{R_\mathrm{s}}{r_0} = \frac{\pi}{\beta_2}.$$

Bulk electric charge q_s & Charge of the electrosphere q_e

$$\frac{Z}{A} q_\mathrm{s} = \frac{1-\Lambda_\mathrm{m}\Lambda_\mathrm{i}}{1+\Lambda_\mathrm{i}} - \frac{(1+\Lambda_\mathrm{m})\Lambda_\mathrm{i}}{(1+\Lambda_\mathrm{i})^2}\pi\Theta_\mathrm{eb},$$

$$\frac{Z}{A} q_\mathrm{e} = -\frac{1+\Lambda_\mathrm{m}}{1+\Lambda_\mathrm{i}}\left[-\sqrt{\left(1-\frac{\Lambda_\mathrm{i}}{1+\Lambda_\mathrm{i}}\pi\Theta_\mathrm{eb}\right)^2 - \frac{\Lambda_\mathrm{i}}{1+\Lambda_\mathrm{i}}\pi^2\Theta_\mathrm{eb}^2} + \left(1-\frac{\Lambda_\mathrm{i}}{1+\Lambda_\mathrm{i}}\pi\Theta_\mathrm{eb}\right)\right]$$

where

$$\Theta_\mathrm{eb} = \beta_1\theta_\mathrm{eb}/\beta_2 \sim 1$$

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

3. Charge-mass-radius relation

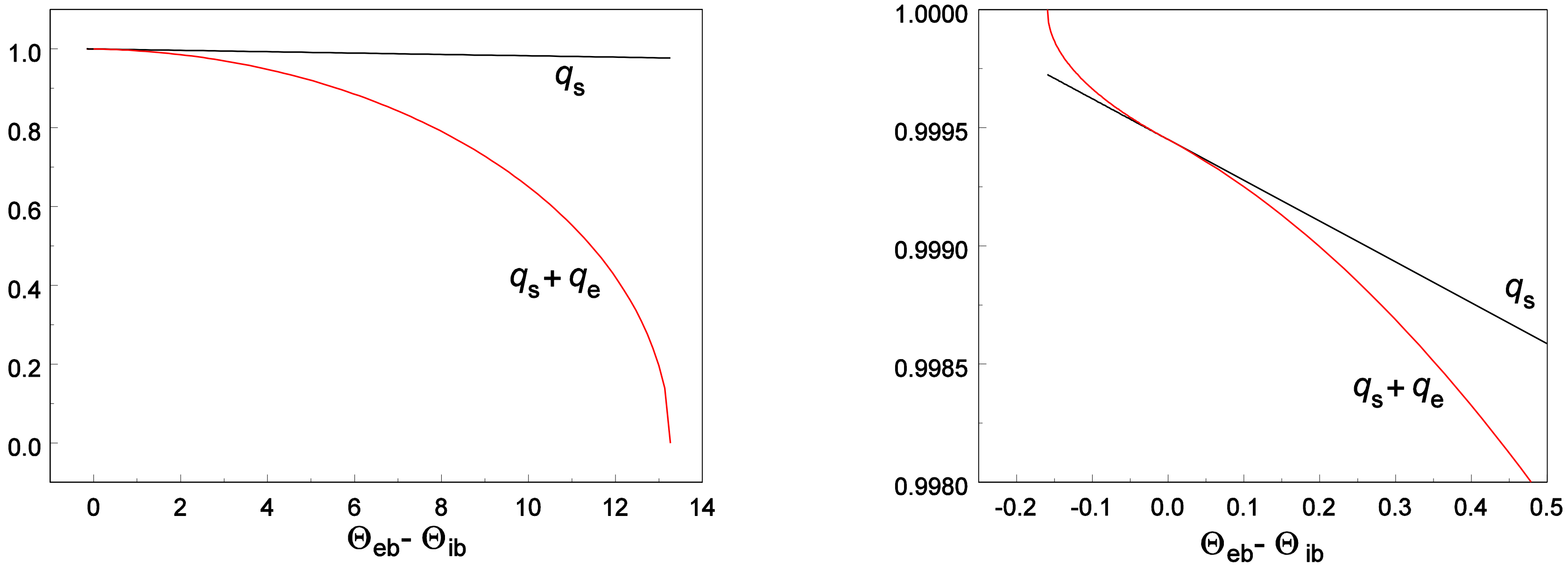


FIG. 4: (color online) The bulk charge $q_{\rm s}$ and the total charge $q_{\rm s} + q_{\rm e}$ as functions of the difference $\Theta_{\rm eb} - \Theta_{\rm ib}$ for a mixture of electrons and protons ($Z = A = 1$) and for $\Lambda_{\rm i} = \Lambda_{\rm m}$.

NB: $(q_{\rm s}+q_{\rm e})^{\rm max} = A/Z$

IV. Exactly solvable model

B. Two-fluid model with unit polytropic indices

- In the two-fluid model with unit polytropic indices, it is possible to explicitly construct the general solution describing charged stars with electro- and ionospheres.

- The guiding parameter of the problem is Λ_e

- In an exponentially small neighborhood of $\Lambda_e = \Lambda_e^{reg}$, the electron and ion densities can be varied without restrictions.

- The stellar envelope is sensitive to exponentially small deviations of Λ_e from Λ_e^{reg}.

V. General solution to the unconstrained hydrostatic equilibrium equations

The general solution is of the form:

$$\theta_{\rm k} = \theta_{\rm k0} + \chi_{\rm k},$$

regular   irregular

In inner layers of the star $\chi_k \ll \theta_{k0}$

Small deviations in:

$$\Delta\Lambda_{\rm e} = \Lambda_{\rm e} - \Lambda_{\rm e}^{\rm reg},\ |\Delta\Lambda_{\rm e}| \ll 1$$

Linearization of the main ODE system gives

$$\begin{aligned}
\Delta_x(\chi_{\rm e} + \Lambda_{\rm i}\chi_{\rm i}) &= -(\eta_{\rm i}\theta_{\rm i0}^{\eta_{\rm i}-1}\chi_{\rm i} \\
&\quad + \Lambda_{\rm m}\Lambda_{\rm e0}\eta_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}-1}\chi_{\rm e} + \Lambda_{\rm m}\Delta\Lambda_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}}), \\
\Delta_x(\chi_{\rm e} - \Lambda_{\rm m}\Lambda_{\rm i}\chi_{\rm i}) &= -\Lambda_{\rm G}(\eta_{\rm i}\theta_{\rm i0}^{\eta_{\rm i}-1}\chi_{\rm i} \\
&\quad - \Lambda_{\rm e0}\eta_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}-1}\chi_{\rm e} - \Delta\Lambda_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}}).
\end{aligned}$$

$\chi_{\rm k}(0) = \chi'_{\rm k}(0) = 0$ ← initial conditions

V. General solution to the unconstrained hydrostatic equilibrium equations

A. Irregular component in the WKB approximation

A similar case occurs in the Schrödinger equation

$$(A_\varepsilon :) \qquad E\Psi(x) = \left(-\frac{\hbar^2}{2m}\Delta + V(x) \right)\Psi(x), \quad \hbar \to 0,$$

which makes the analogy quite obvious. In quantum mechanics, the Wentzel, Kramers, and Brillouin approximation is used, known as **the WKB METHOD**

V. General solution to the unconstrained hydrostatic equilibrium equations

A. Irregular component in the WKB approximation

To restore the initial conditions: $\chi_{\mathrm{k}} = \chi_{\mathrm{k}}^{\mathrm{irr}} + \chi_{\mathrm{k}0}$

Общее решение однородного ур-я

Частное решение неоднородного ур-я

Following the WKB method:

$$\chi_{\mathrm{k}}^{\mathrm{irr}}(x) = g_{\mathrm{k}}(x) \exp\left(\sqrt{\Lambda_{\mathrm{G}}} S(x)\right)$$

where k = i,e,

$$g_{\mathrm{k}}(x) = g_{\mathrm{k}0}(x) + \frac{g_{\mathrm{k}1}(x)}{\sqrt{\Lambda_{\mathrm{G}}}} + \frac{g_{\mathrm{k}2}(x)}{\Lambda_{\mathrm{G}}} + \ldots .$$

- To the lowest order

$$S_{\pm}(x) = \pm \int_0^x \sqrt{\frac{\eta_{\mathrm{i}} \theta_{\mathrm{i}0}^{\eta_{\mathrm{i}}-1}(x') + \Lambda_{\mathrm{i}} \Lambda_{\mathrm{e}}^{\mathrm{reg}} \eta_{\mathrm{e}} \theta_{\mathrm{e}0}^{\eta_{\mathrm{e}}-1}(x')}{\Lambda_{\mathrm{i}}(1+\Lambda_{\mathrm{m}})}} dx' .$$

Finally (cf.: $1/\Lambda_G \leftarrow\rightarrow \hbar^2$)

$$\chi_{\mathrm{k}0}^{\mathrm{irr}}(x) = C_{\mathrm{k}0} \frac{\sinh\left(\sqrt{\Lambda_{\mathrm{G}}} S_{+}(x)\right)}{x\sqrt{S'_{+}(x)}} + O(\Lambda_{\mathrm{G}}^{-\frac{1}{2}})$$

V. General solution to the unconstrained hydrostatic equilibrium equations

B. Correction to the irregular component

как частное решение неоднородного уравнения

$$\begin{aligned}
\triangle_x(\chi_\mathrm{e}+\Lambda_\mathrm{i}\chi_\mathrm{i}) &= -(\eta_\mathrm{i}\theta_\mathrm{i0}^{\eta_\mathrm{i}-1}\chi_\mathrm{i} \\
&\quad +\Lambda_\mathrm{m}\Lambda_\mathrm{e0}\eta_\mathrm{e}\theta_\mathrm{e0}^{\eta_\mathrm{e}-1}\chi_\mathrm{e}+\Lambda_\mathrm{m}\triangle\Lambda_\mathrm{e}\theta_\mathrm{e0}^{\eta_\mathrm{e}}), \\
\triangle_x(\chi_\mathrm{e}-\Lambda_\mathrm{m}\Lambda_\mathrm{i}\chi_\mathrm{i}) &= -\Lambda_\mathrm{G}(\eta_\mathrm{i}\theta_\mathrm{i0}^{\eta_\mathrm{i}-1}\chi_\mathrm{i} \\
&\quad -\Lambda_\mathrm{e0}\eta_\mathrm{e}\theta_\mathrm{e0}^{\eta_\mathrm{e}-1}\chi_\mathrm{e}-\triangle\Lambda_\mathrm{e}\theta_\mathrm{e0}^{\eta_\mathrm{e}}).
\end{aligned}$$

The functions

$$\widehat{\theta}_\mathrm{k0}(x)=\theta_\mathrm{k}(0)\theta_\mathrm{k0}\left(x\sqrt{\frac{\theta_\mathrm{i}^{\eta_\mathrm{i}}(0)}{\theta_\mathrm{e}(0)}},\Lambda_\mathrm{i}\frac{\theta_\mathrm{i}(0)}{\theta_\mathrm{e}(0)}\right)$$

are the regular solutions of the main ODE system with the modified initial condition $\theta_k(0) \neq 1$ and

$$\Lambda_\mathrm{e}^{\mathrm{reg}\prime}(\Lambda_\mathrm{i})=\frac{\theta_\mathrm{i}^{\eta_\mathrm{i}}(0)}{\theta_\mathrm{e}^{\eta_\mathrm{e}}(0)}\Lambda_\mathrm{e}^{\mathrm{reg}}\left(\Lambda_\mathrm{i}\frac{\theta_\mathrm{i}(0)}{\theta_\mathrm{e}(0)}\right).$$

V. General solution to the unconstrained hydrostatic equilibrium equations

B. Correction to the irregular component

The functions

$$\chi_{\mathrm{k}0} = \widehat{\theta}_{\mathrm{k}0} - \theta_{\mathrm{k}0}$$

satisfy the non-linear eqs. for

$$\triangle\Lambda_{\mathrm{e}} = \Lambda_{\mathrm{e}}^{\mathrm{reg}\prime} - \Lambda_{\mathrm{e}}^{\mathrm{reg}}$$

with $|\Delta\Lambda_e| << 1$.

Finally,

$$\chi_{\mathrm{k}0}(x) = \theta_{\mathrm{k}0}(x)\triangle\theta_{\mathrm{k}} + x\frac{\partial\theta_{\mathrm{k}0}(x)}{\partial x}\frac{\eta_{\mathrm{i}}\triangle\theta_{\mathrm{i}} - \triangle\theta_{\mathrm{e}}}{2} + \Lambda_{\mathrm{i}}\frac{\partial\theta_{\mathrm{k}0}(x)}{\partial\Lambda_{\mathrm{i}}}(\triangle\theta_{\mathrm{i}} - \triangle\theta_{\mathrm{e}}).$$

where $|\Delta\theta_k(0)| << 1$ $(k = i, e)$.

V. General solution to the unconstrained hydrostatic equilibrium equations

C. General solution

The initial conditions $\theta_k(0) = 1$ are satisfied for

$$C_{k0}\sqrt{\Lambda_G S'_+(0))} + \triangle\theta_k = 0$$

The ion and electron components are related by

$$C_{e0} + \Lambda_i C_{i0} = 0$$

- Finally, the irregular part takes the form

$$\chi_i(x) = \frac{C_{i0}}{x\sqrt{S'_+(x)}}\sinh\left(\sqrt{\Lambda_G}S_+(x)\right) - C_{i0}\sqrt{\Lambda_G S'_+(0)}\left[\theta_{i0}(x) + x\frac{\partial\theta_{i0}(x)}{\partial x}\frac{\eta_i+\Lambda_i}{2} + \Lambda_i\frac{\partial\theta_{i0}(x)}{\partial\Lambda_i}(1+\Lambda_i)\right],$$

$$\chi_e(x) = -\frac{C_{i0}\Lambda_i}{x\sqrt{S'_+(x)}}\sinh\left(\sqrt{\Lambda_G}S_+(x)\right) + C_{i0}\sqrt{\Lambda_G S'_+(0)}\left[\Lambda_i\theta_{e0}(x) - \frac{\partial\theta_{e0}(x)}{\partial x}\frac{\eta_i+\Lambda_i}{2} - \Lambda_i\frac{\partial\theta_{e0}(x)}{\partial\Lambda_i}(1+\Lambda_i)\right]$$

Conclusions

Найдено общее решение уравнений гидростатического равновесия в отсутствие локальной электронейтральности для двух-компонентного вещества в Ньютоновой теории гравитации.

- **Подробно рассмотрены две точно решаемые модели звезд для политропных уравнений состояния вещества.**

- **Описана структура и условия формирования электросферы и ионосферы звезд.**

- **Электрический заряд звезд варьируется в пределах - 0.1 ÷ 100 Кулон.**

Conclusions

Нерешенные вопросы

- Регулярные решения выделены: общее решение сконцентрировано вокруг регулярного. Почему?

- Почему функция $\Lambda_e^{reg}(\Lambda_G)$ единственна?

- В точно решаемой модели есть объяснение превращению параметра Λ_e в функцию $\Lambda_e^{reg}(\Lambda_G)$. Насколько общим можно считать объяснение?

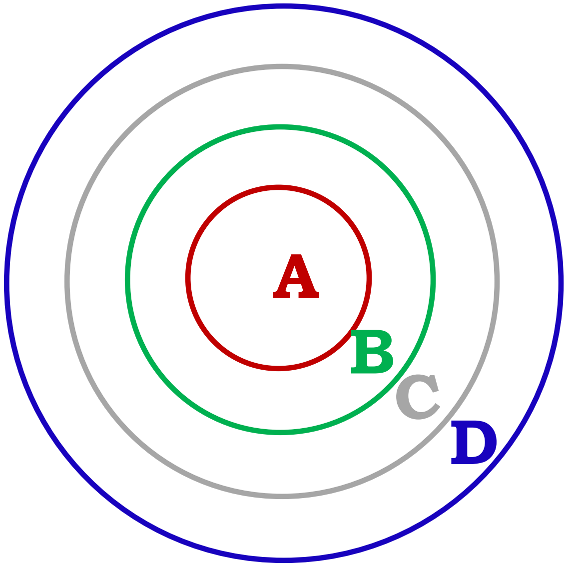
Conclusions

Нерешенные вопросы

- Обобщение на общую теорию относительности (ОТО).

- Обобщение на многокомпонентные вещества ($n > 2$). Ожидаемая картина – сепарация элементов

(в каждом слое 2-х компонентное в-во?)



A – очень тяжелые
B – тяжелые
C - легкие
D – очень легкие

- Заряженные странные звезды: $Q_s \sim 10^{20}$ C ?

СПАСИБО ЗА ВНИМАНИЕ!