Holographic models of QCD in the strong coupling regime

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Outline

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The Setup

- AdS/QCD – a new approach to studying strong interactions.
- Based on AdS/CFT:
  $\mathcal{N} = 4$ $U(N_c)$ SYM $\leftrightarrow$ Superstring theory IIB in $AdS_5 \times S^5$.
- Geometry created by a stack of N_c $D3$-branes. Radius $\ell \gg \ell_s$.
- Isometries of $AdS_5 \times S^5$ correspond to global symmetries of SYM:
- $S^5 \to SO(6) \cong SU(4)$ – R-symmetry;
- $AdS_5 \to SO(2,4) \cong$ Superconformal + Poincaré.
- We use these to match $AdS_5 \times S^5$ fields and SYM:
  $m^2\ell^2 = (\Delta - s)(\Delta + s - 4)$

The Setup

According to the AdS/CFT prescription:

Quantum Field Theory	Classical Gravity in 5D
Source $J(x_\mu)$ of an operator $\mathcal{O}$	Boundary value $\Phi(x_\mu, 0)$ of a 5D field $\Phi(x_\mu, z)$
Effective action with sources	Action on classical trajectories
Isometries	Global symmetries

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Approaches to holographic QCD – “top-down”

- This form of AdS/CFT can be adapted to include matter as fundamental $\mathcal{N} = 2$ hypermultiplets, and we can break supersymmetry completely – Sakai, Sugimoto.
- However we can simply start with 5D models that preserve the essential features of QCD.
- In order to understand the underlying structure of the QCD dual we need to develop:
- ten-dimensional “top-down” models [S.Sakai, T. Sugimoto, 2004, 2005]
- simpler “bottom-up” AdS/QCD models [J. Erlich, E. Katz, D. T. Son, M. A. Stephanov, 2005; A. Karch, E. Katz, D. T. Son, M. A. Stephanov, 2006]

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Approaches to holographic QCD – “bottom-up”

Operators under consideration in "bottom-up" AdS/QCD:

- Symmetry currents $J^a_{L,R\ \mu}$ where a is an adjoint $SU(N_f)_{L,R}$ index;
- Chiral symmetry violation order parameter $\Sigma^{\alpha\beta} = \langle \bar{q}^\alpha q^\beta \rangle$ where $\alpha, \beta = 1...N_f$ are fundamental flavor indices.

Radial coordinate of the AdS is interpreted as the energy scale with UV region near the boundary.
QCD is asymptotically conformal in the UV $\Rightarrow$ our 5D space is asymptotically AdS near the boundary.
We have to modify the geometry in the IR region to reflect the confinement.

AdS/QCD Models

- KK-decompose all the fields and integrate out the z-axis, get an effective action for mesons - a chiral Lagrangian.
- An “expansion” of χPT
- In order to sharpen the model we have to test its consistency – one has to calculate quantities known in QCD from the AdS point of view.

Outline

Low-Energy Theorems

Expressed in terms of the quark currents $S_i(x)$, $P_j(y)$, Chiral Lagrangian (L_i – parameters of the NLO Lagrangian of order of $\mathcal{O}(p^4)$, $B = G_\pi/F_\pi$), spectral density $\rho(\lambda, m)$ and topological charge density $Q(x) \sim \mathrm{tr} F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)$:

$$i\int d^4x \left\langle \delta_{ij}S_0(x)S_0(0) - P_i(x)P_j(0)\right\rangle = -\frac{G_\pi^2\delta_{ij}}{m_\pi^2} + \delta_{ij}\frac{B^2}{8\pi^2}(L_3 - 4L_4 + 3)$$

$$= 2\delta_{ij}\int d\lambda \left(\frac{m\frac{\partial}{\partial m}\rho(\lambda,m)}{(\lambda^2+m^2)} - \frac{2m^2\rho(\lambda,m)}{(\lambda^2+m^2)^2}\right), \quad (1)$$

$$i\int d^4x \left\langle S_i(x)S_j(0) - \delta_{ij}P_0(x)P_0(0)\right\rangle = \delta_{ij}\int d\lambda \frac{4m^2\rho(\lambda,m)}{(\lambda^2+m^2)^2} - 2\delta_{ij}\frac{\int d^4x\,\langle Q(x)Q(0)\rangle}{m^2V}, \quad (2)$$

$$i\int d^4x\,\langle P_3(x)P_0(0)\rangle = 2(m_u - m_d)m\int d\lambda \frac{\rho(\lambda,m)}{(\lambda^2+m^2)^2} - (m_u - m_d)\frac{\int d^4x\,\langle Q(x)Q(0)\rangle}{m^3V}. \quad (3)$$

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Five-Dimensional Effective Action

$$S_{5D} = \int d^5x\sqrt{g}e^{-\Phi}\mathrm{tr}\left\{g_X^2\left(|DX|^2+\frac{3}{\ell^2}|X|^2+\frac{\kappa}{\ell^2}|X|^4\right) - \frac{1}{4g_5^2}(F_L^2+F_R^2)\right\}$$

with a metric $ds^2 = \frac{\ell^2}{z^2}(-dz^2+dx_\mu dx^\mu)$.

- "Hard-wall": $\Phi(z)\equiv 0,\ \kappa=0,\ 0\leqslant z\leqslant z_m$,
- "Soft-wall": $\Phi(z)\sim\lambda z^2 (z\to\infty),\ \kappa\neq 0,\ 0\leqslant z<\infty$.

$$L_\mu^a(x,z=0) = \text{source of } \bar{q}_L(x)\gamma_\mu t^a q_L(x),$$
$$R_\mu^a(x,z=0) = \text{source of } \bar{q}_R(x)\gamma_\mu t^a q_R(x),$$
$$\lim_{z\to 0}\frac{2}{z}X^{\alpha\beta}(x,z) = \text{source of } \bar{q}_L^\alpha(x)q_R^\beta(x)$$
$$= m\delta^{\alpha\beta} \text{ in the absence of (pseudo)scalar currents.}$$

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Zero momentum correlation functions of QCD currents

Correlation functions are calculated via the AdS/CFT prescription:

$$\mathcal{Z}_{QCD}[\mathcal{J}_I(x_\mu)] = \exp\left(iS_{5D\ classical}\right)\big|_{\Phi_I(0,x_\mu)=\mathcal{J}_I(x_\mu)} \Rightarrow$$

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle_{conn} = -\frac{\delta}{\delta\Phi_I(0,x)}\frac{\delta}{\delta\Phi_J(0,y)}S_{5D\ classical}\bigg|_{\Phi_I(0,x_\mu)=0}$$

At zero momentum for N_f quark flavors with equal masses

$$i\langle P_i(0)P_j(0)\rangle = \delta_{ij}\frac{C}{m} = \delta_{ij}\frac{G_\pi^2}{m_\pi^2},$$

where $\left\langle \bar{q}^\alpha q^\beta \right\rangle = C\delta^{\alpha\beta}$ in the chiral limit, $\Sigma = \langle \bar{q}^\alpha q_\alpha \rangle$. One can see that we obtain a singularity corresponding to the pion exchange. The pole residue is the same for the middle and left-hand sides of Eqn. (1).

$N_f = 2$ case

In the particular $N_f = 2,\ m_u \neq m_d$ case

$$i\left\langle P_i(0)P_j(0)\right\rangle = \delta_{ij}\frac{C}{m} - \frac{C\Delta m}{2m^2}(\delta_{i0}\delta_{3j} + \delta_{j0}\delta_{3i} - i\delta_{i1}\delta_{2j} - i\delta_{j1}\delta_{2i}),$$

where $m = \dfrac{m_u + m_d}{2},\ \Delta m = m_u - m_d$.
Scalar current correlators are calculated analogously and are regular in the chiral limit. Thus for a zero momentum

$$i\left\langle \delta_{ij}S_0(0)S_0(0) - P_i(0)P_j(0)\right\rangle$$

$$\sim i\left\langle S_i(x)S_j(0) - \delta_{ij}P_0(x)P_0(0)\right\rangle \sim -\delta_{ij}\frac{C}{m},$$

$$i\left\langle P_3(0)P_0(0)\right\rangle \sim -\frac{C\Delta m}{2m^2}.$$

Chiral Lagrangian from Holography

Another point of view on the holographic models:

- Five-dimensional fields on the AdS boundary are sources of the QCD currents.
- QCD currents are sources of the mesons.
- Kaluza–Klein modes $\propto$ meson wavefunctions with corresponding quantum numbers.
- If we integrate out the dynamics along the z axis, the 5D action will generate an effective chiral Lagrangian.

Similar to the "top-down" models [T. Sakai, S. Sugimoto, 2005].

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Kaluza–Klein expansion

Gauge fields can be combined into

$$V^a_\mu = L^a_\mu + R^a_\mu, \; A^a_\mu = L^a_\mu - R^a_\mu$$

One can fix the gauge $V_z = A_z = \partial^\mu V_\mu = 0$, so that A_μ retains a longitudinal component: $A_\mu = A_{\perp\mu} + \partial_\mu \phi$. $\phi \propto$ the source of the pseudoscalar current.
KK expansion:

$$\phi^a(z,x) = \sum_n f^{(n)}_\phi(z) \phi^{a(n)}(x), f^{(n)}_\phi(z)$$

–E.o.M. solution in AdS, $\phi^{a(0)}(x) \propto \pi^a(x)$.

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Parameters of the Chiral Lagrangian

Having integrated out all z-dependence in the 5D action of the lowest KK-mode we obtain

$$S_{5D} \to A_1 \cdot \frac{1}{2} \partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)} + A_2 \cdot [\partial_\mu \phi^{(0)}, \partial_\nu \phi^{(0)}]^2 + A_3 \cdot m \partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)}.$$

This allows to find the parameters explicitly $L_{1,2,3} \propto \frac{A_2}{A_1^2}$, $L_4 \propto \frac{A_3}{A_1}$. The following (universal for holographic models) equation holds

$$L_3 = -3L_2 = -6L_1.$$

Parameters L_i are regular in the chiral limit.

The Spectral Density of the Dirac operator

The Dirac operator $\hat{D} \equiv \gamma_\mu(\partial_\mu + ig_{YM}A_\mu)$ has no dual AdS description, and its spectral density $\rho(\lambda) = \frac{1}{V}\left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle_A$, where $\{\lambda_n\}$ are the eigenvalues of $i\hat{D}$, has no direct AdS/QCD interpretation similar to the Chiral Lagrangian and QCD partition function.

However one can express $\rho(\lambda)$ via a partition function of a QCD-like theory whith a dual description.

[S. F. Edwards and P. W. Anderson, 1975; J. J. M. Verbaarschot and M. R. Zirnbauer, 1984; K. B. Efetov, 1983]

$$\rho(\lambda) = \frac{1}{V}\left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle_A = \frac{1}{\pi V}\left\langle \lim_{\mu\to 0} \sum_n \frac{\mu}{\mu^2 + (\lambda - \lambda_n)^2} \right\rangle_A$$

$$= \frac{1}{2\pi V} \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \left\langle \log Det[i\hat{D} - \lambda - i\mu] + \log Det[i\hat{D} - \lambda + i\mu] \right\rangle_A$$

The Spectral Density

We use a so-called "replica trick" : $\log z = \frac{\partial}{\partial n} z^n \Big|_{n=0}$

$$\rho(\lambda) = \frac{1}{\pi V} \lim_{\mu\to 0} \frac{\partial}{\partial \mu} \lim_{n\to 0} \frac{\partial}{\partial n} \mathrm{Re} \left\langle Det^n[i\hat{D} - \lambda - i\mu] \right\rangle_A$$

$$= \frac{1}{\pi V} \lim_{\mu\to 0} \frac{\partial}{\partial \mu} \lim_{n\to 0} \frac{\partial}{\partial n} \mathrm{Re} \int DA\, e^{iS_{YM}[A]} \times \mathcal{Z}_{quarks}(m_q = m,\ N_f)[A]$$

$$\times \mathcal{Z}_{ghosts}(m_q = \lambda + i\mu,\ n \cdot N_f)[A].$$

Spectral Density in Holography

This partition function can be calculated in "hard-wall" AdS/QCD, with a boundary condition on the scalar field:

$$\lim_{z\to 0}\frac{2}{z}X = \begin{pmatrix} m & & & & & \\ & \ddots & & & 0 & \\ & & m & & & \\ & & & \lambda+i\mu & & \\ & 0 & & & \ddots & \\ & & & & & \lambda+i\mu \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ N_f \\ N_f+1 \\ \vdots \\ N_f(n+1) \end{matrix}$$

Result is the following:

$$\rho(\lambda) = -\frac{1}{\pi}\left(\Sigma(m) + m\frac{d}{dm}\Sigma(m)\right)\Big|_{m=\lambda}.$$

Spectral Density in Holography

A QCD result [V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, 1981]

$\Sigma(m) = \Sigma(0)\left(1 - \frac{3m_\pi^2 \log m_\pi^2/\mu_{hadr}^2}{32\pi^2 F_\pi^2}\right)$ leads to a formula

$$\rho(\lambda) = -\frac{1}{\pi}\Sigma(0)\left(1 - \frac{3\Sigma(0)}{8\pi^2 N_f F_\pi^4}\lambda - \frac{3\Sigma(0)}{4\pi^2 N_f F_\pi^4}\lambda \log\lambda\right), \lambda > 0.$$

So:

- the result agrees with the Casher–Banks identity $\rho(0) = -\Sigma(0)/\pi$ [T. Banks and A. Casher, 1980],
- up to a $\propto N_f^2 - 4$ factor reproduces the result of Smilga and Stern [A. Smilga and J. Stern, 1993] for the term linear in λ,
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Spectral Density in Holography

A QCD result [V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, 1981]

$\Sigma(m) = \Sigma(0)\left(1 - \frac{3m_\pi^2 \log m_\pi^2/\mu_{hadr}^2}{32\pi^2 F_\pi^2}\right)$ leads to a formula

$$\rho(\lambda) = -\frac{1}{\pi}\Sigma(0)\left(1 - \frac{3\Sigma(0)}{8\pi^2 N_f F_\pi^4}\lambda - \frac{3\Sigma(0)}{4\pi^2 N_f F_\pi^4}\lambda \log \lambda\right), \lambda > 0.$$

So:

- the result agrees with the Casher–Banks identity $\rho(0) = -\Sigma(0)/\pi$ [T. Banks and A. Casher, 1980],
- up to a $\propto N_f^2 - 4$ factor reproduces the result of Smilga and Stern [A. Smilga and J. Stern, 1993] for the term linear in λ,
- has terms $\propto \lambda \log \lambda$.

Spectral Density in Holography

Drawbacks of this result are the following:

- No dependence on mass;
- No terms $\propto \lambda^2$ and higher.

It seems that we might obtain a more precise result for $\rho'(0)$ if we either manage to formulate a consistent "soft-wall" model with different flavors or take into account the N_f–dependence of the 5D metric ($\sim$ to the flavor brane back-reaction).

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Compatibility of the AdS/QCD models with low-energy theorems

Integrals with $\rho(\lambda)$ in the right-hand sides of the theorems Eqn. $(1-3)$ yield

$$\int d\lambda \, \frac{m^2 \rho(\lambda)}{(\lambda^2 + m^2)^2} \sim \frac{C}{m}, \int d\lambda \, \frac{m \Delta m \rho(\lambda)}{(\lambda^2 + m^2)^2} \sim \frac{C \Delta m}{m^2},$$

and in the $m \to 0$ limit for each theorem Eqn. (1, 2, 3) we get equal pole residues on all sides. Thus, AdS/QCD models are compatible with low-energy theorems in the chiral limit.

Summary

- We have demonstrated the agreement of the AdS/QCD model in the chiral limit with the low-energy theorems.
- We have proposed a way to calculate the spectral density of the Dirac operator in AdS/QCD.
- Result in the "hard-wall" model agrees with the Casher–Banks identity and up to a numerical N_f–depending factor reproduces the Smilga and Stern result.
- Testing the theorems beyond the chiral limit might probably bring us closer to understanding the structure of the AdS/QCD models.

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Outline

Essence of CME

Generation of an electric current parallel to a magnetic field in a topologically nontrivial background
[D. E. Kharzeev, L. D. McLerran and H. J. Warringa, 2008;
K. Fukushima, D. E. Kharzeev, and H. J. Warringa, 2008].

- Chirally symmetric phase of QCD with massless quarks q_L, q_R of a unit electromagnetic charge
- $q_L \to (q^+_{-1/2}, q^-_{+1/2})$ and $q_R \to (q^+_{+1/2}, q^-_{-1/2})$ (charge and helicity), magnetic moment $\propto$ charge $\times$ spin
- External magnetic field aligns magnetic moments $\Rightarrow$ spins and momenta are correlated with the magnetic field
- Components $(q^+_{-1/2}, q^-_{+1/2})$ move in the opposite direction to the field, $(q^+_{+1/2}, q^-_{-1/2})$ – along the field.

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- An electromagnetic current $J \propto$ l.h.s. - r.h.s.

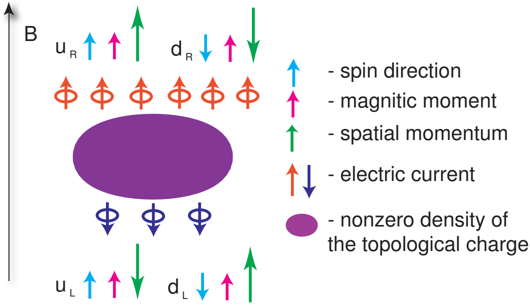

Figure: CME as a simple rearrangement of momenta.

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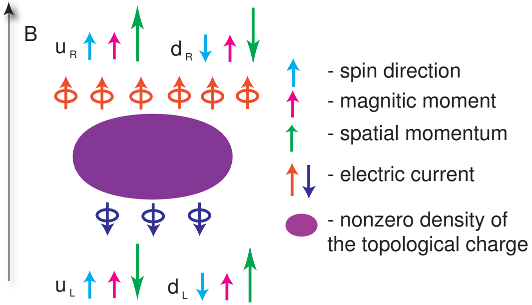

Figure: CME as a simple rearrangement of momenta.

CME at weak coupling

In the weak-coupling limit [K. Fukushima, D. E. Kharzeev, and H. J. Warringa, 2008] the resulting current is

$$\mathbf{J}^V = \frac{\mu_5 \mathbf{B}}{2\pi^2} \equiv \mathcal{J}_{FKW}$$

A temperature-dependent expression for the susceptibility $\chi \propto T$ has also been obtained [K. Fukushima, D. E. Kharzeev, and H. J. Warringa, 2009].

CME in the gauge/gravity framework

The CME has been studied in the dual models to shed light into its properties in the strong-coupling limit:

- [H.-U. Yee, 2009] – in a model of Einstein gravity with a $U(1)_L \times U(1)_R$ Maxwell theory in the AdS_5 space and in the Sakai-Sugimoto model. Results agree with those in the weak-coupling limit.
- [A. Rebhan, A. Schmitt and S. A. Stricker, 2010] – in the Sakai-Sugimoto model. Result is $2/3$ of the weak-coupling result in the absence of the Bardeen counterterm and is zero with the counterterm.

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Experimental status of CME and CME on lattices

- Experimental status is discussed in [**STAR** collaboration and S. A. Voloshin, 2009, 2011; **P**HENIX] collaboration, 2010].
- Lattice calculations by the ITEP Lattice group in the quenched approximation [P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya and M. I. Polikarpov, 2009; P. V. Buividovich, M. N. Chernodub, T. Kalaydzhyan, D. E. Kharzeev, E. V. Luschevskaya and M. I. Polikarpov, 2010] ...
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Gauge sector of the soft-wall AdS/QCD

In the chirally symmetric phase we only need to consider the gauge sector, $|X| = 0$ (its phase – the 5D pion – is a more involved issue).

$$
\begin{aligned}
S_{5D} &= S_{YM}[L] + S_{YM}[R] + S_{CS}[L] - S_{CS}[R] \\
S_{YM}[A] &= -\frac{1}{4g_5^2}\int e^{-\phi} F \wedge *F = -\frac{1}{4g_5^2}\int dz\, d^4x\, e^{-\phi}\sqrt{g} F_{MN} F^{MN} \\
S_{CS}[A] &= -\frac{N_c}{24\pi^2}\int A \wedge F \wedge F = -\frac{N_c}{24\pi^2}\int dz\, d^4x\, \epsilon^{MNPQR} A_M F_{NP} F_{QR}
\end{aligned}
$$

with a metric tensor

$$
ds^2 = g_{MN} dX^M dX^N = \frac{\ell^2}{z^2}\eta_{MN} dX^M dX^N = \frac{\ell^2}{z^2}(-dz^2 + dx_\mu dx^\mu).
$$

Symmetry Currents from the On-shell Action

Chemical potentials $\mu = \frac{1}{2}(\mu_L + \mu_R), \mu_5 = \frac{1}{2}(\mu_L - \mu_R)$ and a magnetic field are incorporated as boundary conditions for the gauge fields.

Boundary condition $L_\mu(\infty) = R_\mu(\infty), \partial_z L_\mu(\infty) = -\partial_z R_\mu(\infty)$ at $z = \infty$ is an adaptation of an analytical continuation of an analogous condition in the chirally broken Sakai-Sugimoto model. Action, estimated on-shell for the solutions of the E.o.M.'s for the gauge fields in the chirally symmetric phase ($|X| = 0$), yields:

$$\frac{\delta S[L,R]}{\delta L_3(z=0)} = \mathcal{J}_L, \quad \frac{\delta S[L,R]}{\delta R_3(z=0)} = \mathcal{J}_R,$$

$$\mathcal{J} = \mathcal{J}_L + \mathcal{J}_R = \frac{N_c}{3\pi^2} B\mu_5.$$

Anomalies and the Bardeen counterterm

In our setup there are two external gauge fields on the boundary – $V_\mu(z=0)$ and $A_\mu(z=0)$. $V_\mu(z=0)$ corresponds to $e\times$ an external electromagnetic field and provides μ, while a nonzero $A_\mu(z=0)$ accounts for μ_5. It has been pointed out in [A. Rebhan, A. Schmitt and S. A. Stricker, 2010] that the divergence of the vector current

$$\partial_\mu \mathcal{J}^\mu = -\frac{N_c}{24\pi^2} F^V_{\mu\nu} \tilde{F}^{A\,\mu\nu}$$

has to be compensated for by a local counterterm

$$S_{Bardeen} = c \int d^4x \epsilon^{\mu\nu\rho\sigma} L_\mu R_\nu (F^L_{\rho\sigma} + F^R_{\rho\sigma}), \qquad (4)$$

with an appropriate choice of the constant c.
$S_{Bardeen}$ may be considered as a product of holographic renormalization. Whether it needs to be taken into account remains unclear.
In our model $c = -\dfrac{N_c}{12\pi^2}$ and

$$\mathcal{J}_{subtracted} = \mathcal{J} + \mathcal{J}_{Bardeen} = \frac{N_c}{3\pi^2} B\mu_5 + \left(-\frac{N_c}{12\pi^2}\right) \times 4B\mu_5 = 0. \qquad (5)$$

Discussion of the relevance of the Bardeen counterterm

It was suggested by V. Rubakov [arXiv: 1005.1888[hep-ph]] that the counterterm has to be excluded from the calculation, since it fixes the anomaly of the vector current only in the presence of a real dynamical axial gauge field, while in our case we are dealing with a constant axial chemical potential, which is different from a constant temporal component of an axial gauge field.
In the absence of this counterterm the CME current in the strong coupling regime agrees exactly with the weak coupling limit (as it will be demonstrated below).

How to formally distinguish between the two aforementioned cases in holography is a problem still open to discussions.

Scalar sector

The 5D scalar field $X = |X|e^{i\pi/f_\pi}$ interacts with the gauge fields via the covariant derivatives, thus inducing an interaction between π and the gauge field A_M.

- Usually the 4D pion is associated with a holonomy $\int A_z dz$ and the $\pi \to \gamma\gamma$ decay is determined by a part of the CS action $\int dz d^4x\, A_z F^V_{\mu\nu} F^{V\mu\nu}$. In the $A_z = 0$ gauge we have to reintroduce pion into the CS term.
- CS action is gauge invariant up to a surface term which is nonzero in our setup. In order to make it explicitly invariant we introduce 2 scalars:

$$S_{CS} = \frac{N_c}{24\pi^2}\left(\int L \wedge dL \wedge dL - \int R \wedge dR \wedge dR\right)$$

$$\to \quad \frac{N_c}{24\pi^2}\left(\int (L + d\phi_L) \wedge dL \wedge dL - \int (R + d\phi_R) \wedge dR \wedge dR\right)$$

Pseudoscalar contribution to the effect

- which under gauge transformations $L \to L + d\alpha_L$, $R \to R + d\alpha_R$ transform as $\phi_{L,R} \to \phi_{L,R} - \alpha_{L,R}$.
- $f_\pi \left(\phi_R - \phi_L\right)$ may be associated with the five-dimensional pion in the gauge in which A_z is set to zero.
- In the D3/D7 models an R-symmetry chemical potential causes the D7 branes to rotate with an angular speed μ_R, so that the phase of the scalar field is $i\mu_R t$.
- $\phi_{L,R}(z = 0) = \mu_{L,R} t$.
- This yields another contribution to the CME:

$$\mathcal{J}_{\phi AA} = \frac{N_c}{6\pi^2} B\mu_5.$$

Summary table

Here is a summary of all the contributions to the CME:

Term in the action	Yang–Mills		Chern–Simons		Scalars in CS
	bulk	boundary	bulk	boundary	
Contribution to the current	$-\frac{1}{3}\frac{N_c}{2\pi^2}B\mu_5$	$\frac{1}{3}\frac{N_c}{2\pi^2}B\mu_5$	$\frac{1}{3}\frac{N_c}{2\pi^2}B\mu_5$	$\frac{1}{3}\frac{N_c}{2\pi^2}B\mu_5$	$\frac{1}{3}\frac{N_c}{2\pi^2}B\mu_5$

Action taken into account	Total	Total without scalars
Resulting current, in terms of $\frac{N_c}{2\pi^2}B\mu_5$	1	$\frac{2}{3}$

Summary

- We have presented a calculation of the CME in soft-wall AdS/QCD.
- The results differ from those in the Sakai-Sugimoto model due to the presence of scalar fields.
- Those scalars act as 'catalysts' for the effect, only triggering it, while its magnitude is defined by the Chern-Simons action.
- The nature of the effect remains topological and it does not depend either on the dilaton or on the metric or on the details of the scalar Lagrangian.
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Outline

The Debye mass

- Let us consider a hot quark-gluon plasma in a deconfined regime but still at strong coupling: ($1\text{ GeV} \gtrsim T \gtrsim 200\text{ MeV} \approx T_c \sim \Lambda_{QCD}$).
- The electric charges are screened: $V(r) \sim \frac{e^{-m_D r}}{r}$.
- One-loop QED result is $m_D^2 = \frac{e^2 T^2}{3}$ [H.A. Weldon, 1982].
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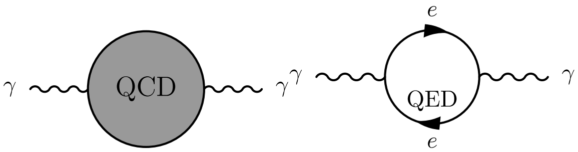
Debye mass definition



$$\Pi_{\mu\nu}(\omega,\vec{k}) = i\int d^4x\ \langle J_\mu(0)J_\nu(x)\rangle_{ret}\, e^{i\omega x_0 - i\vec{k}\vec{x}},$$

$$m_D^2 = e_q^2\Pi_{00}(\omega=0,\vec{k}^2=-m_D^2),$$

$$m_{D\ Mag}^2 = e_q^2\Pi_{33}(\omega=0,\vec{k}=-m_{D\ Mag}^2).$$

Up to α_{em} we are dealing with $\Pi_{\mu\nu}(\omega=0,\vec{k}=0)$.

Nonzero temperatures in holography

We will use the same action as before

$$
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with a metric tensor

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ds^2 = \frac{r^2}{\ell^2}\left(f_{BH}(r)dt^2 - dx_i dx^i\right) - \frac{\ell^2}{r^2}\frac{dr^2}{f_{BH}(r)}.
$$

Here $f_{BH}(r) = 1 - \frac{r_0^4}{r^4}$, $T = \frac{r_0}{\pi \ell^2}$, $r = \frac{\ell^2}{z}$.

Adjusting the model to nonzero temperatures

- We Wick-rotate the 4D space,
- make the time periodical with $\beta = T^{-1}$,
- look at the geometry near the horizon $r = r_0$,
- there will be a conical singularity unless $\beta = \frac{\pi \ell^2}{r_0}$.
- Retarded Green function – only in-falling waves at the horizon: regular at $r = r_0$ in accompanying Eddington–Finkelstein coordinates.
- When $\omega = 0$ for the solution – that is the same as imposing regularity conditions in coordinates at infinity.
- At the horizon $L_0(r_0) = R_0(r_0) = 0$ due to $g_{00}(r_0) = 0$.

Results at zero magnetic field

If **B** $= 0$ the Chern–Simons part of the action is strictly zero and the only nonzero field is:

$$V_0(r) = \frac{1}{2}(L_0(r) + R_0(r)) = \mu\left(1 - \frac{r_0^2}{r^2}\right)$$

and

$$m_D^2 = \frac{N_c}{3} e_q^2 T^2.$$

The non-perturbative QCD calculation is similar to the leading term of the QED perturbative series expression. Similarly

$$m_{D\ Mag} = 0$$

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If $\mathbf{B} = 0$ the Chern–Simons part of the action is strictly zero and the only nonzero field is:

$$V_0(r) = \frac{1}{2}(L_0(r) + R_0(r)) = \mu\left(1 - \frac{r_0^2}{r^2}\right)$$

and

$$m_D^2 = \frac{N_c}{3} e_q^2 T^2.$$

The non-perturbative QCD calculation is similar to the leading term of the QED perturbative series expression. Similarly

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The Debye mass in strong magnetic fields

If $\mathbf{B} = F_{12}^V \neq 0$ the Chern–Simons part of the action plays its role. Due to its Lorentz structure:

- $F_{12}^V(r) \equiv \mathbf{B}$ is a solution to the e.o.m.
- There is a mixing between the pairs V_0 and A_3, V_3 and A_0.
- These mixings are $\propto \mathbf{B}$.
- The Chern–Simons action is effectively bilinear.

Solutions

The solutions of the e.o.m. are expressed in terms of Legendre functions $P_\nu(x), Q_\nu(x)$ of the 1st and 2nd order respectively with a parameter $\nu = \frac{\sqrt{1-9e_q^2\mathbf{B}^2\ell^4/r_0^4}-1}{2}$ that are regular and single-valued at $|x| < 1$, although $Q_\nu(x)$ has a branching point at $x = 1$. Accounting for the boundary conditions we get:

$$
\begin{aligned}
V_0(r) &= \frac{\mu}{\nu\, P_\nu^{-1}(0)} \left(\frac{r_0^2}{r^2}\, P_\nu\left(r_0^2/r^2\right) - P_{\nu+1}\left(r_0^2/r^2\right) \right), \\
A_3(r) &= \frac{\mu}{P_\nu^{-1}(0)\sqrt{-\nu(\nu+1)}} \left(P_\nu(r_0^2/r^2) - P_\nu(0) \right).
\end{aligned}
$$

The Debye mass in a strong field

Variation of the action with respect to the source μ gives

$$m_D^2 = e_q^2 \frac{N_c}{3} \; T^2 \; F\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{9 e_q^2 \mathbf{B}^2}{\pi^4 T^4}} \right), m_{D\ Mag} = 0$$

$$F(\nu) \equiv \frac{P_\nu(0)}{P_\nu^{-1}(0)} = \frac{2\Gamma\left(1-\nu/2\right)\Gamma\left(3/2+\nu/2\right)}{\Gamma\left(1+\nu/2\right)\Gamma\left(1/2-\nu/2\right)}.$$

In a strong magnetic field, $e_q \mathbf{B} \gg T^2$,

$$m_D^2 = e_q^2 \frac{N_c}{2\pi^2} \; |e_q \mathbf{B}|.$$

The Graph

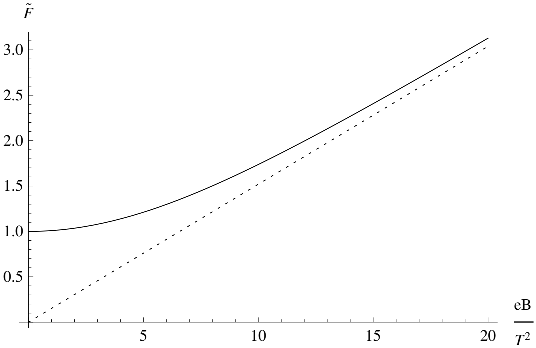

Figure: The function $\tilde{F} = F\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{9e_q^2\mathbf{B}^2}{\pi^4 T^4}}\right)$ (solid) vs its strong field asymptotics (dashed).

Values at the LHC and RHIC

Quark-gluon plasma that is created during heavy-ion collisions at RHIC and at the LHC.
$T \approx 2T_c = 330 \pm 20\ \mathrm{MeV}$ and $|e\mathbf{B}| \approx m_\pi^2 \approx 2 \times 10^4\ \mathrm{MeV}^2$ for RHIC and
$T \approx 4 - 5 \cdot T_c = 750 \pm 120\ \mathrm{MeV}$ and
$|e\mathbf{B}| \approx 15 m_\pi^2 \approx 3 \times 10^5\ \mathrm{MeV}^2$ for the LHC, we obtain

$$
\begin{aligned}
m_D^2 &= (82 \pm 3)^2\ \mathrm{MeV}^2 \text{ at RHIC and} \\
m_D^2 &= (185 \pm 35)^2\ \mathrm{MeV}^2 \text{ at the LHC.}
\end{aligned}
$$

Discussion

- The Debye mass due to strong interactions has been calculated to all orders in the magnetic field.
- In the strong field limit holographic calculation of m_D gives a result similar to the weak-coupling QED [J. Alexandre, 2001].
- $m_{D\ Mag} = 0$ even in the presence of the magnetic field to all orders in **B**.
- This is true, however, only up to $1/N_c$ corrections.
- The same holds for Π_{11} and Π_{22}.
- The similarity of the dynamics of strongly coupled QCD and weakly coupled QED in large external magnetic fields is a nontrivial phenomenon, which was observed also in [Son, Thompson, 2008].

Outline

Magnetic Susceptibility of the Chiral Condensate

- Introduced in the framework of QCD Sum Rules [B. L. Ioffe, A. V. Smilga]
- $\langle \bar{q}\sigma_{\mu\nu}q\rangle_F = \chi \langle \bar{q}q\rangle F_{\mu\nu}$, where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.
- Measures induced tensor current in the QCD vacuum

Different approaches to χ

- $\chi = -\frac{N_c}{4\pi^2 f_\pi^2} = -8.9\,\mathrm{GeV}^{-2}$ - OPE of the $\langle VVA \rangle$ correlator and pion dominance [A. Vainshtein]
- Sum rule fit $\chi = -3.15 \pm 0.30\,\mathrm{GeV}^{-2}$
- Vector dominance $\chi = -(3.38 \div 5.67)\,\mathrm{GeV}^{-2}$ [Balitsky, Yung, Kogan ...]

Possible Alternative Derivations of χ

There is a significant discrepancy in the numerical results. It is natural to start looking elsewhere, e.g. holography, to identify the missing ingredients

- Holographic calculation of $\langle VVA \rangle$: $\chi \sim -11.5\,\mathrm{GeV}^{-2}$ [A. Gorsky, A. Krikun]. Vainshtein relation isn't exact, but fulfilled to good accuracy.
- Holographic Son–Yamamoto relations: χ agrees with Vainshtein. Assumed to be valid at any momentum transfer. No field-theoretical derivation.
- A direct holographic calculation - motivated by enhanced AdS/QCD models [Cappiello, Cata, D'Ambrosio; Domokos, Harvey, Royston; Alvares, Hoyos, Karch] that take into account the 1^{+-} mesons.
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The holographic action

$$\begin{aligned}\mathcal{S}_{5D} &= \int d^5x\sqrt{-g}\,\mathrm{Tr}\left\{-\frac{1}{4g_5^2}\left(F_L^2+F_R^2\right)+g_X^2\left(|DX|^2-m_X^2|X|^2\right)\right. \\ &+ \frac{\lambda}{2}\left(X^+F_LB+BF_RX^+ + \mathrm{c.c.}\right) \\ &- \left. 2g_B\left(\frac{i}{6}\frac{\epsilon^{MNPQR}}{\sqrt{-g}}\left(B_{MN}H^+_{PQR}-B^+_{MN}H_{PQR}\right)+m_B|B|^2\right)\right\}.\end{aligned}$$

$H = DB = dB - iL\wedge B + iB\wedge R,$

$$\bar{q}_{R\bar{f}}\,q_L^f \leftrightarrow X_{\bar{f}}^f\,, \qquad \bar{q}_{R\bar{g}}\gamma_\mu q_R^{\bar{f}} \leftrightarrow R_{\mu\,\bar{g}}^{\bar{f}}\,,$$

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Rearranging the Degrees of Freedom

- In 4D, $\bar{q}\sigma^{\mu\nu}\gamma_5 q = \dfrac{i}{2}\epsilon^{\mu\nu}{}_{\lambda\rho}\bar{q}\sigma^{\lambda\rho}q$
- From the holographic point of view, this condition is ensured by the fact that the kinetic term for $B_{\mu\nu}$ is of the first order in derivatives, which leads to its complex self-duality.
- The "double counting" of the degrees of freedom that arises after we have introduced a complex tensor field is compensated by constraints imposed on half of them.

$$\bar{q}q \leftrightarrow X_+ \,, \qquad \frac{1}{\sqrt{2}}\bar{q}\sigma_{\mu\nu}q \leftrightarrow B_{+\mu\nu} \,,$$

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Classical Equations of Motion and Their Solution

According to the prescription, we have to solve the classical E.o.M.'s

$$\left(\partial_z^2 + \frac{1}{z}\partial_z - \frac{1}{z^2} - \partial_\mu\partial^\mu\right)(B_\pm)_{12} = -\frac{\lambda}{8g_B}\frac{1}{z^2}X_\pm (F_V)_{12}\,,$$

$$\left(\partial_z^2 - \frac{3}{z}\partial_z + \frac{3}{z^2} - \partial_\mu\partial^\mu\right)X_\pm = -\frac{2\lambda}{g_X^2}z^2 (F_V)_{12}(B_\pm)_{12}\,.$$

and calculate

$$\langle\bar{q}q\rangle \propto \left.\frac{\delta\mathcal{S}}{\delta X_+}\right|_{z=0} \qquad \langle\bar{q}\sigma_{\mu\nu}q\rangle \propto \left.\frac{\delta\mathcal{S}}{\delta B_{+\mu\nu}}\right|_{z=0}$$

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The Solution

- A most general property – the scalar and tensor degrees of freedom X_+, B_{+12} decouple from the pseudoscalar and pseudotensor X_-, B_{-12}, thus forming two independent sectors.
- The solutions for X and B are expressed in terms of Bessel and Neumann functions of $|\mathbf{B}|/q^2$ and qz (where $q = \sqrt{q_\mu q^\mu}$ is the momentum and $|\mathbf{B}| = (F_V)_{12}$ is the magnetic field) or their analytical continuations into the complex plane.

Magnetization: results and discussion

We are able to determine the magnetization $\mu(\mathbf{B}) = \dfrac{\langle \bar{q}\sigma_{12}q\rangle}{\langle \bar{q}q\rangle}$:

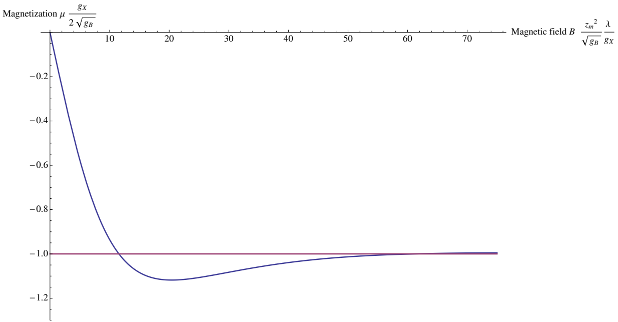

Figure: Magnetization of the chiral condensate $\mu(\mathbf{B})$ as a function of the magnetic field (blue) vs its strong field asymptotics (red).

Magnetization: results and discussion

- is linear in $|\mathbf{B}|$ when the field is weak,
- becomes a negative constant at $\mathbf{B} \sim z_m^{-2} \sim \Lambda_{QCD}^2$.
- Constant asymptotic is to be expected. In large magnetic fields 4D reduces to 2D and the tensor chiral condensate is kinematically reduced to a scalar one.
- The result points to the fact that not only the lowest Landau level plays a significant role: $\lim_{\mathbf{B}\to\infty} \mu(\mathbf{B}) \neq -1$.

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Magnetic susceptibility: results and discussion

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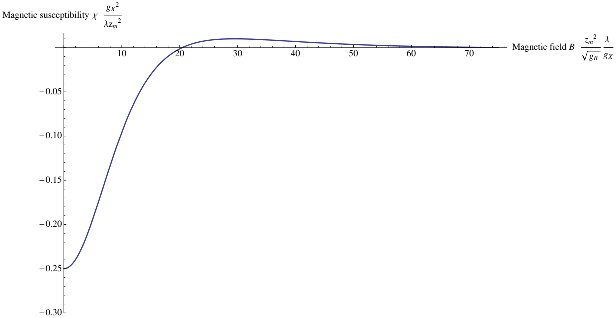

Figure: Magnetic susceptibility of the chiral condensate $\mu(\mathbf{B})$ as a function of the magnetic field.

Susceptibility: results and discussion

- possesses a quadratic behavior $\chi \sim -(const - \mathcal{O}(|\mathbf{B}|^2)$ when the field is weak,
- tends to 0 at $\mathbf{B} \sim z_m^{-2} \sim \Lambda_{QCD}^2$.
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- Some additional unknown factors may contribute, requires more investigation.

Discussion

- A non-perturbative calculation of $\mu(\mathbf{B})$ and $\chi(\mathbf{B})$ to all orders in the magnetic field has been carried out.
- It has been performed in a holographic model enhanced by the inclusion of a tensor field – allows for a direct calculation.
- Our results reproduce the general properties both of the susceptibility and of the magnetization – the weak-field expansion of the former and the negative constant asymptotic of the latter.
- The numerical discrepancy may indicate the incompleteness of the model.

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Outline

Manifestation of the broken conformal symmetry in QCD is the nonzero gluon condensate:

$$\langle \alpha_s \mathrm{tr}(G^2) \rangle \sim (200\ \mathrm{MeV})^4.$$

Measured on lattices by studying small Wilson loops:

$$W(\mathcal{C}) = \frac{1}{N_c} \langle tr\ P \exp[\oint_{\mathcal{C}} igA_\mu dx^\mu] \rangle = 1 - \frac{1}{48} \frac{\langle \alpha_s \mathrm{tr}(G^2) \rangle}{N_c} S^2 + \ldots.$$

[M. A. Shifman, 1980]

In gauge/gravity duality the gluon condensate:

- Is the VEV of an operator, therefore, is the normalizable mode of the dilaton: $\phi = \phi_0 + \phi_4 z^4$.
- Can be read off of the Wison loop VEV.
- We can use Maldacena's prescription: the exponent of the area of a minimal surface in 5D spanning the Wilson contour on the AdS boundary: $\langle W(\mathcal{C})\rangle = e^{-Area(\mathcal{C})}$ [J. Maldacena, 1998].
- The dilaton enters the string frame metric that is used to calculate the surface area.

The Liu–Tseytlin model

A simple supergravity solution with a nontrivial dilaton background – a D-instanton smeared on the D3-branes:

$$ds^2_{D3}|_{str} = \frac{\ell^2}{z^2}\sqrt{h_{-1}}(dx^\mu dx_\mu + dz^2 + z^2 d\Omega_5^2),$$

$$e^\phi = g_s h_{-1}, \quad h_{-1} = 1 + \frac{q}{\lambda} z^4, \quad \phi_4 = \frac{q}{\lambda} = \frac{\pi^2}{\sqrt{2}\lambda}\frac{\langle \alpha_s \mathrm{tr}(G^2)\rangle}{N_c},$$

$$\ell^4 = 4\pi g_s N_c l_s^4, \quad g^2_{YM} = 4\pi g_s, \quad \lambda = g^2_{YM} N_c.$$

The Gross–Ooguri Phase Transition

Let us study the Wilson loop correlators:

- two parallel loops of equal radii R and separated by a distance l in a transversal direction.
- String worldsheet stretched over two Wilson contours:

$$\langle W(\mathcal{C}_1)W(\mathcal{C}_2)\rangle = \exp\left(-S_{NG}\right).$$

- Two types of worldsheets are possible – connected and disconnected.

The Gross–Ooguri Phase Transition

- At small distances l the connected solution is preferable (has a smaller area).
- At $l = l_c$ the disconnected worldsheet becomes preferable: $S_{conn.}(l_c) = S_{disc.}$.
- At this point the correlator undergoes a first order phase transition – and vanishes. (N_c-suppressed propagating gravitational modes make it a crossover.)
- At a greater distance $l = l_*$ the connected solution stops existing.
- For pure $AdS_5 \times S^5$ the critical value equals $l_c = 0.91R$ and the connected solution becomes unstable at $l_* = 1.04R$.

The Gross–Ooguri Phase Transition

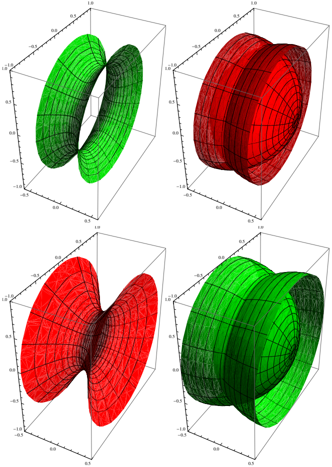

Figure: A schematic representation of the Gross–Ooguri phase transition. Classically preferable worldsheet solutions are shaded green.

Phase transition shift

- Treating the worldsheet solution perturbatively in $\phi_4 R^4$: $S = S^0 + S^1$. Similarly $l_c = l_{c0} + l_{c1}$.
- They are determined by:

$$S^0_{conn.}(l_{c0}) = S^0_{disc.},$$
$$l_{c1} = \frac{S^1_{disc.} - S^1_{conn.}(l_{c0})}{\partial S^0_{conn.}/\partial l(l_{c0})}.$$

- This gives

$$l_{c1} = (6.05 \pm 0.06) \times 10^{-2} \phi_4 R^5.$$

Discussion

- Presence of the gluon condensate drives the point of the Gross–Ooguri transition to larger distances between the loops.
- This is due to the fact that the normalizable mode of the dilaton makes the area of the disconnected surface larger.
- The case of concentric Wilson loops – when condensate is large enough the Gross–Ooguri phase transition changes its order. There is a jump in the area of the minimal surface.