Competition and duality correspondence between chiral and superconducting channels in (2+1)-dimensional four-fermion models

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Introduction

Introduction

Models with four-fermion interactions

It is well known that relativistic quantum field models with four-fermion interactions serve as effective theories for low energy considerations of different real phenomena in a variety of physical branches:

- Meson spectroscopy, neutron star and heavy-ion collision physics are often investigated in the framework of (3+1)-dimensional 4F theories.
- Physics of (quasi)one-dimensional organic Peierls insulators (polyacetylene) is well described in terms of the (1+1)-dimensional 4F Gross-Neveu (GN) model.
- The quasirelativistic treatment of electrons in planar systems like high-temperature superconductors or in graphene is also possible in terms of (2+1)-dimensional GN models.

It is important to note that the low-dimensional versions of the 4F theories provide just a method to describe solid state matter and to check the theoretical mechanism experimentally.

Introduction

Chiral symmetry breaking vs. superconductivity competition and duality correspondence

In this talk we demonstrate that there exists a dual correspondence between chiral symmetry breaking phenomenon and superconductivity in the framework of some (2+1)-dimensional 4F theories.

Before now, such a duality correspondence was a well-known feature of only some (1+1)-dimensional 4F theories:

- In 1977 Ojima and Fukuda mentioned that as a result of Pauli–Gürsey symmetry the chiral phase in (1+1)–dimensional 4F model could be interpreted as a difermion supercondicting phase. [Prog. Theor. Phys. **57**, 1720 (1977)]
- In 2003 Thies showed that in addition to the duality between condensates there is also duality between fermion number-μ and chiral charge-μ_5 chemical potentials. [Phys. Rev. D **68**, 047703 (2003)]
- In 2014 Ebert et al. investigated chiral symmetry breaking vs. superconductivity competition taking into account μ, μ_5 - chemical potentials and inhomogeneous patterns for the condensates. The duality correspondence was also investigated in details. [Phys. Rev. D **90**, 045021 (2014)]

It is worth to note that in recent years properties of media with nonzero chiral chemical potential μ_5, i.e. chiral media, attracted considerable interest. In nature, chiral media might be realized in heavy-ion collisions, compact stars, condensed matter systems, etc.

The model and its thermodynamical potential

Lagrangian of the model

$$\mathcal{L} = \bar\psi_k\Big[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5\Big]\psi_k + \frac{G_1}{N}(4F)_{ch} + \frac{G_2}{N}(4F)_{sc}, \quad \text{where}$$

$$(4F)_{ch} = \left(\bar\psi_k\psi_k\right)^2 + \left(\bar\psi_k i\gamma^5\psi_k\right)^2, \quad (4F)_{sc} = \Big(\psi_k^T C\psi_k\Big)\Big(\bar\psi_j C\bar\psi_j^T\Big).$$

Definitions

- ψ_k $(k = 1, ..., N)$ – fundamental multiplet of the $O(N)$
- ψ_k – four-component (reducible) Dirac spinor
- γ^ν $(\nu = 0, 1, 2)$ and γ^5 – gamma-matrices
- $C \equiv \gamma^2$ – charge conjugation matrix

Lagrangian of the model

$$\mathcal{L} = \bar{\psi}_k \Big[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5\Big]\psi_k + \frac{G_1}{N}(4F)_{ch} + \frac{G_2}{N}(4F)_{sc}, \quad \text{where}$$

$$(4F)_{ch} = \left(\bar{\psi}_k\psi_k\right)^2 + \left(\bar{\psi}_k i\gamma^5\psi_k\right)^2, \quad (4F)_{sc} = \Big(\psi_k^T C\psi_k\Big)\Big(\bar{\psi}_j C\bar{\psi}_j^T\Big).$$

Notations

- μ – fermion number chemical potential
- μ_5 – chiral (axial) chemical potential
- G_1, G_2 – coupling constants

Lagrangian of the model

$$\mathcal{L} = \bar{\psi}_k \Big[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5\Big]\psi_k + \frac{G_1}{N}(4F)_{ch} + \frac{G_2}{N}(4F)_{sc}, \quad \text{where}$$

$$(4F)_{ch} = \big(\bar{\psi}_k\psi_k\big)^2 + \big(\bar{\psi}_k i\gamma^5\psi_k\big)^2, \quad (4F)_{sc} = \Big(\psi_k^T C\psi_k\Big)\Big(\bar{\psi}_j C\bar{\psi}_j^T\Big).$$

Symmetries

- Lagrangian is invariant under transformations from the $U_V(1) \times U_{\gamma_5}(1)$ group
- Fermion number conservation group $U_V(1)$: $\psi_k \to \exp(i\alpha)\psi_k$
- Continuous chiral transformations $U_{\gamma^5}(1)$: $\psi_k \to \exp(i\alpha\gamma^5)\psi_k$
- Lagrangian is also invariant under transformations from the internal auxiliary $O(N)$ group

Gamma matrices in the four-dimensional spinor space

Irreducible representation of the $SO(2,1)$ group

$$\tilde{\gamma}^0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \tilde{\gamma}^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \tilde{\gamma}^2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that the definition of chiral symmetry is slightly unusual in (2+1)-dimensions. The formal reason is simply that there exists no other 2×2 matrix anticommuting with the Dirac matrices $\tilde{\gamma}^\nu$ which would allow the introduction of a γ^5-matrix. The important concept of chiral symmetries and their breakdown by mass terms can nevertheless be realized by considering a four-component reducible representation for Dirac fields:

Reducible representation of the $SO(2,1)$ group

$$\gamma^\mu = \begin{pmatrix} \tilde{\gamma}^\mu & 0 \\ 0 & -\tilde{\gamma}^\mu \end{pmatrix}; \quad \psi(x) = \begin{pmatrix} \tilde{\psi}_1(x) \\ \tilde{\psi}_2(x) \end{pmatrix}.$$

There exist two matrices, γ^3 and γ^5, which anticommute with all γ^μ and with themselves:

$$\gamma^3 = i \begin{pmatrix} 0\,, & I \\ -I\,, & 0 \end{pmatrix}, \quad \gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0\,, & I \\ I\,, & 0 \end{pmatrix}.$$

Duality correspondence and Pauli–Gürsey transformation

Pauli–Gürsey transformation of the fields

$$PG: \quad \psi_k(x) \longrightarrow \frac{1}{2}(1-\gamma^5)\psi_k(x) + \frac{1}{2}(1+\gamma^5)C\bar{\psi}_k^T(x).$$

Taking into account that all spinor fields anticommute with each other, it is easy to see that under the action of the PG-transformation the 4F structures of the Lagrangian are converted into themselves:

$$(4F)_{ch} \stackrel{PG}{\longleftrightarrow} (4F)_{sc},$$

and, moreover, each Lagrangian $\mathcal{L}(G_1, G_2; \mu, \mu_5)$ is transformed into another one according to the following rule:

$$\mathcal{L}(G_1, G_2; \mu, \mu_5) \stackrel{PG}{\longleftrightarrow} \mathcal{L}(G_2, G_1; -\mu_5, -\mu).$$

Semi-bosonized version of the Lagrangian

Let us introduce the semi-bosonized version of the Lagrangian that contains only quadratic powers of fermionic fields as well as auxiliary bosonic fields $\sigma(x)$, $\pi(x)$, $\Delta(x)$ and $\Delta^*(x)$:

$$\widetilde{\mathcal{L}} = \bar{\psi}_k\Big[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5 - \sigma - i\gamma^5\pi\Big]\psi_k - \\ - \frac{N(\sigma^2+\pi^2)}{4G_1} - \frac{N\Delta^*\Delta}{4G_2} - \frac{\Delta^*}{2}[\psi_k^T C\psi_k] - \frac{\Delta}{2}[\bar{\psi}_k C\bar{\psi}_k^T], \quad \text{where}$$

Bosonic fields

$$\sigma = -2\frac{G_1}{N}(\bar{\psi}_k\psi_k), \quad \pi = -2\frac{G_1}{N}(\bar{\psi}_k i\gamma^5\psi_k);$$

$$\Delta = -2\frac{G_2}{N}(\psi_k^T C\psi_k), \quad \Delta^* = -2\frac{G_2}{N}(\bar{\psi}_k C\bar{\psi}_k^T);$$

- σ and π – are real fields
- Δ and Δ^* – are Hermitian conjugated complex fields

Properties of the bosonic fields

Under the chiral $U_{\gamma^5}(1)$ group the fields Δ, Δ^* are singlets, but the fields σ, π are transformed in the following way:

$$U_{\gamma^5}(1): \qquad \sigma \to \cos(2\alpha)\sigma + \sin(2\alpha)\pi, \\ \pi \to -\sin(2\alpha)\sigma + \cos(2\alpha)\pi$$

Clearly, all the fields are also singlets with respect to the auxiliary $O(N)$ group, since the representations of this group are real. Moreover, with respect to the parity transformation P:

$$P: \quad \psi_k(t,x,y) \to i\gamma^5\gamma^1\psi_k(t,-x,y), \qquad k = 1, ..., N,$$

the fields $\sigma(x)$, $\Delta(x)$ and $\Delta^*(x)$ are even quantities, i.e. scalars, but $\pi(x)$ is a pseudoscalar.

- If $\langle\Delta\rangle \neq 0$, then the Abelian fermion number conservation $U_V(1)$ symmetry of the model and parity invariance is spontaneously broken down and the superconducting phase is realized in the model.
- If $\langle\sigma\rangle \neq 0$ then the continuous $U_{\gamma^5}(1)$ chiral symmetry of the model is spontaneously broken.

Effective action

The effective action $\mathcal{S}_{\rm eff}(\sigma, \pi, \Delta, \Delta^*)$ of the considered model is expressed by means of the path integral over fermion fields:

$$\exp(i\mathcal{S}_{\rm eff}(\sigma, \pi, \Delta, \Delta^*)) = \int \prod_{l=1}^{N}[d\bar{\psi}_l][d\psi_l]\exp\Big(i\int \widetilde{\mathcal{L}}\, d^3x\Big),$$

where

$$\mathcal{S}_{\rm eff}(\sigma, \pi, \Delta, \Delta^*) = -\int d^3x \left[\frac{N}{4G_1}(\sigma^2+\pi^2) + \frac{N}{4G_2}\Delta\Delta^*\right] + \widetilde{\mathcal{S}}_{\rm eff}, \quad \text{and}$$

$$e^{(i\widetilde{\mathcal{S}}_{\rm eff})} = \int [d\bar{\psi}_l][d\psi_l] e^{\left\{i\int\left[\bar{\psi}(\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5 - \sigma - i\gamma^5\pi)\psi - \frac{\Delta^*}{2}(\psi^T C\psi) - \frac{\Delta}{2}(\bar{\psi}C\bar{\psi}^T)\right]d^3x\right\}}$$

Henceforth we omit the index k from quark fields.

Effective action

The effective action $\mathcal{S}_{\text{eff}}(\sigma, \pi, \Delta, \Delta^*)$ of the considered model is expressed by means of the path integral over fermion fields:

$$\exp(i\mathcal{S}_{\text{eff}}(\sigma, \pi, \Delta, \Delta^*)) = \int \prod_{l=1}^{N}[d\bar{\psi}_l][d\psi_l] \exp\Big(i \int \widetilde{\mathcal{L}}\, d^3x\Big),$$

The ground state expectation values $\langle\sigma\rangle$, $\langle\Delta\rangle$, etc. of the composite bosonic fields are determined by the saddle point equations:

$$\frac{\delta\mathcal{S}_{\text{eff}}}{\delta\sigma} = 0, \quad \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\pi} = 0, \quad \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\Delta} = 0, \quad \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\Delta^*} = 0.$$

Notations for simplicity:

$$\langle\sigma\rangle \equiv M, \quad \langle\pi\rangle \equiv \pi, \quad \langle\Delta\rangle \equiv \Delta, \quad \langle\Delta^*\rangle \equiv \Delta^*.$$

Thermodynamic potential (TDP)

In the leading order of the large-N expansion TDP is defined by the following expression:

$$\int d^3x\Omega(M,\pi,\Delta,\Delta^*) = -\frac{1}{N}\mathcal{S}_{\rm eff}\{\sigma,\pi,\Delta,\Delta^*\}\Big|_{\sigma=\langle\sigma\rangle,\Delta=\langle\Delta\rangle,\ldots}$$

The TDP is invariant with respect to chiral $U_{\gamma^5}(1)$ symmetry group. So, it depends on the quantities M and π through the combination $M^2+\pi^2$. Moreover, without loss of generality, one can suppose that $\langle\pi\rangle\equiv\pi=0$. Thus, to find the other ground state expectation values $\langle\sigma\rangle$ etc., it is enough to study the global minimum point of the TDP $\Omega(M,\Delta,\Delta^*)$:

$$\Omega(M,\Delta,\Delta^*)\equiv\Omega(M,\pi,\Delta,\Delta^*)\Big|_{\pi=0}$$

Calculation of the TDP

Taking into account all simplifications, we have the following form for the TDP:

$$\int d^3x\Omega(M,\Delta,\Delta^*) = \int d^3x\left(\frac{M^2}{4G_1}+\frac{\Delta\Delta^*}{4G_2}\right) +$$
$$+\frac{i}{N}\ln\left(\int[d\bar{\psi}_l][d\psi_l]\exp\left(i\int d^3x\Big[\bar{\psi}D\psi\ -\frac{\Delta}{2}(\psi^T C\psi)-\frac{\Delta^*}{2}(\bar{\psi}C\bar{\psi}^T)\Big]\right)\right),$$

$$\text{where}\quad D=\gamma^\rho i\partial_\rho+\mu\gamma^0+\mu\gamma^0\gamma^5-M.$$

To proceed further, let us point out that without loss of generality the quantities Δ,Δ^* might be considered as real ones. So, in what follows we will suppose that $\Delta=\Delta^*\equiv\Delta$, where Δ now is already a real quantity.

Calculation of the TDP

After path integration we have for the TDP the following expression:

$$\Omega(M,\Delta)=\frac{M^2}{4G_1}+\frac{\Delta^2}{4G_2}+\frac{i}{2}\sum_{\eta=\pm}\int\frac{d^3p}{(2\pi)^3}\ln P_\eta(p_0),\quad \text{where}$$

$$P_\eta(p_0)=a+\eta b p_0-2cp_0^2+p_0^4,\quad \text{and}$$

$$a=(\mu_5^2-\mu^2+M^2-\Delta^2)^2-2|\vec{p}|(\mu_5^2+\mu^2-M^2-\Delta^2)+|\vec{p}|^4$$
$$b=8\mu\mu_5|\vec{p}|,\quad c=\mu_5^2+|\vec{p}|^2+\mu^2+M^2+\Delta^2.$$

It is clear that the TDP is an even function of each of the quantities μ, μ_5, M, and Δ, i.e. without loss of generality we can consider in the following only $\mu\geq 0$, $\mu_5\geq 0$, $M\geq 0$, and $\Delta\geq 0$ values of these quantities.

Calculation of the TDP

Also, as a consequence of the Pauli–Gürsey transformation of the spinor fields, the TDP is invariant with respect to the so-called duality transformation:

$$\mathcal{D}: \quad G_1 \longleftrightarrow G_2, \quad M \longleftrightarrow \Delta, \quad \mu \longleftrightarrow \mu_5$$

According to the general theorem of algebra, the polynomial $P_\eta(p_0)$ can be presented in the form:

$$P_\eta(p_0) \equiv (p_0 - p_{01}^\eta)(p_0 - p_{02}^\eta)(p_0 - p_{03}^\eta)(p_0 - p_{04}^\eta), \quad \text{where}$$

p_{01}^η, p_{02}^η, p_{03}^η and p_{04}^η are the roots of this polynomial. In particular at $\Delta = 0$

$$\left(p_{01}^\eta, p_{02}^\eta\right)\Big|_{\Delta=0} = \eta\mu \pm \sqrt{M^2 + (\mu_5 - |\vec{p}|)^2}, \quad \text{and}$$
$$\left(p_{03}^\eta, p_{04}^\eta\right)\Big|_{\Delta=0} = -\eta\mu \pm \sqrt{M^2 + (\mu_5 + |\vec{p}|)^2}.$$

To obtain the roots at $M = 0$ one should simply substitute $M \to \Delta$ and $\mu \to \mu_5$.

Calculation of the TDP

The fourth-order polynomial with similar coefficients a, b, c was studied in our previous paper [Phys.Rev. **D90** (2014), 045021], where it was shown that all its roots p^η_{0i} $(i = 1, ..., 4)$ are real quantities. The roots p^η_{0i} are the energies of quasiparticle or quasiantiparticle excitations of the system.

It is possible to integrate TDP over p_0 and present it in the following form:

Unrenormalized TDP

$$\Omega^{un}(M,\Delta) \;\; = \;\; \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \frac{1}{4}\sum_{\eta=\pm}\int\frac{d^2p}{(2\pi)^2}\Big(|p^\eta_{01}| + |p^\eta_{02}| + |p^\eta_{03}| + |p^\eta_{04}|\Big).$$

The TDP is an ultraviolet divergent quantity, so one should renormalize it, using a special dependence of the bare quantities, such as the bare coupling constants $G_1 \equiv G_1(\Lambda)$ and $G_2 \equiv G_2(\Lambda)$ on the cutoff parameter Λ (Λ restricts the integration region in the divergent integrals, $|\vec{p}| < \Lambda$).

Renormalization of the TDP in the vacuum case: $\mu = 0, \mu_5 = 0$

At $\mu = 0$ and $\mu_5 = 0$ TDP (which is usually called effective potential) looks like:

$$V^{un}(M,\Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \int \frac{d^2p}{(2\pi)^2}\left(\sqrt{|\vec{p}|^2 + (M+\Delta)^2} + \sqrt{|\vec{p}|^2 + (M-\Delta)^2}\right).$$

It is useful to take into account the following asymptotic expansion at $|\vec{p}| \to \infty$:

$$\sqrt{|\vec{p}|^2 + (M+\Delta)^2} + \sqrt{|\vec{p}|^2 + (M-\Delta)^2} = 2|\vec{p}| + \frac{(M^2+\Delta^2)}{|\vec{p}|} + \mathcal{O}\left(1/|\vec{p}|^3\right).$$

Using the asymptotic expansion and integrating the effective potential over p_1 and p_2 term-by-term one can show that:

$$\begin{aligned} V^{reg}(M,\Delta) = \quad & M^2 \quad \left[\frac{1}{4G_1} - \frac{2\Lambda\ln(1+\sqrt{2})}{\pi^2}\right] + \\ & \Delta^2 \quad \left[\frac{1}{4G_2} - \frac{2\Lambda\ln(1+\sqrt{2})}{\pi^2}\right] - \frac{2\Lambda^3(\sqrt{2}+\ln(1+\sqrt{2}))}{3\pi^2} + \mathcal{O}(\Lambda^0), \end{aligned}$$

Renormalization of the TDP in the vacuum case: $\mu = 0, \mu_5 = 0$

Clearly, to cancel ultraviolet divergency the bare couples should have the following form:

$$\frac{1}{4G_1} \equiv \frac{1}{4G_1(\Lambda)} = \frac{2\Lambda\ln(1+\sqrt{2})}{\pi^2} + \frac{1}{2\pi g_1},$$
$$\frac{1}{4G_2} \equiv \frac{1}{4G_2(\Lambda)} = \frac{2\Lambda\ln(1+\sqrt{2})}{\pi^2} + \frac{1}{2\pi g_2}, \quad \text{where}$$

$g_{1,2}$ are finite and Λ-independent model parameters with dimensionality of inverse mass. Since bare couplings G_1 and G_2 do not depend on a normalization point, the same property is also valid for $g_{1,2}$.

After calculating the finite term $\mathcal{O}(\Lambda^0)$ and taking the limit $\Lambda \to \infty$, we have for the renormalized effective potential $V^{ren}(M,\Delta)$ the following expression:

$$V^{ren}(M,\Delta) \equiv \Omega^{ren}(M,\Delta)\Big|_{\mu=0,\mu_5=0} = \frac{M^2}{2\pi g_1} + \frac{\Delta^2}{2\pi g_2} + \frac{(M+\Delta)^3}{6\pi} + \frac{|M-\Delta|^3}{6\pi}$$

Renormalization of the TDP in the general case

Using the same method, after tedious but straightforward calculations, the TDP (reduced on the M-axis) can be presented in the following form:

$$F_1(M) = \frac{M^2}{2\pi g_1} + \frac{(\mu_5^2+M^2)^{3/2}}{3\pi} - \frac{\theta\left(\mu - \sqrt{M^2+\mu_5^2}\right)}{6\pi}\left[\mu^3 - 3\mu(M^2-\mu_5^2) + 2(\mu_5^2+M^2)^{3/2}\right]$$
$$-\frac{\theta\left(\sqrt{M^2+\mu_5^2}-\mu\right)}{2\pi}\left[\mu_5^2\sqrt{\mu_5^2+M^2} + \mu_5 M^2 \ln\left(\frac{\mu_5+\sqrt{\mu_5^2+M^2}}{M}\right)\right]$$
$$-\frac{\theta(\mu - M)\,\theta\left(\sqrt{M^2+\mu_5^2}-\mu\right)}{2\pi}\left[\mu_5\mu\sqrt{\mu^2-M^2} - \mu_5 M^2 \ln\left(\frac{\mu+\sqrt{\mu^2-M^2}}{M}\right)\right]$$

$$F_1(M) \equiv \Omega^{ren}(M,\Delta=0).$$

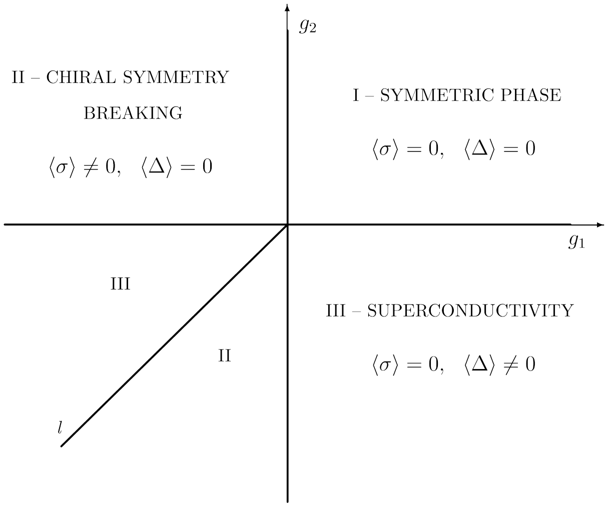
To obtain TDP reduced to Δ-axis, one should simply substitute $M \to \Delta, \mu \leftrightarrow \mu_5$:

$$F_2(\Delta) \equiv \Omega^{ren}(M=0,\Delta) = F_1(\Delta)\Big|_{g_1\to g_2,\mu\leftrightarrow\mu_5}$$

Numerical calculations

Numerical calculations of the model (Vacuum case: $\mu = 0, \mu_5 = 0$)

The (g_1, g_2)-phase portrait:

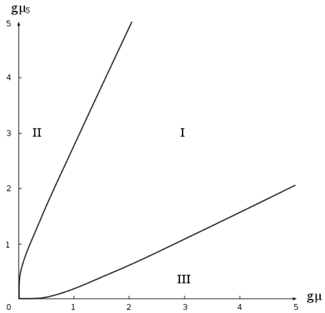


At $g_{1,2} < 0$ the line l is defined by the relation $l \equiv \{(g_1, g_2) : g_1 = g_2\}$.

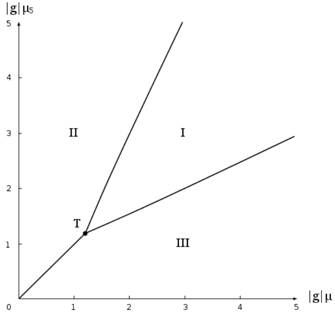
Numerical investigation of the model (Self-dual case: $g_1 = g_2$)

The (μ, μ_5)-phase portraits at fixed coupling constants:

$g_1 = g_2 \equiv g > 0$



$g_1 = g_2 \equiv g < 0$

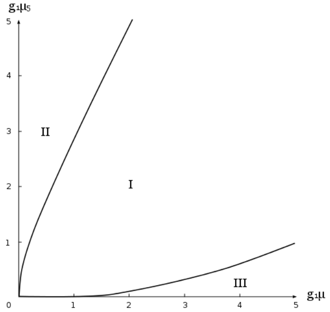


The notations I, II and III mean the symmetric, the chiral symmetry breaking (CSB) and the superconducting (SC) phases, respectively. T denotes a triple point.

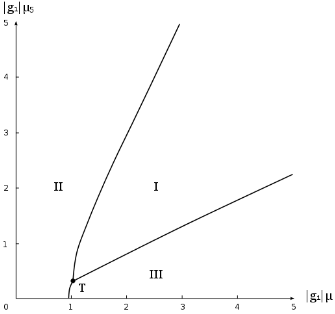
Numerical investigation of the model (General case)

The (μ, μ_5)-phase portraits at fixed coupling constants:

$g_1 > 0$ and $g_2 = 0.2 g_1$



$g_1 < 0$ and $g_2 = -2 g_1$



The notations I, II and III mean the symmetric, the chiral symmetry breaking (CSB) and the superconducting (SC) phases, respectively.

Discussions/Summary

Alternative model symmetric under $U_{\gamma^3}(1)$ - group

Each of the matrices γ^5 and $\gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^5$ can be selected as a generator for the corresponding $U_{\gamma^3}(1)$ and $U_{\gamma^5}(1)$ chiral group of spinor field transformations.

Alternatively, it is possible to construct a 4F model symmetric under $U_{\gamma^3}(1)$ continuous chiral transformations, $\psi(x) \to \exp(i\alpha\gamma^3)\psi(x)$:

$$L = \bar{\psi}_k \left[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^3\right]\psi_k + \frac{G_1}{N}(4F)_{ch} + \frac{G_2}{N}(4F)_{SC}, \quad \text{where}$$

$$(4F)_{ch} = \left(\bar{\psi}_k\psi_k\right)^2 + \left(\bar{\psi}_k i\gamma^3\psi_k\right)^2, \quad (4F)_{sc} = \left(\psi_k^T \tilde{C}\psi_k\right)\left(\bar{\psi}_j \tilde{C}\bar{\psi}_j^T\right).$$

Here $\tilde{C} = iC\gamma^3\gamma^5$ and μ is the usual particle number chemical potential. Since this Lagrangian is invariant under $U_{\gamma^3}(1)$, there exist a corresponding conserved density of chiral charge $n_3 = \sum_{k=1}^{N} \bar{\psi}_k\gamma^0\gamma^3\psi_k$ as well as its thermodynamically conjugate quantity, the chiral (or axial) chemical potential μ_3.

Alternative model symmetric under $U_{\gamma^3}(1)$ - group

Using the modified Pauli–Gürsey transformation of spinor fields:

$$\widetilde{PG}: \quad \psi_k(x) \longrightarrow \frac{1}{2}(1-\gamma^3)\psi_k(x) + \frac{1}{2}(1+\gamma^3)\widetilde{C}\bar{\psi}_k^T(x),$$

one can easily show that there is similar duality:

$$(4F)_{ch} \stackrel{\widetilde{PG}}{\longleftrightarrow} (4F)_{SC} \quad \text{and} \quad L_{\gamma^3}(G_1, G_2; \mu, \mu_3) \stackrel{\widetilde{PG}}{\longleftrightarrow} L_{\gamma^3}(G_2, G_1; -\mu_3, -\mu).$$

We have shown that the TDP for the alternative model has the following form:

$$\Omega_{\gamma^3}(M,\Delta) = \Omega_{\gamma^5}(M,\Delta)\Big|_{\mu_5 \to \mu_3}.$$

It is clear that the TDP $\Omega_{\gamma^3}(M,\Delta)$ is invariant under the following dual transformation:

$$G_1 \longleftrightarrow G_2, \quad M \longleftrightarrow \Delta, \quad \mu \longleftrightarrow \mu_3.$$

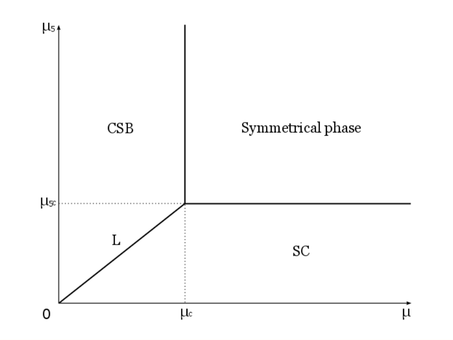
To find phase portraits of the model, it is sufficient to perform the replacement $\mu_5 \to \mu_3$.

Numerical calculations of the NJL model in (1+1) dimensions

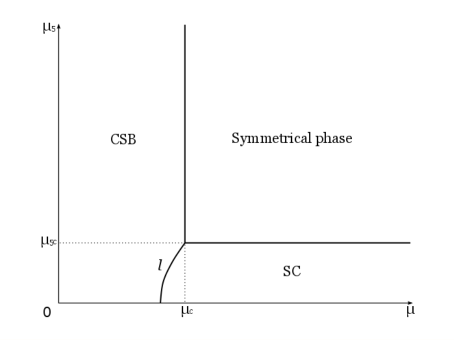
Results from PRD90 (2014), 045051 (D.Ebert et al.)

In 2014 we investigated a very similar problem in (1+1)-dimensions. But there we implied that both condensates (chiral and superconducting) have a spatial wave-like dependence. Here are two characteristic phase portraits, comparable to (2+1)-dimensional case:

Selfdual case: $g_1 = g_2$ (homogeneous case)



$g_{CSB} > g_{SC}$ (homogeneous case)



Conclusions

- Duality correspondence between CSB and SC demonstrated for (2+1)-dimensional 4F models
- For comparison and illustrations, a variety of phase portraits in the (μ, μ_5)- and (g_1, g_2) planes is shown
- Selfdual (at $\mu = \mu_5$ or at $g_1 = g_2$) phase diagrams which transform into themselves under the duality mapping
- Non-selfdual phase portraits
- The growth of the chiral chemical potential μ_5 promotes the chiral symmetry breaking, whereas particle number chemical potential μ induces superconductivity in the system.

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