Non-extensive Statistics in High Energy Physics

Trambak Bhattacharyya

BLTP JINR & University of Cape Town

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Boltzmann-Gibbs (BG) distribution is not universal.

Many anomalous natural, and social systems exist for which *BG* statistical concepts appear to be inapplicable

Some of them can be handled using the techniques of Statistical Mechanics by introducing a more general entropy called the Tsallis (a.k.a Tsallis non-extensive) entropy.

Formally, the Tsallis statistics starts by defining the generalized entropy with the help of the normalized probability of the micro-states of the system.

A S Parvan Eur. Phys. J A 53, 53 (2017)

Following is the lowest order version obtained from the exact Tsallis statistics

$$f = \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{1}{q-1}}$$

$$q \to 1 \Rightarrow f \to f_{\text{Boltzmann}}$$

I Bediaga, E M F Curado and J M de Miranda, Phys. A 286, 156 (2000); C Beck, Phys. A 286, 164 (2000); J Cleymans, D Worku, Eur. Phys. J A 48, 160 (2012)

Exponential and Tsallis: Difference in the expressions of the transverse momentum (p_{T}) spectra:

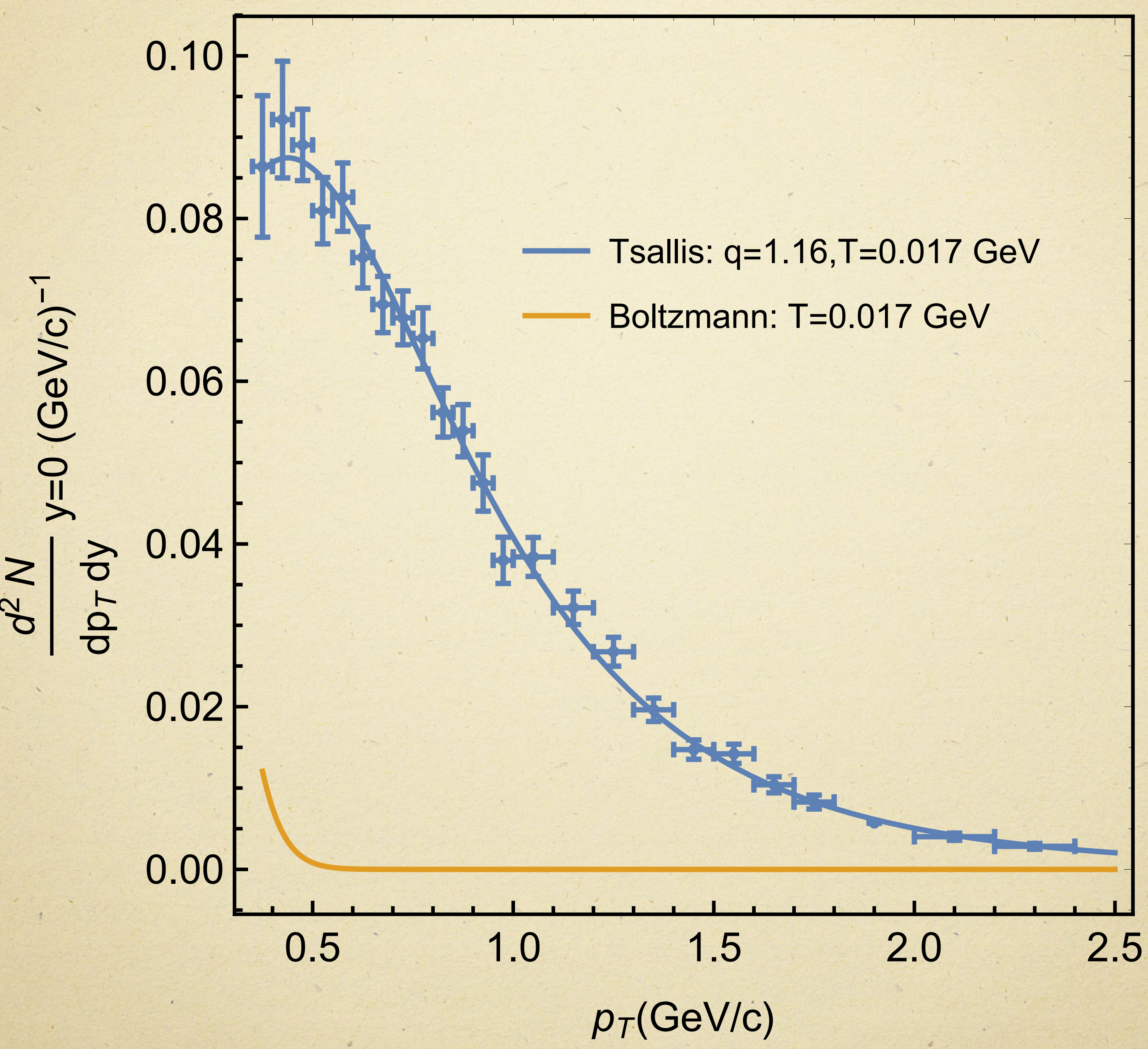
Tsallis

$$\left.\frac{d^2N}{p_T\, dp_T\, dy}\right|_{y=0} = gV\frac{m_T}{(2\pi)^2}\left[1+(q-1)\frac{m_T}{T}\right]^{-q/(q-1)}$$

Exponential (Boltzmann like)

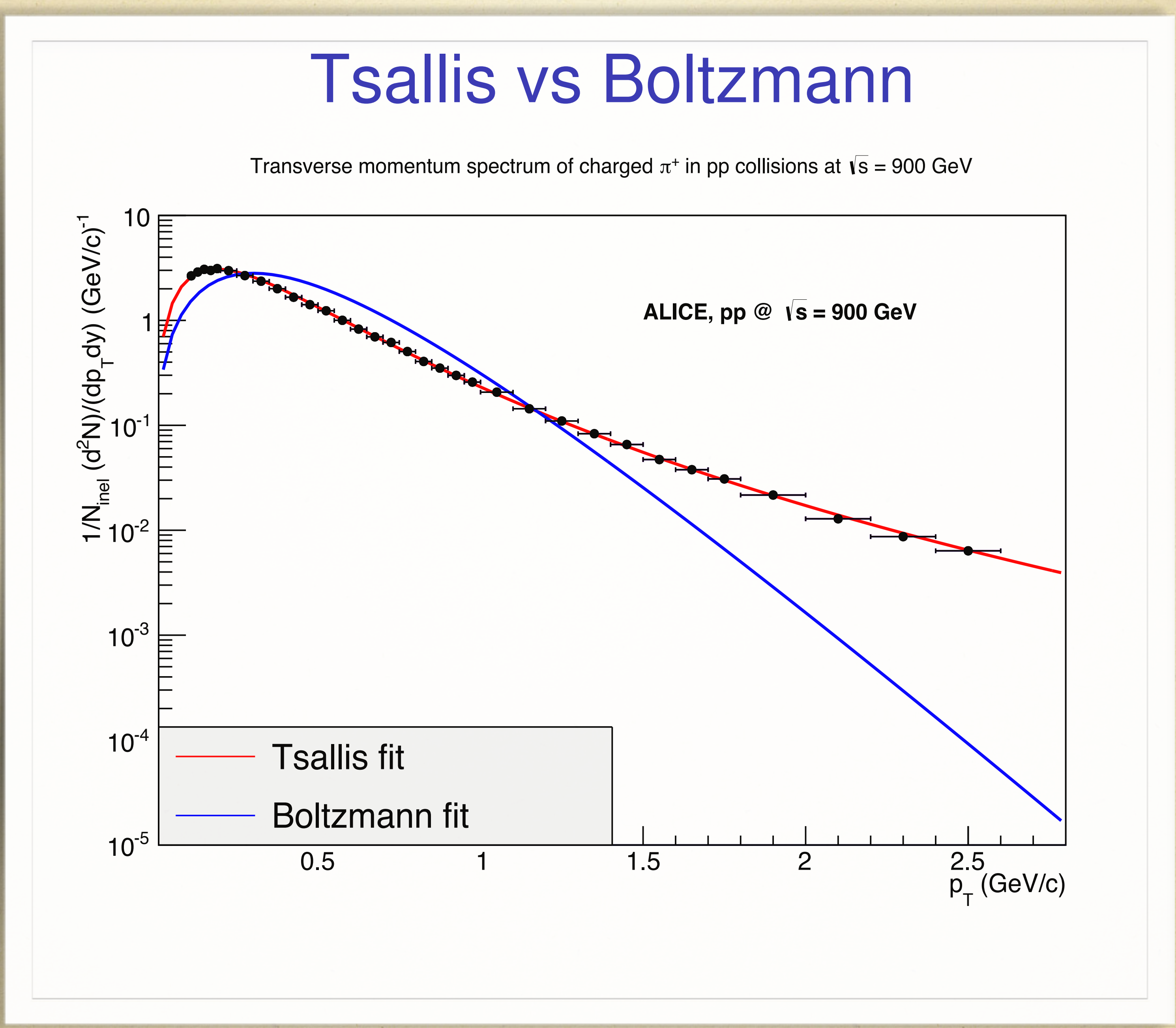
$$\left.\frac{d^2N}{p_T\, dp_T\, dy}\right|_{y=0} = gV\frac{m_T}{(2\pi)^2}e^{-\frac{m_{\mathrm{T}}}{T}}$$

Exponential and Tsallis: Difference in the descriptions of the transverse momentum (p_{T}) spectra (protons p-p 900 GeV) for a particular value of q and T:



T and q values taken fromTB, J Cleymans, L Marquez, S Mogliacci, M Paradza, J Phys G 45 no.5, 055001 (2018)

π^+ spectrum for p-p 900 GeV fitted with Tsallis and Boltzmann



Presentation by J Cleymans @ CERN Heavy Ion Forum 2014

So, the Tsallis distribution is distinctly different from the Boltzmann distribution which is a limiting case of the former.

Inference: Tsallis distribution is a generalisation of the Boltzmann distribution

If so, what physical situation does the Tsallis distribution describe ?

‘Effective’ Boltzmann factor

$$k_{eff} = \int f(\beta)\, e^{-\beta E}\, d\beta$$

Replace $f(\beta)$ by a χ^2 distribution

Boltzmann statistics: with a single value of Boltzmann temperature $T = 1/\beta$

Distribution of T

$$k_{eff} = \Big(1 + \langle\beta\rangle(q-1)\; E\Big)^{\frac{1}{1-q}} \equiv exp_q(-\langle\beta\rangle E)$$

$$q - 1 = \frac{\langle\beta^2\rangle - \langle\beta\rangle^2}{\langle\beta\rangle^2}$$

Tsallis (inverse) temperature, average of all Boltzmann (inverse) temperatures

Tsallis parameter, relative variance in Boltzmann (inverse) temperature

G. Wilk and Z. Włodarczyk, Phys. Rev. Lett. 84, 2770(2000)

Tsallis thermodynamic variables à la J. Cleymans and D. Worku J. Phys. G 39, 025006 (2016)

$$S = -gV \int \frac{d^3p}{(2\pi)^3} \left[f^q \ln_q f - f \right],$$

$$N = gV \int \frac{d^3p}{(2\pi)^3} f^q,$$

$$\epsilon = g \int \frac{d^3p}{(2\pi)^3} \; E \; f^q,$$

$$P = g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} \; f^q.$$

$$T = \left. \frac{\partial \epsilon}{\partial s} \right|_n,$$

$$\mu = \left. \frac{\partial \epsilon}{\partial n} \right|_s,$$

$$n = \left. \frac{\partial P}{\partial \mu} \right|_T,$$

$$s = \left. \frac{\partial P}{\partial T} \right|_\mu.$$

Tsallis Thermodynamic Variables: closed analytical forms

Massless

Massive

- Taylor's Series approximation
- Mellin-Barnes

Tsallis pressure of a d-dimensional massless gas

$$P_d = \int_0^\infty \frac{d^{d-1}p}{(2\pi)^d} \frac{p}{\left(1 + \delta q \frac{p-\mu}{T}\right)^{\frac{1+\delta q}{\delta q}}} = \frac{gT^4}{6\pi^2} \frac{1}{(1-\delta q)(1/2-\delta)(1/3-\delta q)}$$

$$\delta q = q - 1; d < \frac{1+\delta q}{\delta q} \Rightarrow q < 1 + \frac{1}{d-1} \Rightarrow q < \frac{4}{3} : d = 4$$

T Bhattacharyya, J Cleymans, Sylvain Mogliacci Phys. Rev. D 94, 094026 (2016)

Tsallis pressure of a massive gas: Taylor's series approximation

$$P^B + (q-1)P^1$$

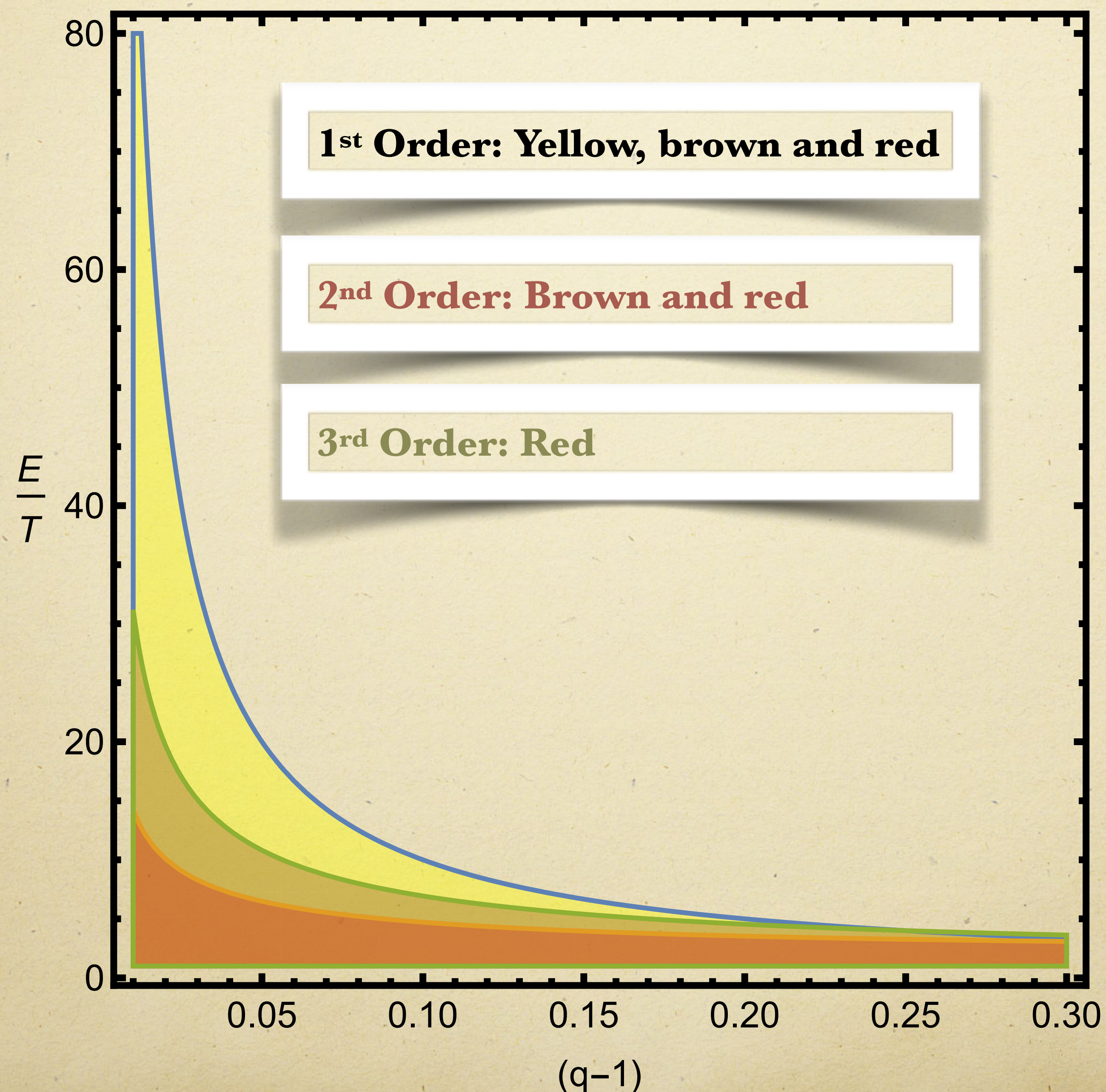
$$P^B = \frac{g e^{\frac{\mu}{T}} T^4 a^2 K_2(a)}{2\pi^2} \quad ; \; a = \frac{m}{T}$$

$$P^1 = \frac{g e^{\frac{\mu}{T}} T^4}{4\pi^2} \left[a^4 K_2(a) + 3a^3 K_3(a) - 2a^3 b K_3(a) + a^2 b^2 K_2(a) + 2a^2 b K_2(a)\right]$$

T Bhattacharyya, J Cleymans, A Khuntia, P Pareek, R Sahoo Eur. Phys. J A 30, 52 no.2 (2016)

Tsallis pressure of a massive gas

Taylor's series approximation of the Tsallis distribution. Leads to severe constraints on the allowed region.



Tsallis pressure of a massive gas

Second attempt: Mellin-Barnes contour integration representation of the Tsallis distribution

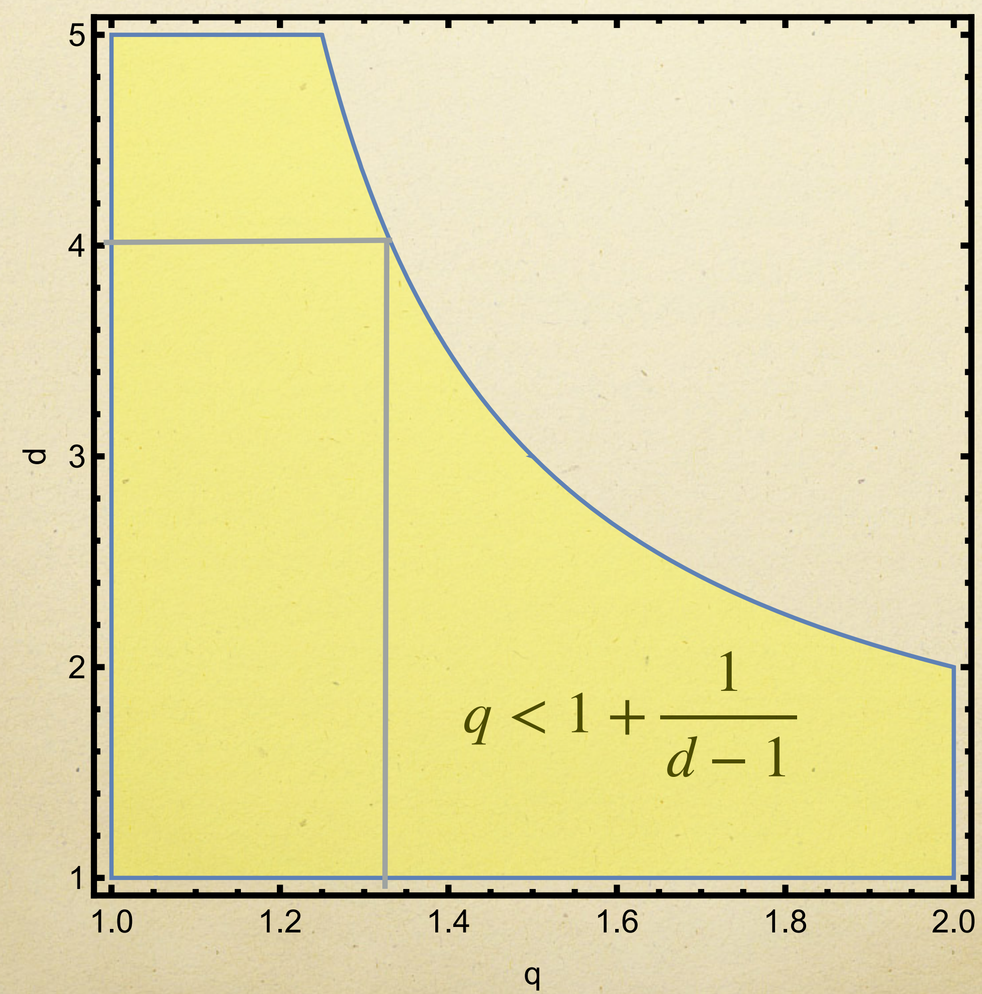
$$P_U = \frac{gm^4}{16\pi^{\frac{3}{2}}}\left(\frac{T}{\delta qm}\right)^{\frac{1+\delta q}{\delta q}}\left[\frac{\Gamma(\frac{1-3\delta q}{2\delta q})}{\Gamma(\frac{1+2\delta q}{2\delta q})}\times {}_2F_1\left(\frac{1+\delta q}{2\delta q},\frac{1-3\delta q}{2\delta q},\frac{1}{2};\left(\frac{\delta q\mu - T}{\delta qm}\right)^2\right)\right.$$
$$\left. + 2\left(\frac{\delta q\mu - T}{\delta qm}\right)\times\frac{\Gamma(\frac{1-2\delta q}{2\delta q})}{\Gamma(\frac{1+\delta q}{2\delta q})}\times {}_2F_1\left(\frac{1+2\delta q}{2\delta q},\frac{1-2\delta q}{2\delta q},\frac{3}{2};\left(\frac{\delta q\mu - T}{\delta qm}\right)^2\right)\right],$$

T Bhattacharyya, J Cleymans, Sylvain Mogliacci Phys. Rev. D 94, 094026 (2016)

Summary so far

Taylor's expansion of the Tsallis distribution leads to constraints.

Unapproximated calculations display the existence of the pole structure which was missed in the approximated calculations.



Pressure, energy density etc. ⇒ energy momentum tensor $T^{\mu\nu}$ ⇒ hydrodynamic evolution equation ⇒ which dictates the evolution of the medium (quark-gluon plasma) created in high energy collisions

Medium particles: low energy particles (average momentum ~ temperature)

High energy particles: probes (e.g. charm/bottom quarks etc.) ⇒ they also evolve in the evolving background of the medium

An exercise: Boltzmann transport equation (BTE) in the relaxation time approximation (RTA) as the evolution equation of a particle homogeneously distributed and subjected to no external force.

$$\frac{\partial f}{\partial t} = -\frac{f - f_{eq}}{\tau}$$

$$f_{in} = \frac{gV}{(2\pi)^2} p_T m_T \left[1 + (q-1)\frac{m_T}{T}\right]^{-\frac{q}{q-1}} \qquad f_{eq} = \frac{gV}{(2\pi)^2} p_T m_T e^{-\frac{m_T}{T_{eq}}}$$

$$R_{AA} = \frac{e^{-\frac{m_T}{T_{eq}}}}{(1 + (q-1)\frac{m_T}{T})^{-\frac{q}{q-1}}} + \left[1 - \frac{e^{-\frac{m_T}{T_{eq}}}}{(1 + (q-1)\frac{m_T}{T})^{-\frac{q}{q-1}}}\right] e^{-\frac{t_f}{\tau}}$$

S Tripathy, T Bhattacharyya, P Garg, P Kumar, R Sahoo and J Cleymans Eur. Phys. J A 52, 289 (2016)

Fitting nuclear suppression factor of the hadrons

Pb+Pb 2.76 TeV					
Particle	Centrality (%)	χ^2/ndf	t_f/τ	q	T (GeV)
$\pi^+ + \pi^-$	0–5	0.364461	2.07313 ± 0.061906	1.00151 ± 0.00149	0.17854 ± 0.00143
$K^+ + K^-$	0–5	0.390686	2.24302 ± 0.098624	1.00406 ± 0.00125	0.16803 ± 0.00133
K_s^0	0–5	0.284477	1.88499 ± 0.100713	1.00410 ± 0.00373	0.17320 ± 0.00379
$p + \bar{p}$	0–5	0.267087	1.72079 ± 0.163273	1.00378 ± 0.00079	0.15771 ± 0.00111
$\Lambda + \bar{\Lambda}$	0–5	0.017809	1.85201 ± 0.649860	1.00600 ± 0.00369	0.15411 ± 0.00512
D^0	30–50	0.262131	0.84223 ± 0.099520	1.01915 ± 0.01202	0.13538 ± 0.01568
J/ψ	0–90	0.155083	1.06248 ± 0.157049	1.01252 ± 0.00998	0.14676 ± 0.01408

The quark-gluon plasma medium through which particles are moving is barely ideal.

Existence of the quasi-stationary states like the one given by the Tsallis distribution as opposed to the Boltzmann distribution

Indications that we need to have a modified kinetic equation (i.e. a modified Boltzmann transport equation) to deal with such situations.

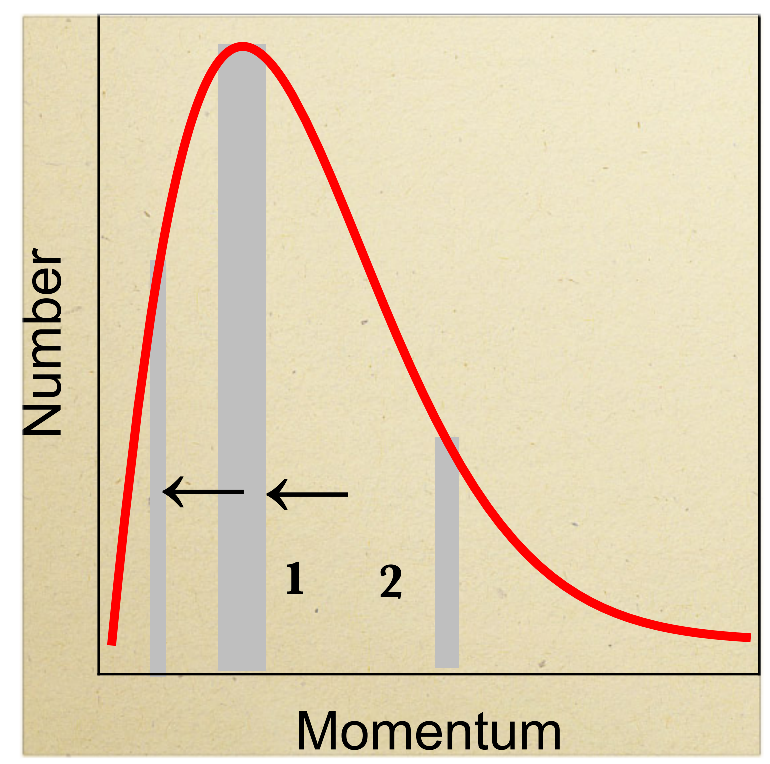
Boltzmann Transport Equation: Recap

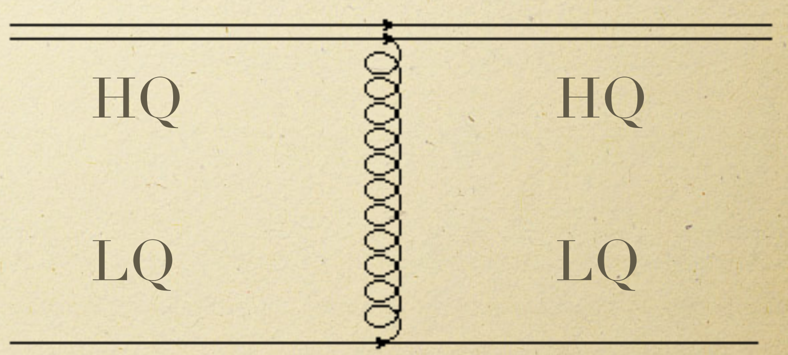
$$\frac{d}{dt} f(\vec{x}, \vec{p}; t) = C[f]$$

Molecular chaos ☛ $f_2(h, l) \equiv f_1(h) \times f_1(l)$

$C[f]$

Kinematics ⊗ Kinetics





Molecular chaos (MC)

$$f_1(h) \times f_1(l) = e^{\ln[f_1(h)]} \times e^{\ln[f_1(l)]}$$

Generalized Molecular chaos (GMC)

Ansatz

$$f_1(h) \otimes_q f_1(l) = e_q^{\ln_q[f_1(h)]} \times e_q^{\ln_q[f_1(l)]}$$

$$e_q(x) = [1-(q-1)x]^{1/(1-q)} \; ; \; \ln_q(x) = \frac{1-x^{1-q}}{q-1}$$

$$f_2(h,l) = f_1(h)\, f_1(l) + (q-1)\, f_1(h)\, f_1(l)\, \ln[f_1(h)]\, \ln[f_1(l)] + \mathcal{O}[(q-1)^2] + \ldots .$$

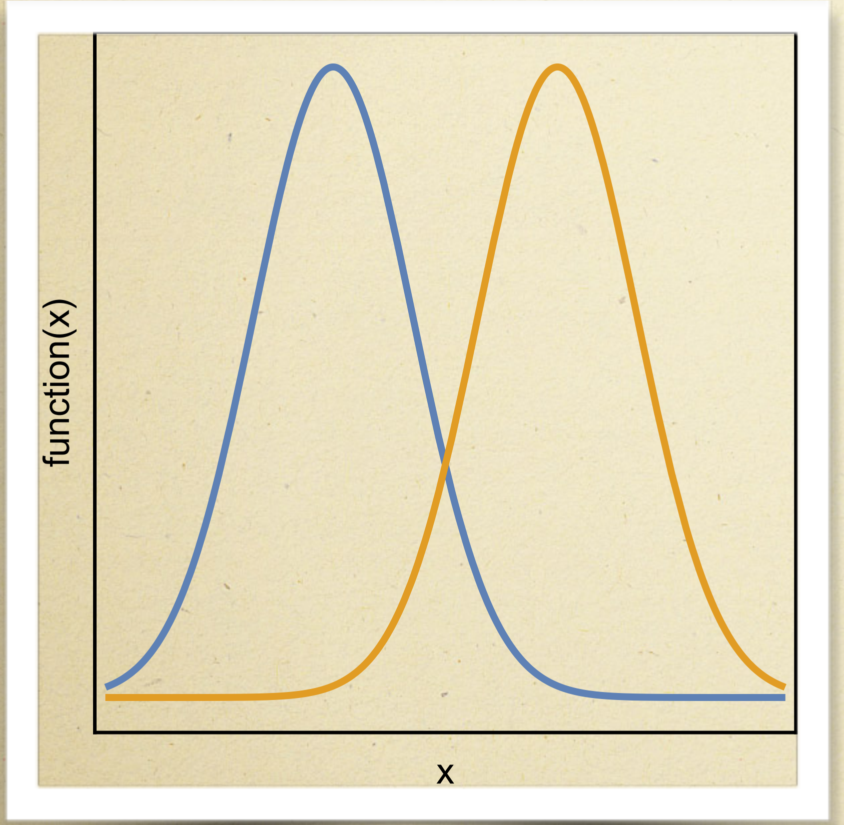
Low momentum transfer

Molecular chaos ⇒ BTE → Fokker-Planck

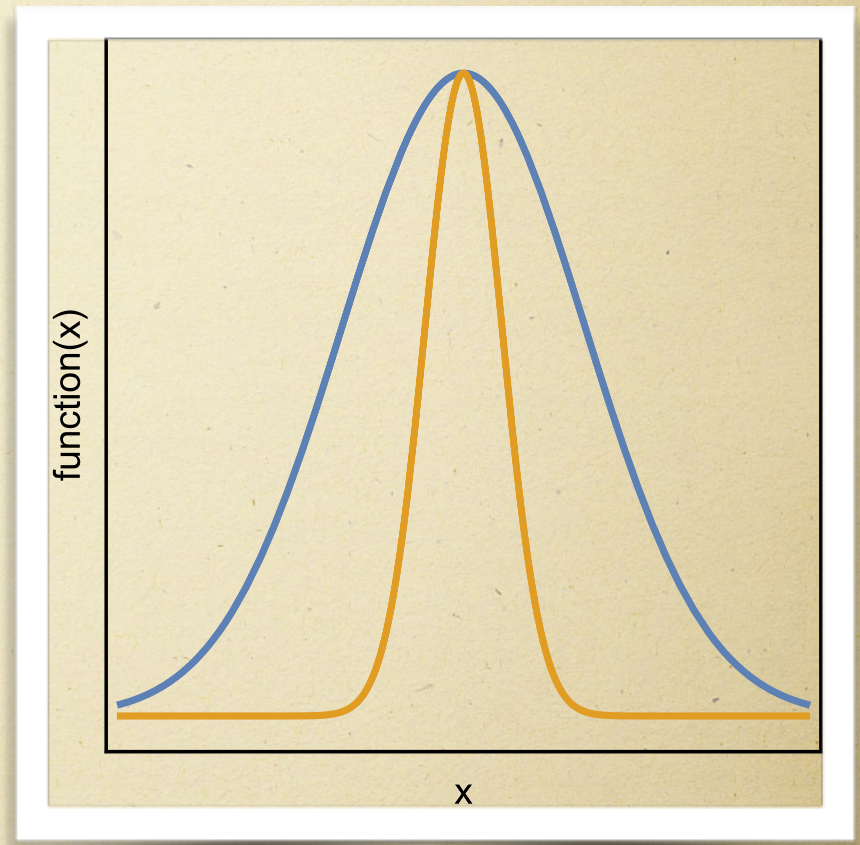
Generalized Molecular chaos
⇒ non-linear BTE → non-linear Fokker-Planck

$$\frac{\partial f}{\partial t} = -\frac{\partial [A_{i,q} f]}{\partial p_i} + \frac{\partial [B_{ij,q} f^{2-q}]}{\partial p_i \partial p_j}$$

Drag



Diffusion



G Wolschin, Phys. Lett. B 569, 67(2003), A. Lavagno Braz. Jour. of Phys. **35**, 516 (2005)

$$A_i = \frac{1}{2E_{\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{d^3\mathbf{q}'}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} |\overline{M}|^2 (2\pi)^4 \times \delta^4(p+q-p^{'}-q^{'}) \ f(\mathbf{q}) \ (\mathbf{p}-\mathbf{p}')_i$$

$$B_{ij} = \frac{1}{2E_{\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{d^3\mathbf{q}'}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} |\overline{M}|^2 (2\pi)^4 \times \delta^4(p+q-p^{'}-q^{'}) \ f(\mathbf{q}) \ \frac{1}{2}(\mathbf{p}'-\mathbf{p})_i(\mathbf{p}'-\mathbf{p})_j$$

Linear Fokker-Planck drag and diffusion coefficients of the heavy quarks

B. Svetitsky
Phys Rev D 37, 2484 (1988)

S Mazumder, TB, J Alam, S K Das
Phys Rev C 84, 044901 (2011)

S Mazumder, TB, J Alam
Phys. Rev. D 89, 014002 (2014)

D B Walton and J Rafelski
Phys. Rev. Lett. 84, 31 (2014)

Non-linear Fokker-Planck drag and diffusion coefficients of the heavy quarks

TB and J Cleymans arXiv: 1707.08425

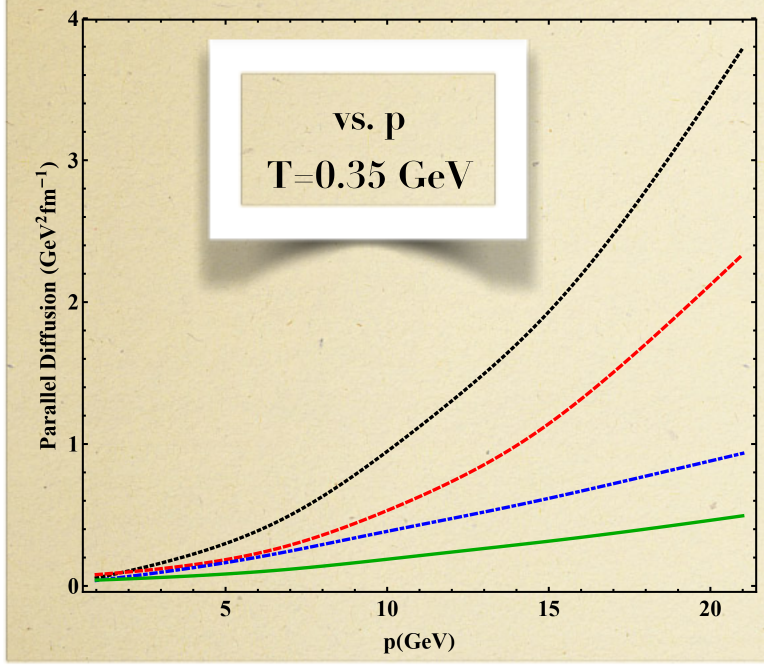
$$A_i^{\mathrm{NE}} = \frac{1}{2E_{\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{d^3\mathbf{q}'}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} |\overline{M}|^2 (2\pi)^4 \times \delta^4(p+q-p^{'}-q^{'}) \times \mathcal{R}^1_{\mathbf{p},\mathbf{q}} \ (\mathbf{p}-\mathbf{p}')_i$$

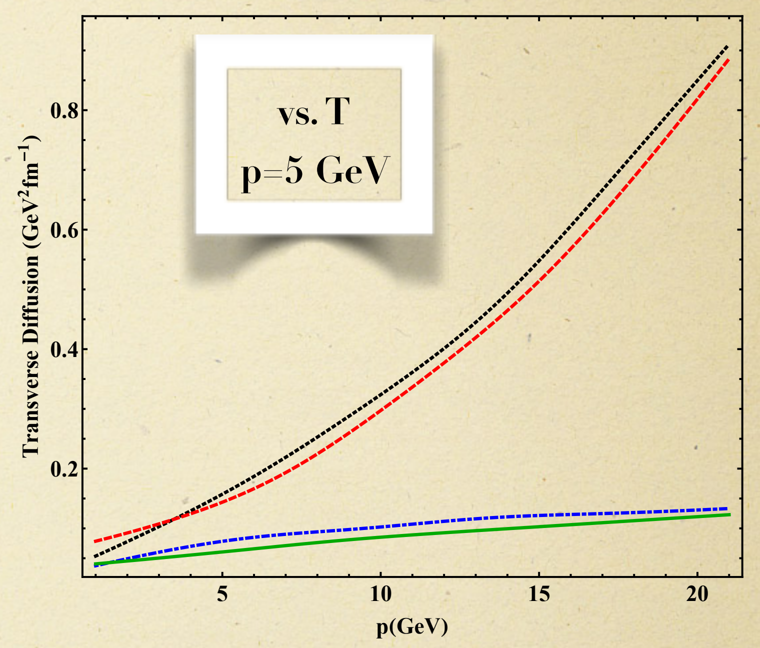
$$B_{ij}^{\mathrm{NE}} = \frac{1}{2E_{\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{d^3\mathbf{q}'}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} |\overline{M}|^2 (2\pi)^4 \times \delta^4(p+q-p^{'}-q^{'}) \mathcal{R}^2_{\mathbf{p},\mathbf{q}} \ \frac{1}{2}(\mathbf{p}'-\mathbf{p})_i(\mathbf{p}'-\mathbf{p})_j$$

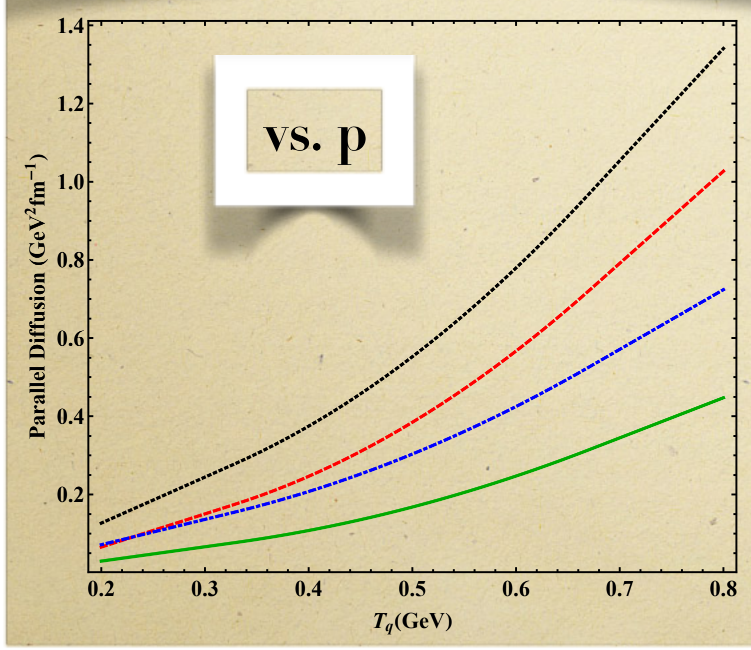
Parallel diffusion

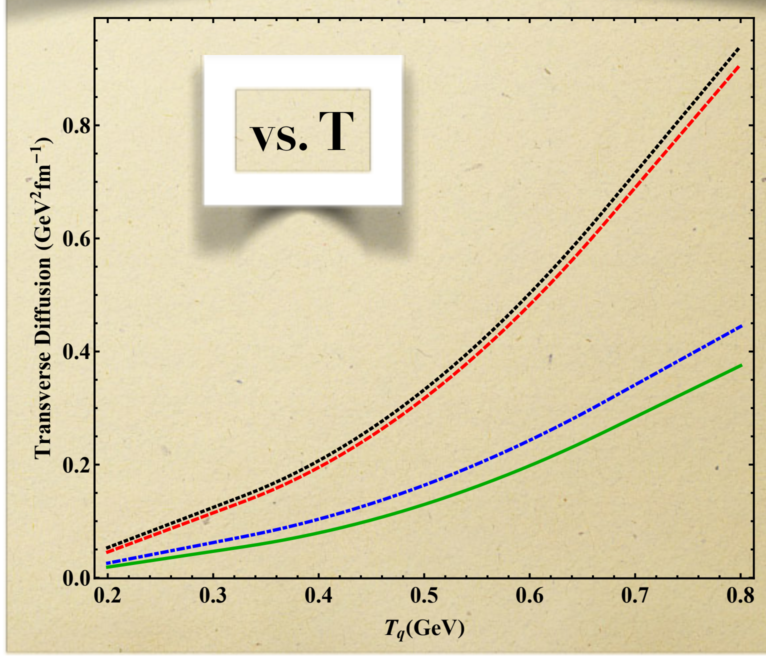
c: NL
b: NL
c: L
b:L

Transverse diffusion









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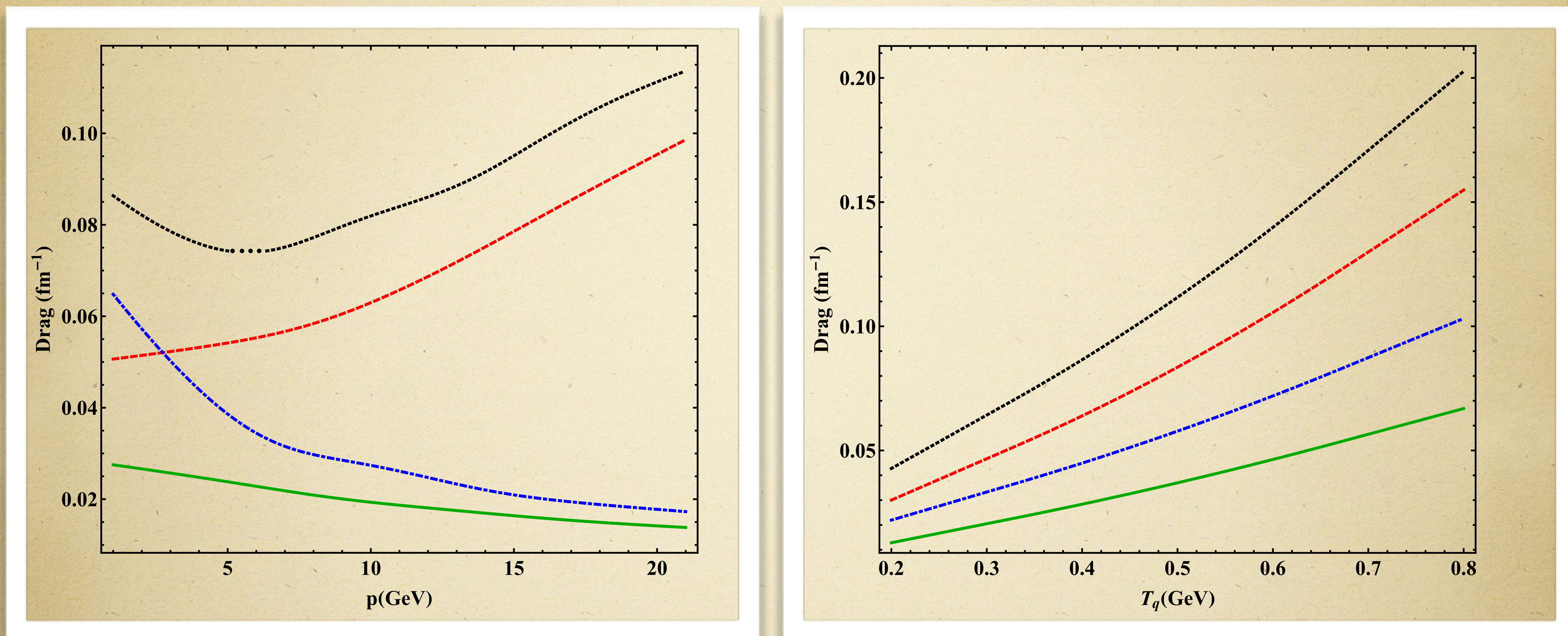

c: NL; b:NL; c:L; b:L

Summary, conclusion and outlook

Tsallis distribution is a generalisation of the Boltzmann distribution

Fluctuation, non-ideal plasma effects can be dealt with with the help of the Tsallis statistics

Inclusion of radiation

S Mazumder, TB, J Alam Phys. Rev. D 89, 014002 (2014)

Dokshitzer and Kharzeev, PLB 519, 199 (2001) JPG 17, 1481 (1991)

TB, Surasree Mazumder and Raktim Abir
Advances in High Energy Physics 2016 , 1298986 (2016)

Extension to dense systems

Connection with the experimental observables and finding out the values of Tsallis q-parameter from the nuclear suppression factor data.

Thank you !!