

Universal Correlations at Ultralow Energies



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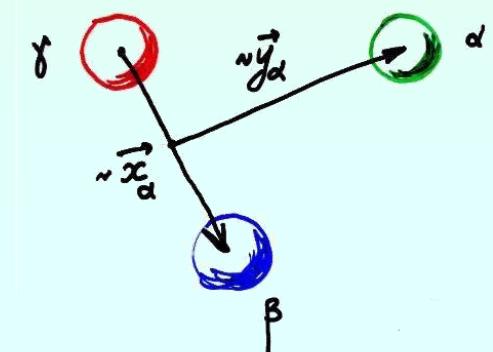
Outline

- ◆ Formalism (Faddeev equations)
- ◆ Phillips line and modified Phillips line
- ◆ Helium trimer and nnp system

In describing the three-body system we use the standard Jacobi coordinates [4] x_α, y_α , $\alpha = 1, 2, 3$, expressed in terms of the position vectors of the particles $\mathbf{r}_i \in \mathbb{R}^3$ and their masses m_i ,

$$\mathbf{x}_\alpha = \left[\frac{2m_\beta m_\gamma}{m_\beta + m_\gamma} \right]^{1/2} (\mathbf{r}_\beta - \mathbf{r}_\gamma)$$

$$\mathbf{y}_\alpha = \left[\frac{2m_\alpha(m_\beta + m_\gamma)}{m_\alpha + m_\beta + m_\gamma} \right]^{1/2} \left(\mathbf{r}_\alpha - \frac{m_\beta \mathbf{r}_\beta + m_\gamma \mathbf{r}_\gamma}{m_\beta + m_\gamma} \right)$$



where (α, β, γ) stands for a cyclic permutation of the indices $(1, 2, 3)$. The coordinates $\mathbf{x}_\alpha, \mathbf{y}_\alpha$ fix the six-dimensional vector $X \equiv (\mathbf{x}_\alpha, \mathbf{y}_\alpha) \in \mathbb{R}^6$. The vectors $\mathbf{x}_\beta, \mathbf{y}_\beta$ corresponding to the same point X as the pair $\mathbf{x}_\alpha, \mathbf{y}_\alpha$ are obtained using the transformations

$$\mathbf{x}_\beta = c_{\beta\alpha} \mathbf{x}_\alpha + s_{\beta\alpha} \mathbf{y}_\alpha \quad \mathbf{y}_\beta = -s_{\beta\alpha} \mathbf{x}_\alpha + c_{\beta\alpha} \mathbf{y}_\alpha$$

where the coefficients $c_{\beta\alpha}$ and $s_{\beta\alpha}$ fulfil the conditions $-1 < c_{\beta\alpha} < +1$ and $s_{\beta\alpha}^2 = 1 - c_{\beta\alpha}^2$ with $c_{\alpha\beta} = c_{\beta\alpha}$, $s_{\alpha\beta} = -s_{\beta\alpha}$, $\beta \neq \alpha$ and depend only on the particle masses [4]. For equal masses $c_{\beta\alpha} = -\frac{1}{2}$.

[4] - L.D.Faddeev,S.P.Merkuriev, 1993, *Quantum scattering theory for several particles*

${}^4\text{He}_2 - {}^4\text{He}$

Faddeev integro-differential equations after angular partial-wave analysis

$$\left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + l(l+1) \left(\frac{1}{x^2} + \frac{1}{y^2} \right) - E \right] F_l(x, y) = \begin{cases} -V(x) \Psi_l(x, y), & x > c \\ 0, & x < c \end{cases}$$

At $L=0$ the partial angular momentum l corresponds both to the dimer and an additional atom. x, y stand to the standard Jacobi variables.

$$\Psi_l(x, y) = F_l(x, y) + \sum_{l'} \int_{-1}^1 d\eta h_{ll'}(x, y, \eta) F_{l'}(x', y'),$$

$$x' = (1/4x^2 + 3/4y^2 - \sqrt{3}/2xy\eta)^{1/2}, \quad y' = (3/4x^2 + 1/4y^2 + \sqrt{3}/2xy\eta)^{1/2}, \quad \eta = \hat{x} \cdot \hat{y}$$

The kernel $h_{ll'}$ depend only on hyperangles - see L.D.Faddeev,S.P.Merkuriev, 1993.

nnp system

S=3/2

$$(H_L - E)\Phi_L^q(x, y) = V_t(x)\Psi_L^q(x, y)$$

$$\Psi_L^q(x, y) = \Phi_L^q - \frac{1}{2} \int_{-1}^1 du h^L(x, y, u) \Phi_L^q(x', y')$$

S=1/2

$$(H_L - E)\Phi_{1(2),L}^d(x, y) = V_{t(s)}(x)\Psi_{1(2),L}^d(x, y)$$

$$\Psi_L^d = \Phi_L^d + \frac{1}{2} \int_{-1}^1 du h^L(u) B \Phi_L^d$$

$$\Phi_{i,L}^{q,d}(x, y)|_{x=0} = 0 \quad \quad \quad \Phi_{i,L}^{q,d}(x, y)|_{y=0} = 0$$

L.D.Faddeev, S.P. Merkuriev, 1993

Boundary conditions

$$F_l(x, y)_{x=0} = F_l(x, y)_{y=0} = 0,$$

Hard-core boundary conditions:

$$\Psi_l(x, y)_{x=c} = F_l(c, y) + \sum_l \int_{-1}^1 d\eta h_{ll'}(c, y, \eta) F_{l'}(x', y') = 0,$$

The asymptotic condition for the helium trimer bound states (as $\rho \rightarrow \infty$ and/or $y \rightarrow \infty$)

$$\begin{aligned} F_l(x, y) &= \delta_{l0} \psi_d(x) \exp(i\sqrt{E_t - \varepsilon_d} y) [a_0 + o(y^{-1/2})] \\ &\quad + \frac{\exp(i\sqrt{E_t} \rho)}{\sqrt{\rho}} [A_l(\theta) + o(\rho^{-1/2})] \end{aligned}$$

Here, ψ_d is the dimer wave function, ε_d stands for the dimer energy,

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \arctan(x/y)$$



*Three-body, theory
formalism*

Boundary conditions

The asymptotic condition for the partial-wave Faddeev components of the $(2 + 1 \rightarrow 2 + 1 ; 1 + 1 + 1)$ scattering wave function reads, (as $\rho \rightarrow \infty$ and/or $y \rightarrow \infty$)

$$\begin{aligned}\Phi_l(x, y; p) = & \delta_{l0} \psi_d(x) \{ \sin(py) + \exp(ipy) [a_0(p) + o(y^{-1/2})] \} \\ & + \frac{\exp(i\sqrt{E}\rho)}{\sqrt{\rho}} [A_l(\theta) + o(\rho^{-1/2})].\end{aligned}$$

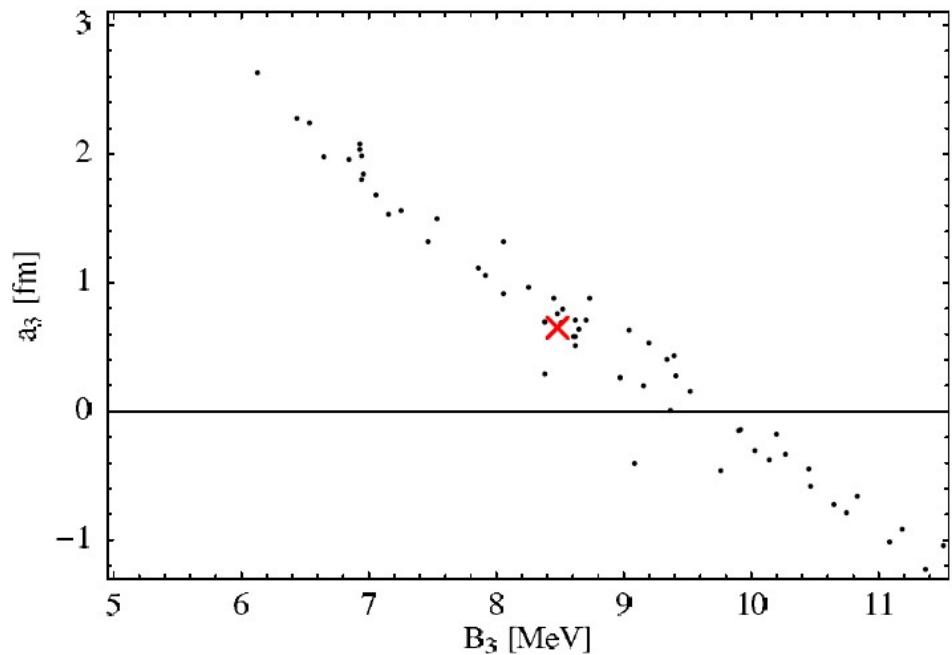
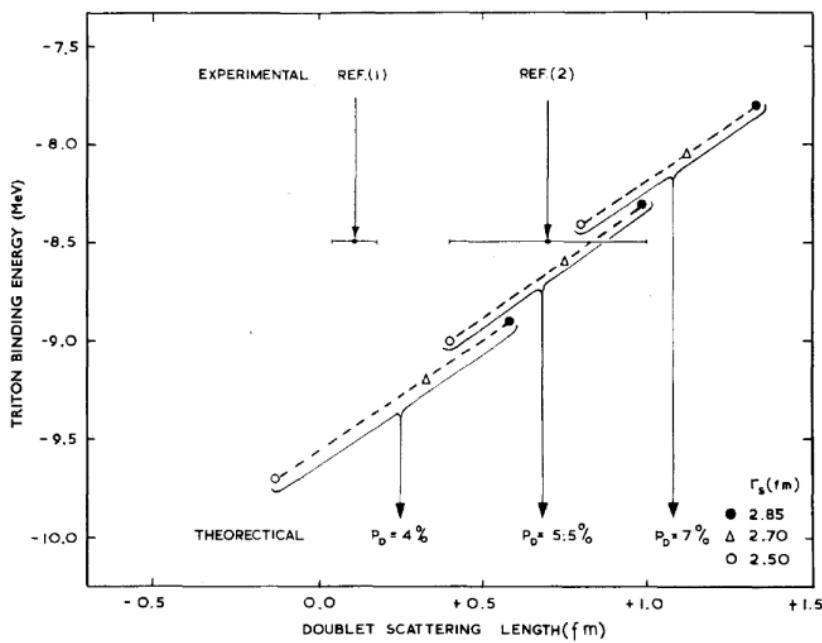
Here ψ_d is the dimer wave function, E stands for the scattering energy given by $E = \varepsilon_d + p^2$ with ε_d the dimer energy, and p is the relative momentum conjugate to the variable y . The coefficient $a_0(p)$ is nothing but the elastic scattering amplitude, while the functions $A_l(\theta)$ provides us, at $E > 0$, with the corresponding partial-wave Faddeev breakup amplitudes. The scattering length is given by

$$l_{sc} = -\frac{\sqrt{3}}{2} \lim_{p \rightarrow 0} \frac{a_0(p)}{p}$$

E.K, A.Motovilov,S.Sofianos
J.Phys.B 31, 1279 (1998)

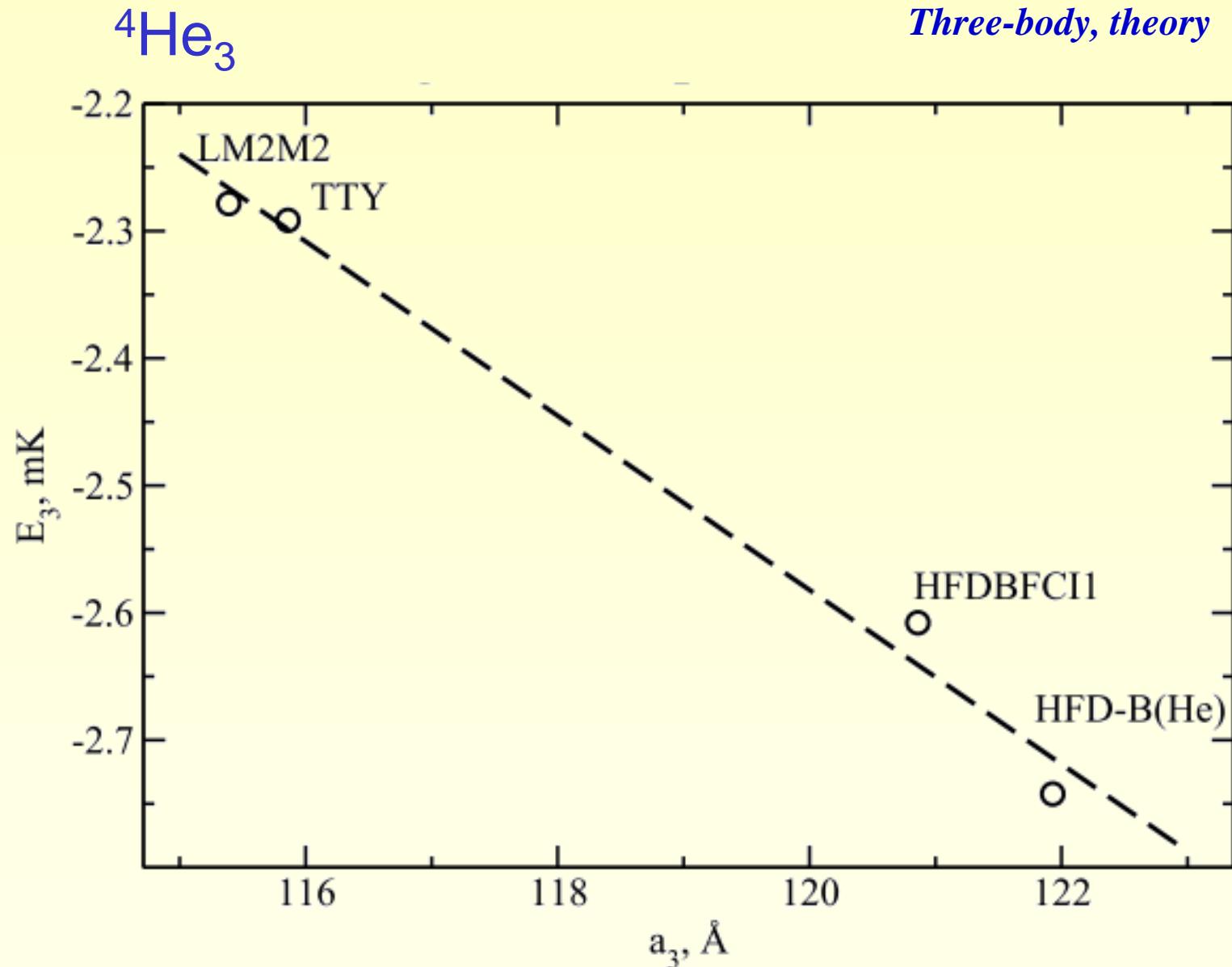
^3H - system

Three-body, theory



Phillips line from the original paper,
showing the unexpected linear correlation

A.C. Phillips Nucl. Phys A **107**, 209 (1968)

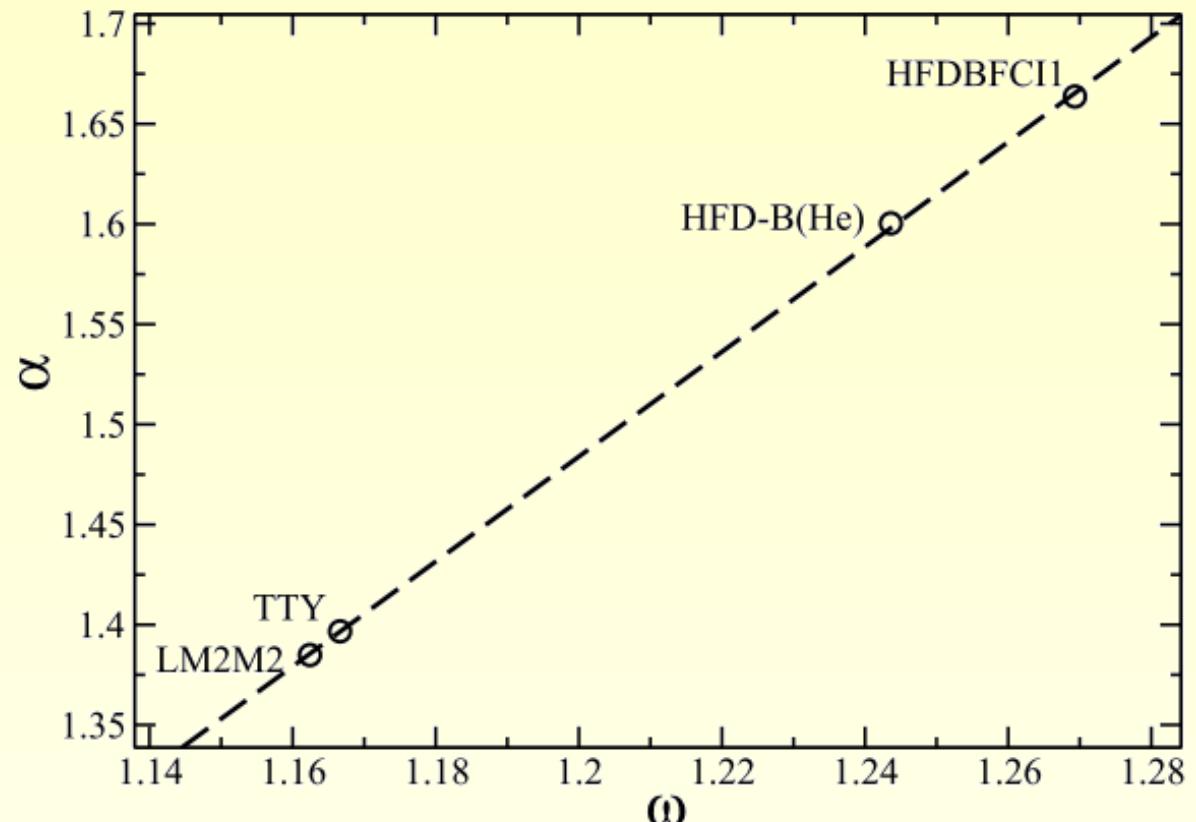


${}^4\text{He}_2$ - ${}^4\text{He}$

Three-body, theory

$$E_2 - E_3 \approx 1/(2m_{12}a_3^2)$$

V.Efimov, E.G.Tkachenko,
Phys.Lett. B **157**, 108 (1985)



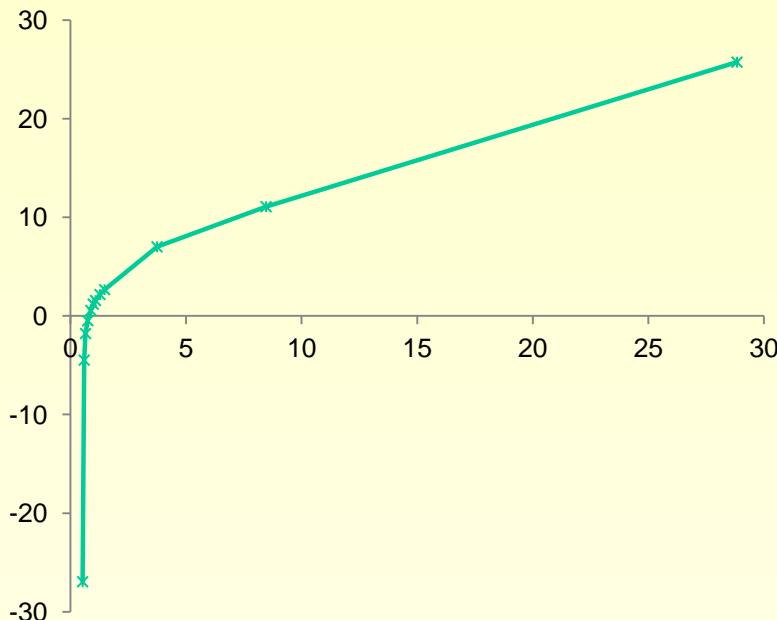
$$\alpha \equiv a_3 \sqrt{-\mu E_2} \propto 1/\sqrt{E_3/E_2 - 1} \equiv \omega$$

V.Roudnev, M.Cavagnero
Phys.Rev.Lett. **108**, 110402 (2012)

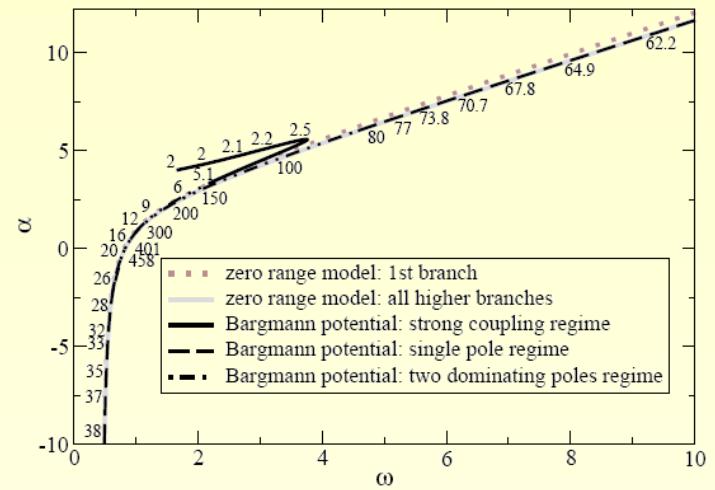
${}^4\text{He}_2 - {}^4\text{He}$

Three-body, theory

$$V(x) = \lambda V_{HFD-B}(x)$$



$$\alpha \equiv a_3 \sqrt{-\mu E_2} \propto 1/\sqrt{E_3/E_2 - 1} \equiv \omega$$

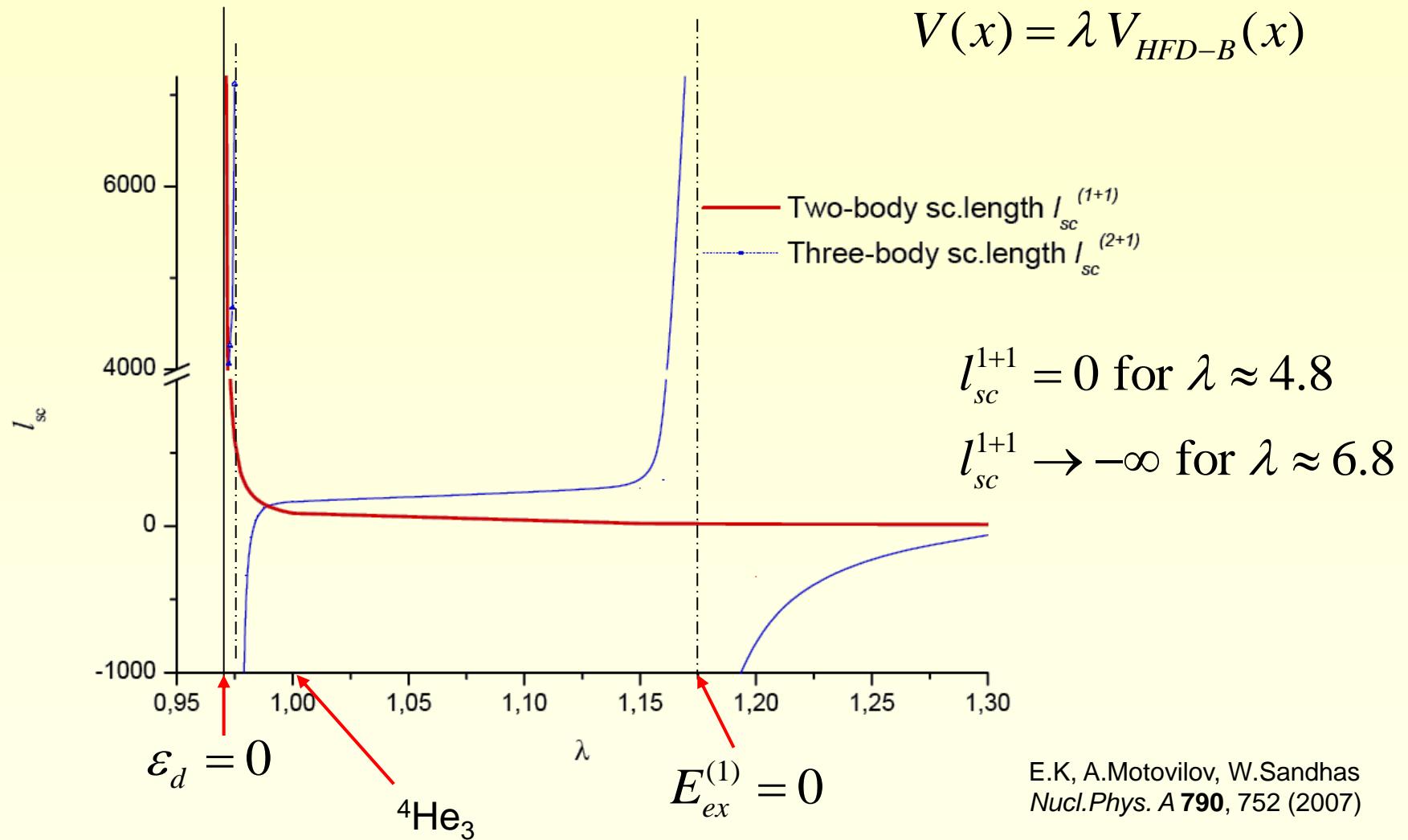


V.Roudnev, M.Cavagnero
Phys.Rev.Lett. **108**, 110402 (2012)

${}^4\text{He}_2 - {}^4\text{He}$

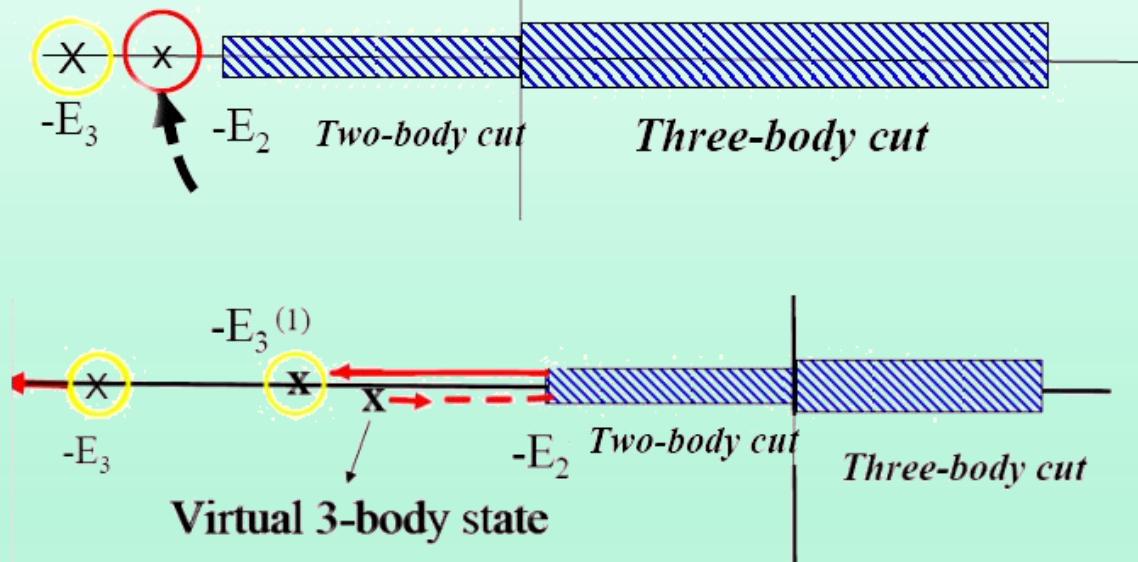
*Three-body, theory
scattering*

Efimov-type properties of excited state



$^4\text{He}_3$

*Three-body, theory
resonances*



$$V(x) = \lambda V_{HFD-B}(x)$$

λ	ϵ_d (mK)	$\epsilon_d - E_t^{(1)}$ (mK)	ρ_{\max} (Å)
1.05	-12.244	0.873	300
1.10	-32.222	0.450	200
1.15	-61.280	0.078	150
1.16	-68.150	0.028	120
1.17	-75.367	0.006	120

λ	ϵ_d	$\epsilon_d - E_{\text{ex}}^{(1)}$	$\epsilon_d - E_{\text{virt}}$	$\epsilon_d - E_{\text{ex}}^{(2)}$	$l_{\text{sc}}^{(1+2)}$	$l_{\text{sc}}^{(1+1)}$	ρ_{\max} (Å)
1.0	-1.685	0.773	—	—	160	88.6	700
0.995	-1.160	0.710	—	—	151	106	900
0.990	-0.732	0.622	—	—	143	132	1050
0.9875	-0.555	0.222	—	—	125	151	1200
0.985	-0.402	0.518	0.097	—	69	177	1300
0.982	-0.251	0.447	0.022	—	-75	223	1700
0.980	-0.170	0.396	0.009	—	-337	271	2000
0.9775	-0.091	0.328	0.003	—	-6972	370	3000
0.975	-0.036	0.259	—	0.002	7120	583	4500
0.973	-0.010	0.204	—	0.006	4260	1092	10000

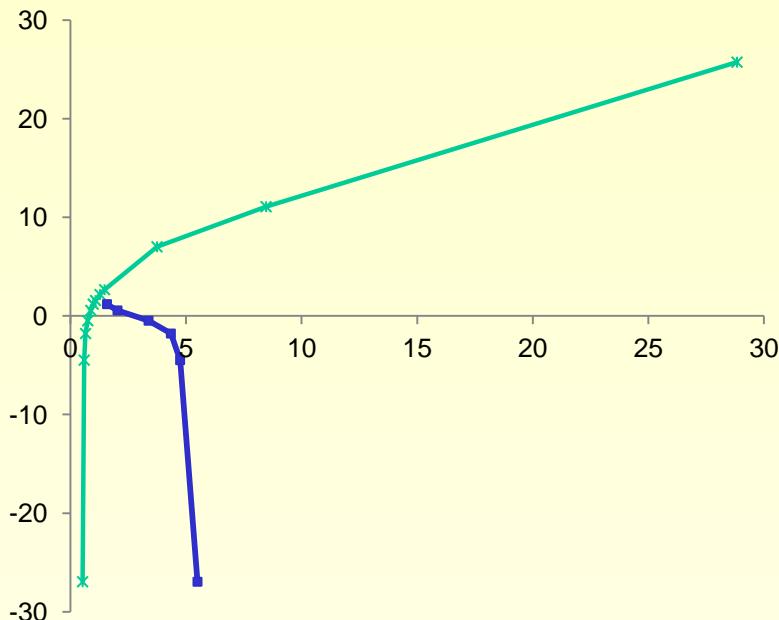
λ	ϵ_d (mK)	$\epsilon_d - E_t^{(1)*}$ (mK)	ρ_{\max} (Å)
1.18	-82.927	0.001	110
1.19	-90.829	0.016	110
1.20	-99.068	0.057	100
1.25	-145.240	0.588	85
1.30	-199.457	1.831	70
1.35	-261.393	3.602	70
1.40	-330.737	6.104	55
1.50	-490.479	12.276	50

E.A.K, Motovilov A.K.
Phys.At.Nucl. **60**, 235 (1997)

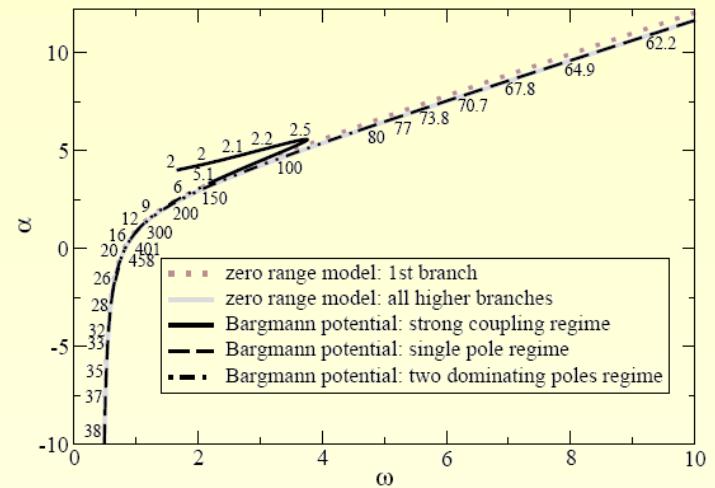
${}^4\text{He}_2 - {}^4\text{He}$

Three-body, theory

$$V(x) = \lambda V_{HFD-B}(x)$$



$$\alpha \equiv a_3 \sqrt{-\mu E_2} \propto 1/\sqrt{E_3/E_2 - 1} \equiv \omega$$

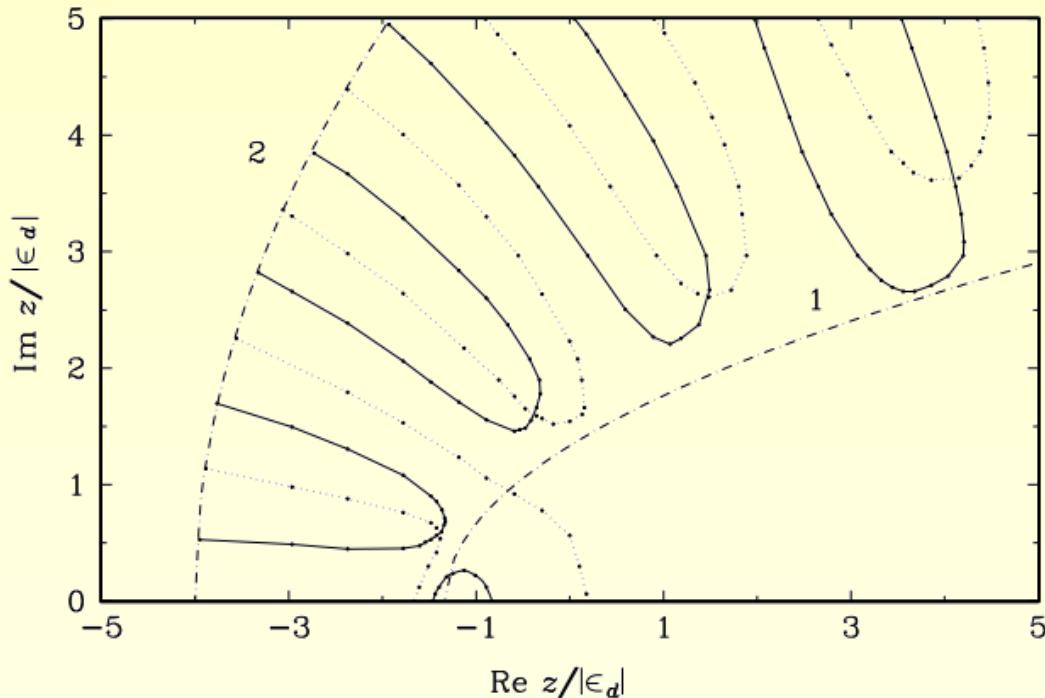


Roudnev, Cavagnaro
Phys.Rev.Lett. **108**, 110402 (2012)

${}^4\text{He}_2 - {}^4\text{He}$

Three-body, theory

S-matrix



$$\lambda = 1$$

A.K.Motovilov, *Math.Nachrichten*
187, 147(1997)

E.A.K, A.K.Motovilov
Phys.At.Nucl. **60**, 235 (1997)

E.A.K., A.K.Motovilov,Y.K.Ho
Nucl.Phys.A **684**, 623 (2001)

Fig. 1. Root locus curves of the real and imaginary parts of the scattering matrix $S_0(z)$ in case of helium trimer. The solid lines correspond to $\text{Re } S_0(z) = 0$, while the tiny dashed lines, to $\text{Im } S_0(z) = 0$. The Numbers 1, 2 denote the boundaries of the domains $\Pi^{(S)}$ and $\Pi^{(\Psi)}$, respectively. Complex roots of the function $S_0(z)$ are represented by the crossing points of the curves $\text{Re } S_0(z) = 0$ and $\text{Im } S_0(z) = 0$ and are located at $(-2.34 + i 0.96) \text{ mK}$, $(-0.59 + i 2.67) \text{ mK}$, $(2.51 + i 4.34) \text{ mK}$ and $(6.92 + i 6.10) \text{ mK}$.

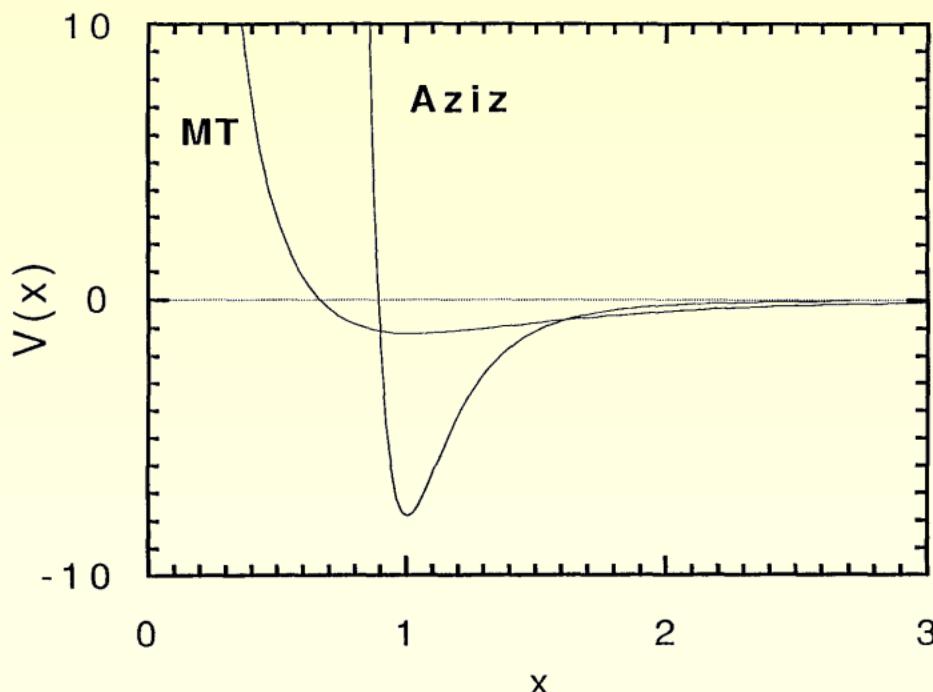
^3H - system

Three-body, theory

MT I-III

$$E_d = -2.224 \text{ MeV}$$

Virtual state of (nnp) -2.69 MeV (0.47 MeV)



$$A=3$$

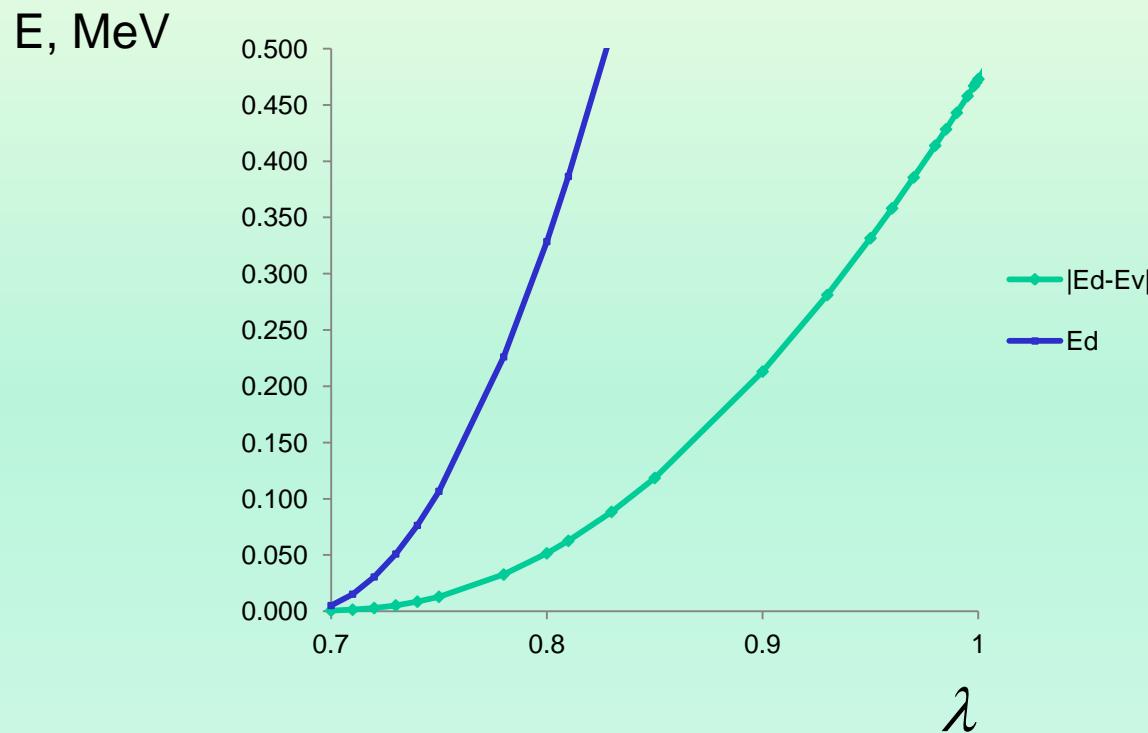
Nucl. Phys. A **848**, 1 (2010) 0.48 MeV

Orlov, Nikitina *Yad.Fiz.* **69** (2006)

Fig. 1. The dimensionless MT I-III $^3\text{S}_1$ nuclear and the HFD-B atomic potentials. The length unit for each potential is the position of its minimum (L). The potentials are made dimensionless according to the Schrödinger equation, i.e. $v = L^2 V / (\hbar^2 / m)$

Carbonell, Gignoux, Merkuriev *FBS* **15**, 15 (1993)

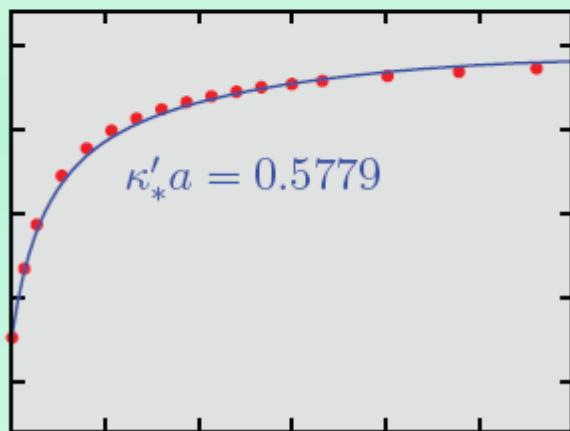
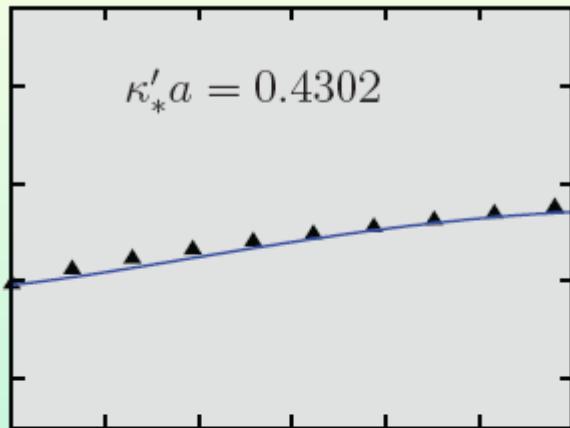
$$V(x) = \lambda V_{MT}(x)$$



$$(ka)^2$$

$$V(x) = \lambda V_c(x)$$

$ka_B \cot \delta(ka)$



A.Kievsky, M.Gattobigio,
Phys. Rev. A **87**, 052719 (2013)

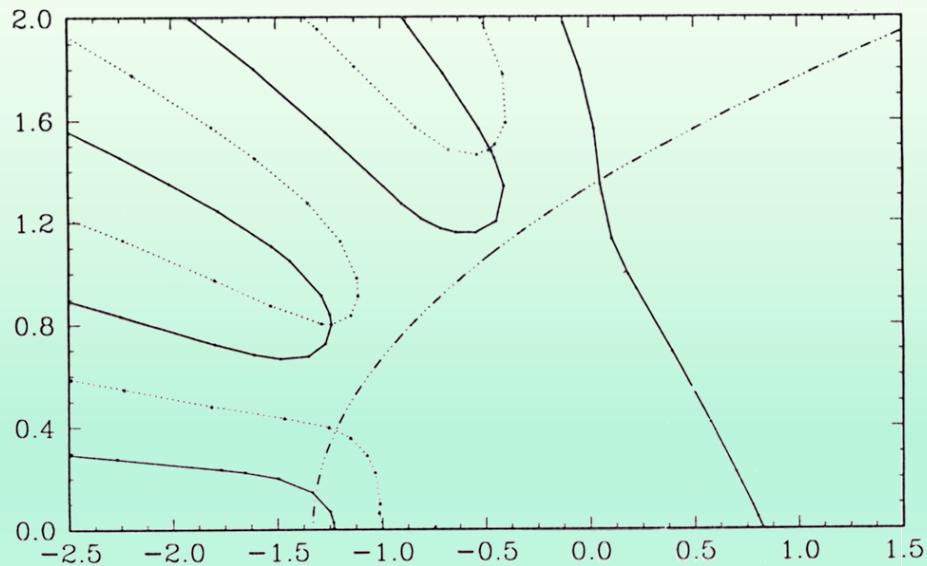
${}^4\text{He} - {}^4\text{He}_2$ scattering

n – d scattering
in the doublet channel

$$ka_B \cot \delta = c_1(ka) + c_2(ka) \cot[s_0 \ln(\kappa'_* a) + \phi(ka)].$$

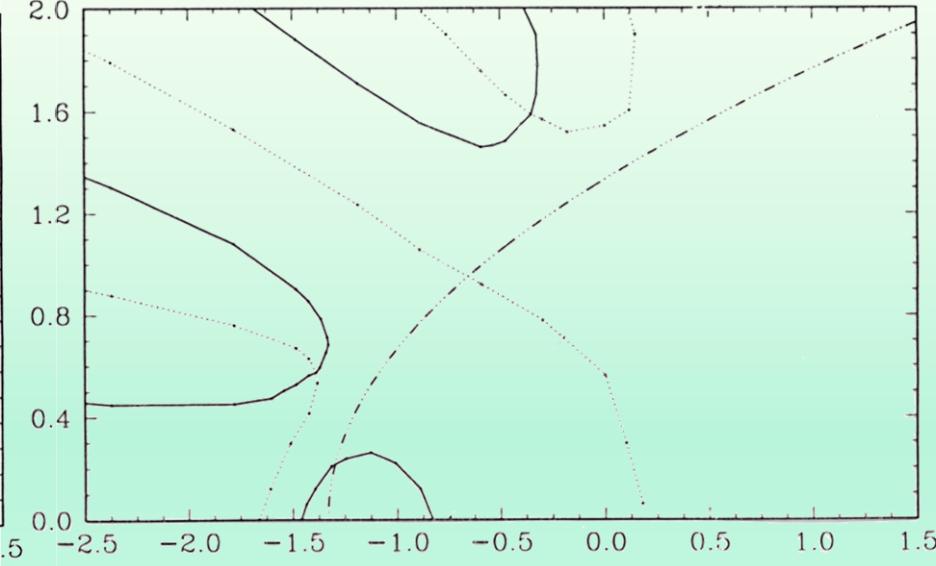
Root locus curve of scattering matrix

$\text{Im } z / |\varepsilon_d|$



^3H

S-matrix root lines in nnp system



$\text{Re } z / |\varepsilon_d|$

$^4\text{He}_3$

S-matrix root lines in $^4\text{He}_3$ system

Solid line - $\text{Re}(S)=0$, tiny dashed line – $\text{Im}(S)=0$

Conclusion

- ◆ We employed formalism which is suitable for three-body atomic systems interacted via hard-core potential. This method let us calculate bound states and scattering observables.
- ◆ It was demonstrated how the Efimov states emerge from the virtual ones when decreasing the strength of the interaction.
- ◆ It was shown similar properties of a very different systems: helium trimer and nnd system