Anomalous statistics of extreme random processes

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- Random walks on “supertrees”: KPZ scaling in a “mean-field approximation”

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Random walks on “supertrees”: KPZ scaling in a “mean-field approximation”

In collaboration with:

Alexander Gorsky
Alexander Valov

Consider ensemble of random matrices A

$$A = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & \cdots \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & \\ \vdots & & & & & \ddots \end{pmatrix}$$

where c_{ij} are taken from one and the same Gaussian distribution $\mathcal{N}(0,1/2)$ with $\langle c_{ij} \rangle \equiv \mu = 0$ and $\langle c_{ij}^2 \rangle \equiv \sigma^2 = 1/2$

The eigenvalue distribution in the ensemble is:

$$f_\beta(\lambda) = c_H^\beta \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^n \lambda_i^2/2}$$ (β =2 – GUE)

The same joint distribution $f_\beta(\lambda)$ appears in ensemble of 3-diagonal random operators (A. Edelman, I. Dumitriu, 2002)

$$M = \begin{pmatrix} a_{11} & b_{12} & 0 & 0 & 0 & \cdots \\ b_{21} & a_{22} & b_{23} & 0 & 0 & \\ 0 & b_{32} & a_{33} & b_{34} & 0 & \\ 0 & 0 & b_{43} & a_{44} & b_{45} & \\ 0 & 0 & 0 & b_{54} & a_{55} & \\ \vdots & & & & & \ddots \end{pmatrix}$$

where all $a_{n,n}$ are normally distributed and $b_{n,n+1}=b_{n+1,n}$ are χ_n-distributied:

$\mu = 0$ and $\sigma^2 = 1/2$

$$\begin{cases} f(x|\mu,\sigma) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ f(x|n) = \dfrac{x^{n-1} e^{-\frac{x^2}{2}}}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)}, \quad x \geq 0 \end{cases}$$

For averaged matrix $<M>$ we get:

$$\left\langle \hat{M} \right\rangle \approx \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

$$\mathbf{E}_{\chi_{(k)}}(x) = \frac{\sqrt{2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

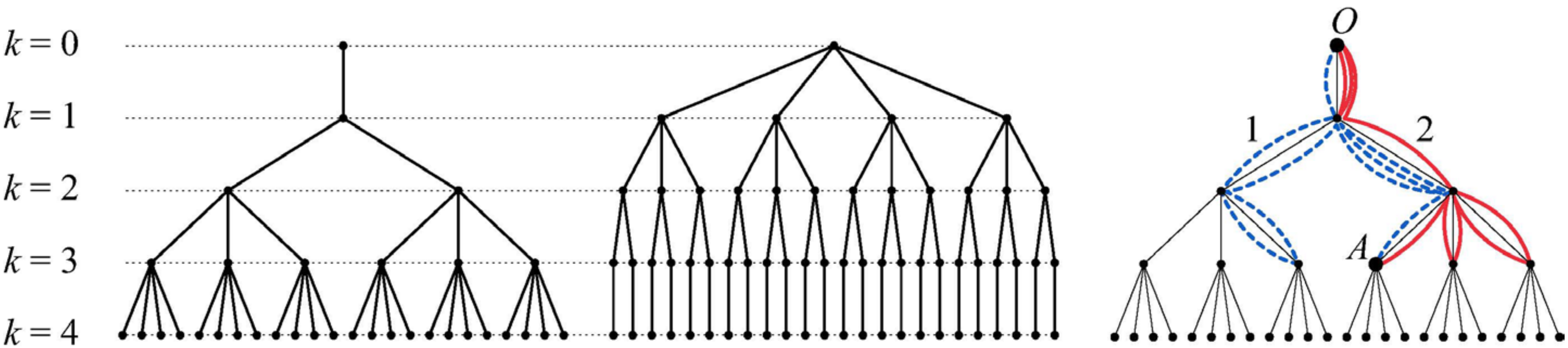
$$\mathbf{E}_{\chi_{(k)}}(x)\Big|_{k\gg 1} = \sqrt{k}$$

Instead of M one can take M', $(\det \hat{M} = \det \hat{M}')$

$$\hat{M}' = \begin{pmatrix} a_{11} & 1 & 0 & 0 & 0 & \dots \\ b_{21}^2 & a_{22} & 1 & 0 & 0 & \\ 0 & b_{32}^2 & a_{33} & 1 & 0 & \\ 0 & 0 & b_{43}^2 & a_{44} & 1 & \\ 0 & 0 & 0 & b_{54}^2 & a_{55} & \\ \vdots & & & & & \ddots \end{pmatrix}$$

$$\left\langle \hat{M}' \right\rangle \approx \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \\ 0 & 2 & 0 & 1 & 0 & \\ 0 & 0 & 3 & 0 & 1 & \\ 0 & 0 & 0 & 4 & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

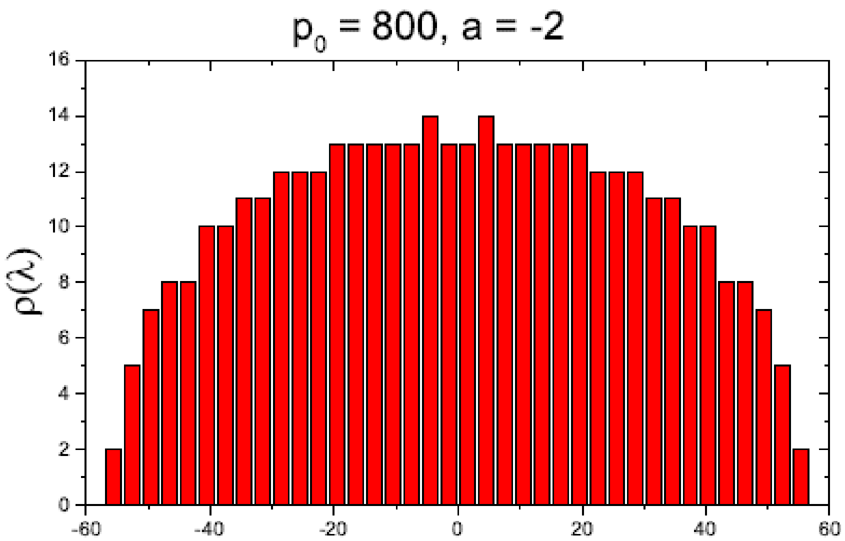
Random walk and Brownian bridges on supertrees



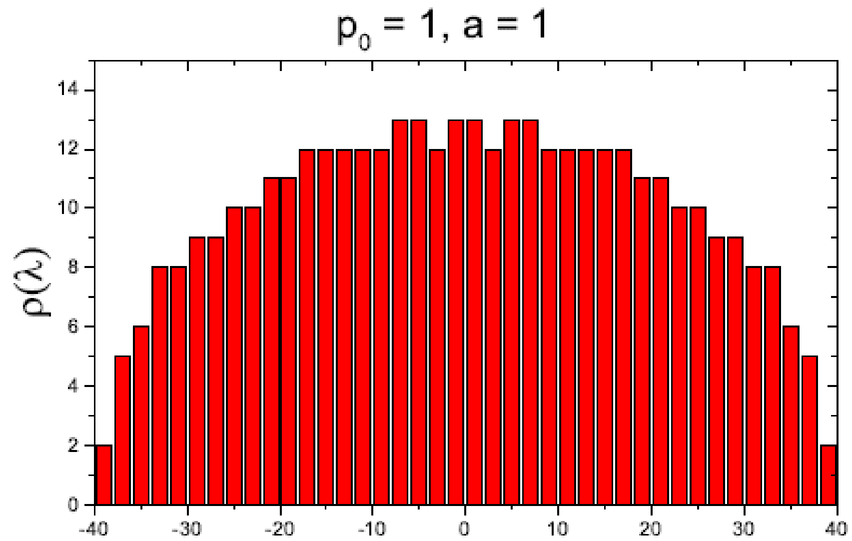
$$p_k = \begin{cases} p_0, & \text{for } k = 0 \\ 2 + ak, & \text{for } k \geq 1,\ a \geq 0 \end{cases} \qquad p_k = \begin{cases} p_0, & \text{for } k = 0 \\ p_0 + ak, & \text{for } k \geq 1,\ a \leq 0 \end{cases}$$

$$\mathbf{Z}_{N+1} = \hat{T}\mathbf{Z}_N; \quad \hat{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ p_0 & 0 & 1 & 0 & & \\ 0 & p_1 - 1 & 0 & 1 & & \vdots \\ 0 & 0 & p_2 - 1 & 0 & & \\ \vdots & & & & \ddots & \\ 0 & & \dots & & p_{K-2} - 1 & 0 \end{pmatrix} \quad \mathbf{Z}_{N=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Spectral density of transfer matrix



(a)



(b)

Spectral density of the transfer matrix on supertree

$$\rho(\lambda) = \frac{1}{2\pi K}\sqrt{4K - \lambda^2}$$ – Wigner semicircle

- Characteristic polynomials = Hermite polynomials

$$P_k(\lambda) \equiv \mathcal{H}_k(\lambda) = (-1)^k e^{\frac{\lambda^2}{2}} \frac{d^k}{d\lambda^k} e^{-\frac{\lambda^2}{2}}; \qquad \mathcal{H}_k(\lambda) = 2^{-k/2} H_k(\lambda/\sqrt{2})$$

- Asymptotic behavior near the spectral edge

  at $K >> 1$, $\lambda = 2K - \varepsilon$, $|\varepsilon| << 1$

$$\mathcal{H}_K(\lambda) \approx \sqrt{2\pi}\, 2^{-K/2} \exp\left(\frac{K\ln(2K)}{2} - \frac{3K}{2} + \lambda\sqrt{K}\right) K^{1/6} \mathrm{Ai}\left(\frac{\lambda - 2\sqrt{K}}{K^{-1/6}}\right)$$

- Finite size scaling of largest eigenvalue

$$\lambda_{\max} = 2\sqrt{K} + a_1 K^{-1/6}$$

  where $a_1 \approx 2.3381$ is the first zero of Airy function

Brownian bridges on finite supertrees

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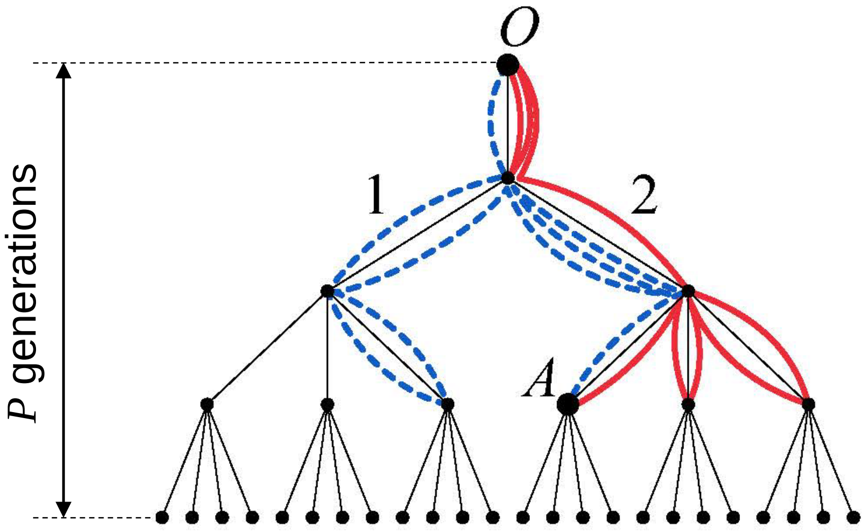

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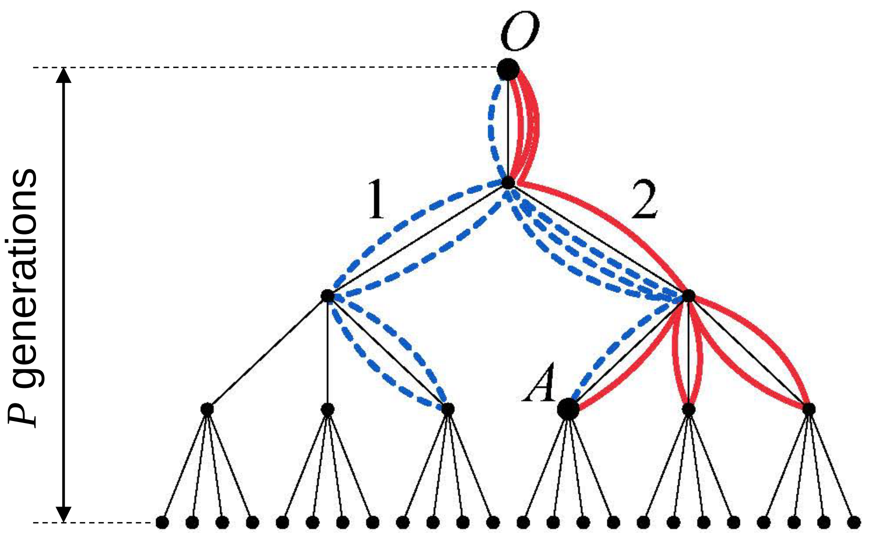
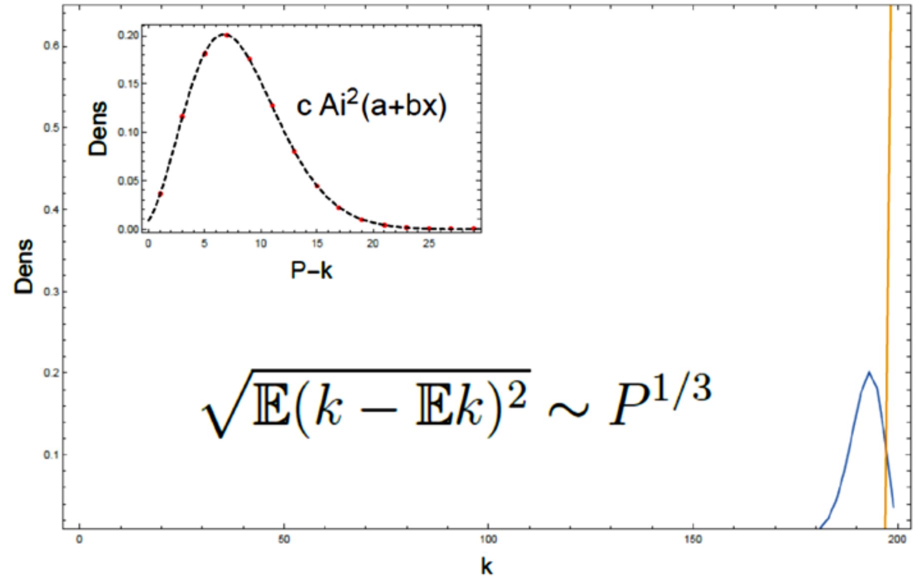
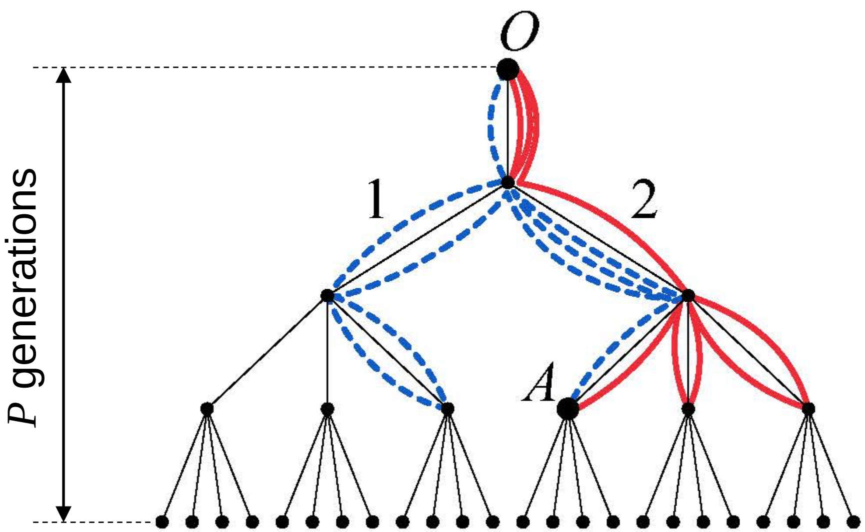


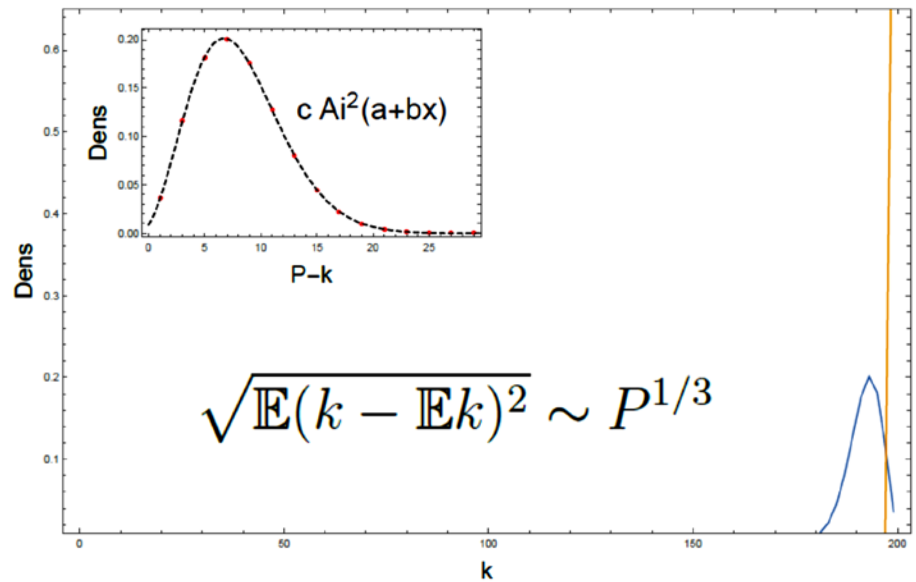


$$Q(P-i) = c\frac{H^2_{P-i-1}(\lambda_{max})}{(P-i)!}$$

$$\approx C\,Ai^2\left(a_1 + \frac{1}{P^{1/3}} + \frac{i}{P^{1/3}}\right)$$

Brownian bridges on finite supertrees





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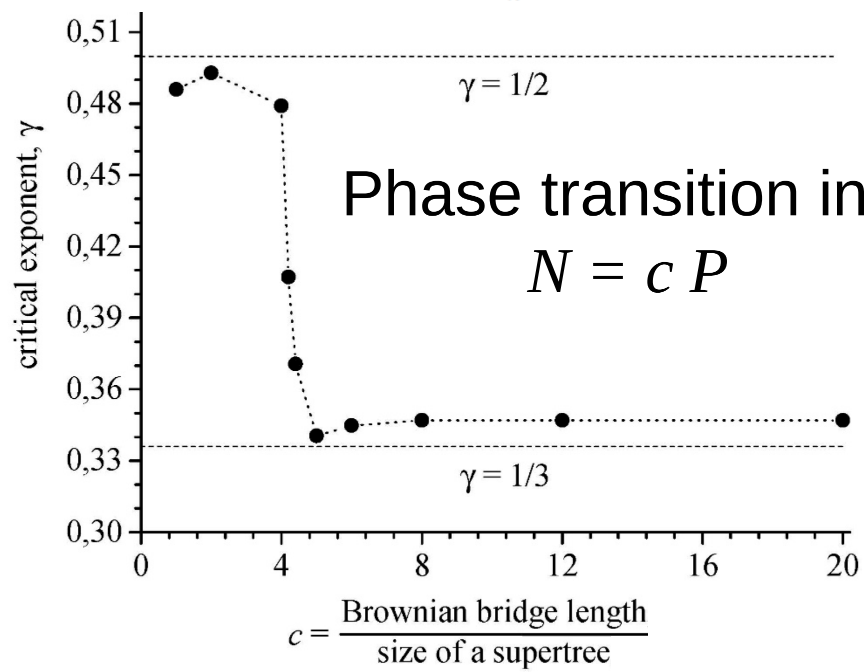

Phase transition in c

$N = c\,P$

Anomalous scaling for fluctuations of “elongated” 2D paths above curved domains: KPZ as a manifestation of large deviations

In collaboration with:

Kirill Polovnikov
Senya Shlosman
Alexander Valov
Alexander Vladimirov

2D Random paths above voids of various shapes

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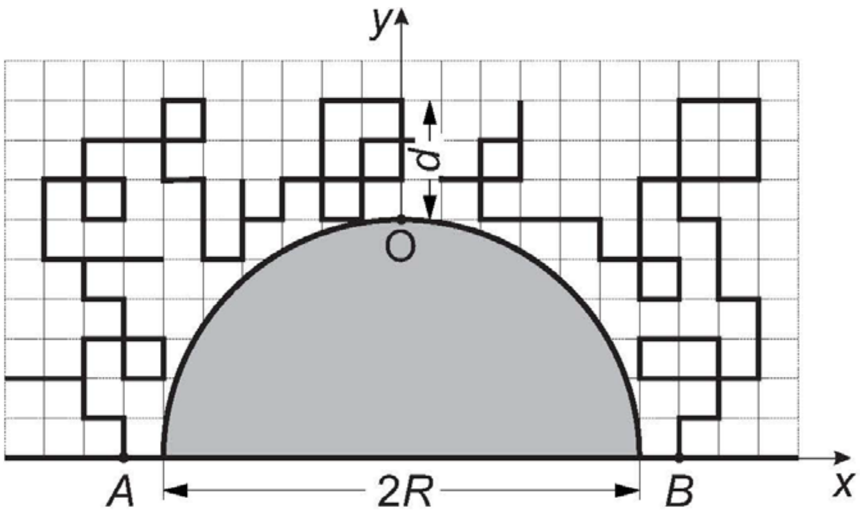
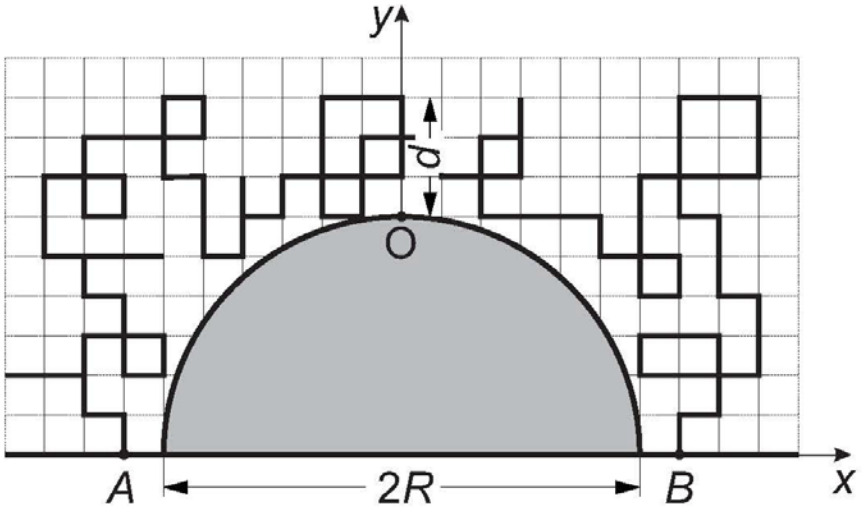


Semicircle:

2D Random paths above voids of various shapes



Semicircle:

$$N \sim R^2 \quad \rightarrow \quad d(N) \sim \sqrt{N}$$

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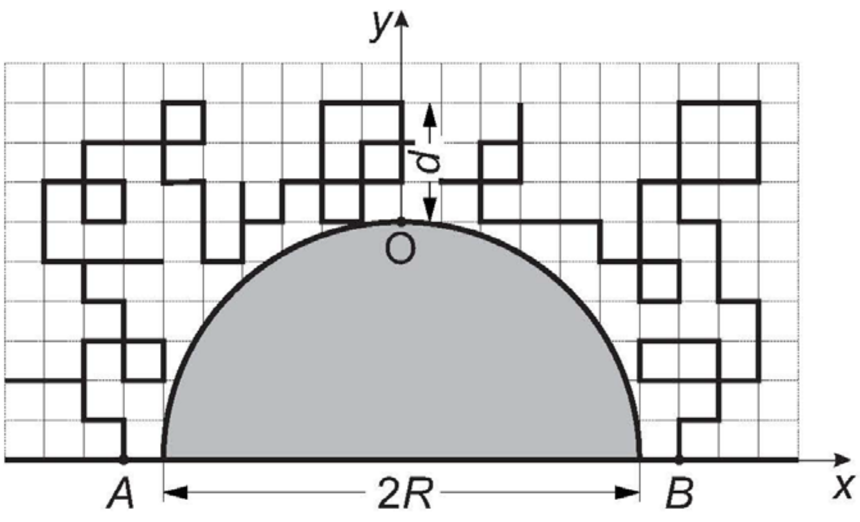

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$$N = cR \quad \to \quad d(N) \sim N^{1/3}$$

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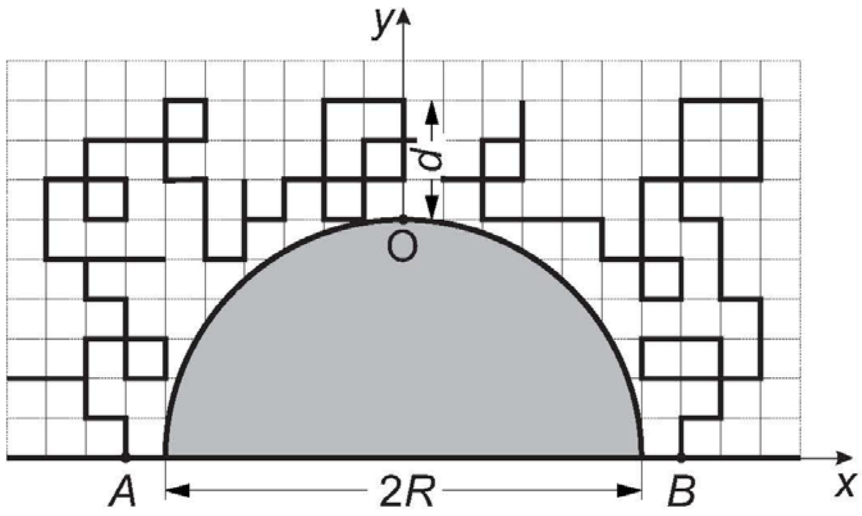

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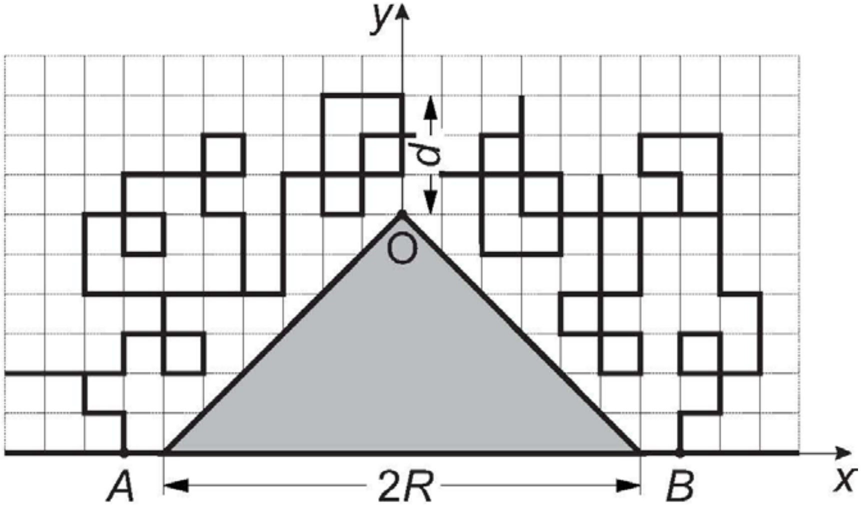
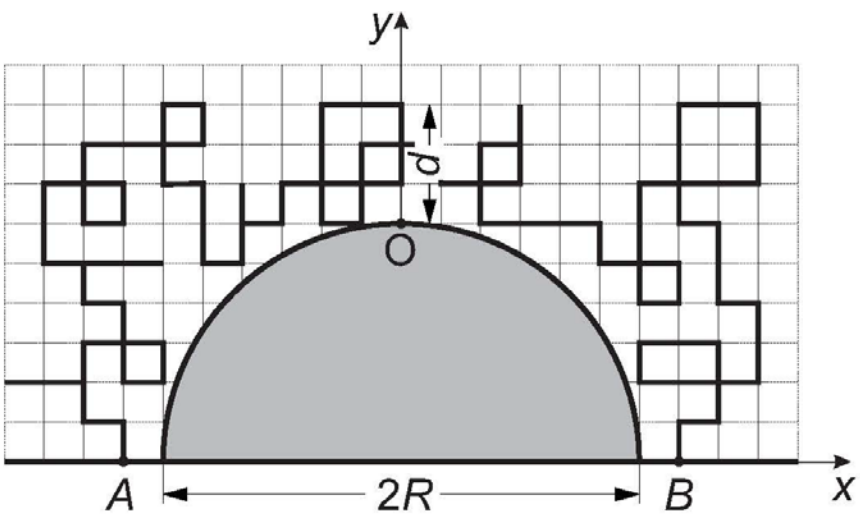


Triangle:

2D Random paths above voids of various shapes



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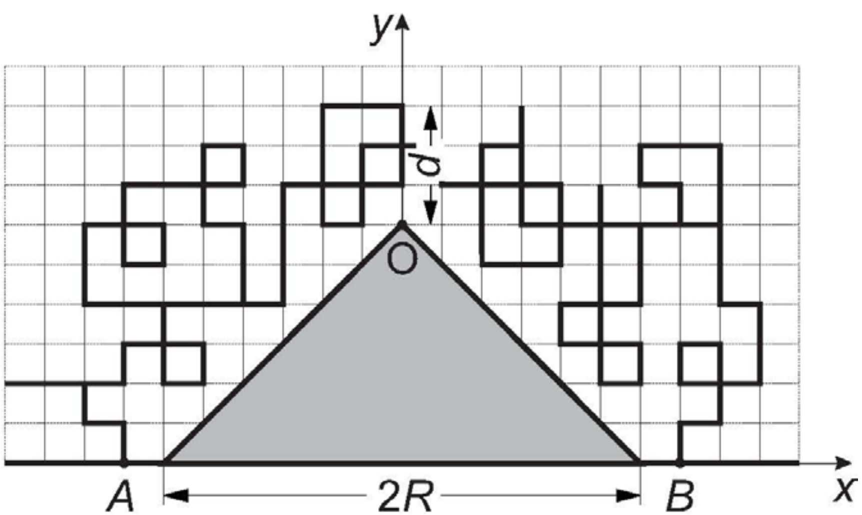

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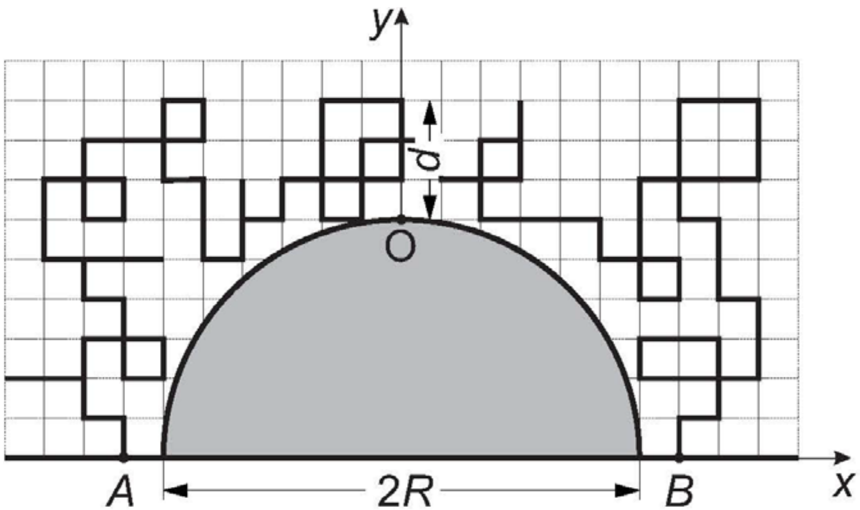

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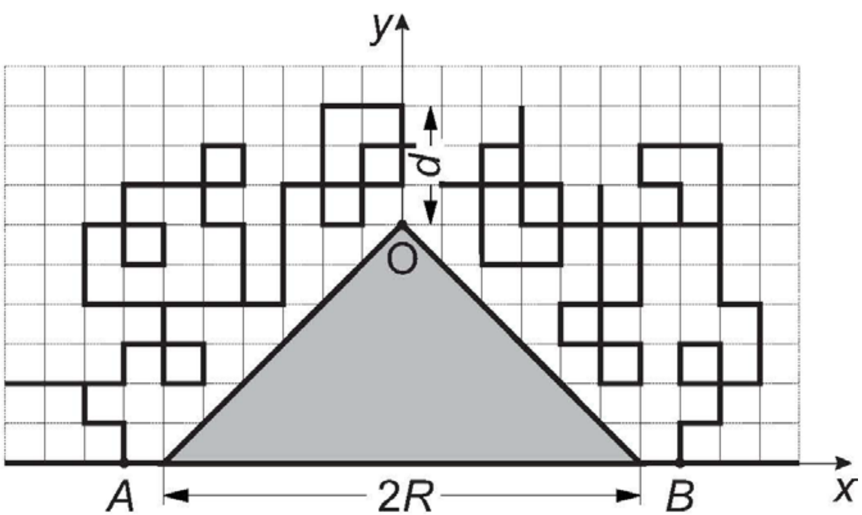

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Stretched ($N = c\,R$) paths above the semicircle

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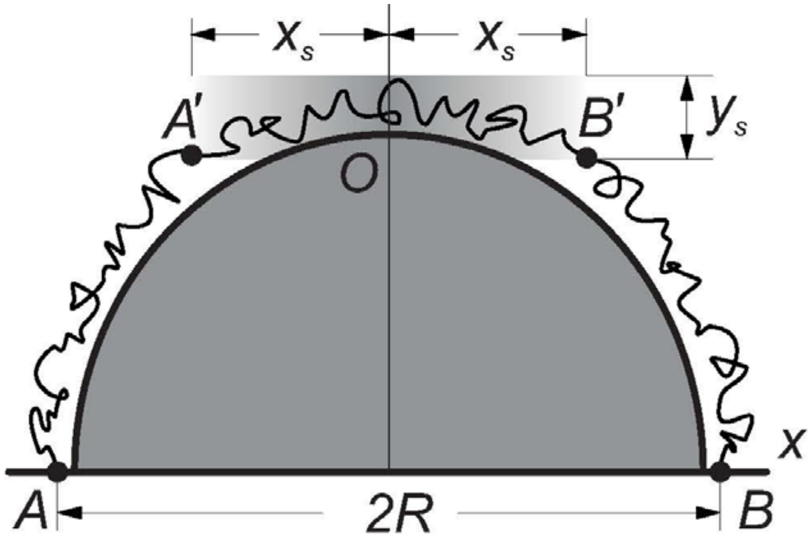
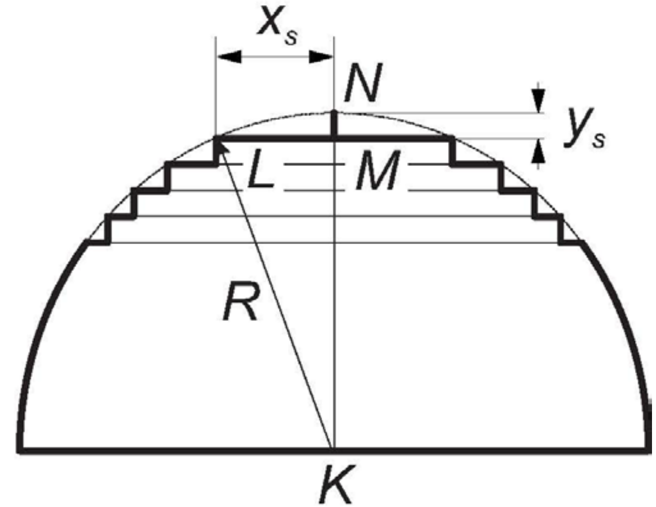
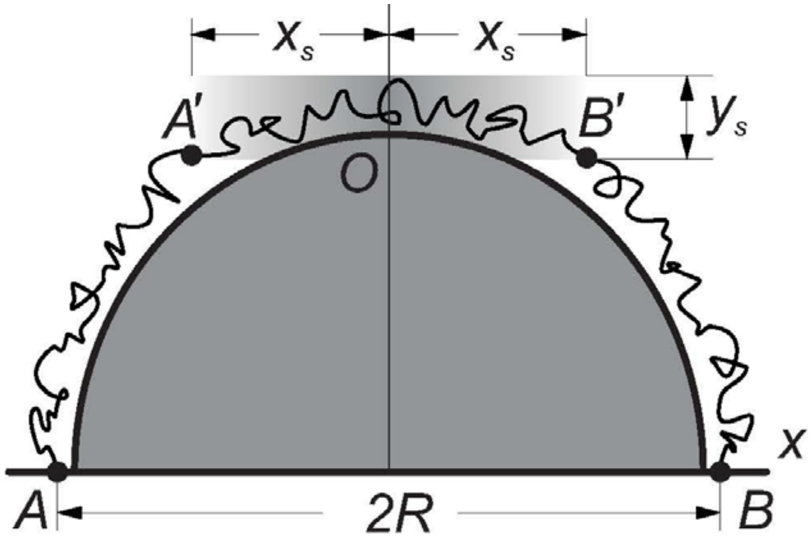
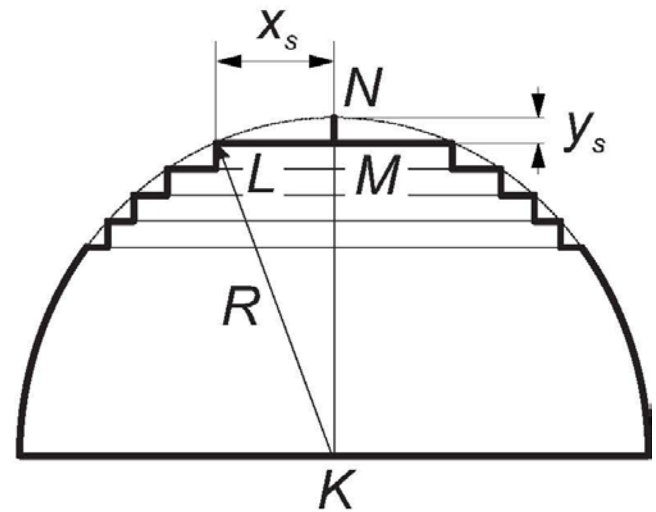




Stretched ($N = c\,R$) paths above the semicircle





Linearizing curved shape, we get: $x = \sqrt{R^2 - (R-y)^2} \approx \sqrt{2Ry}$

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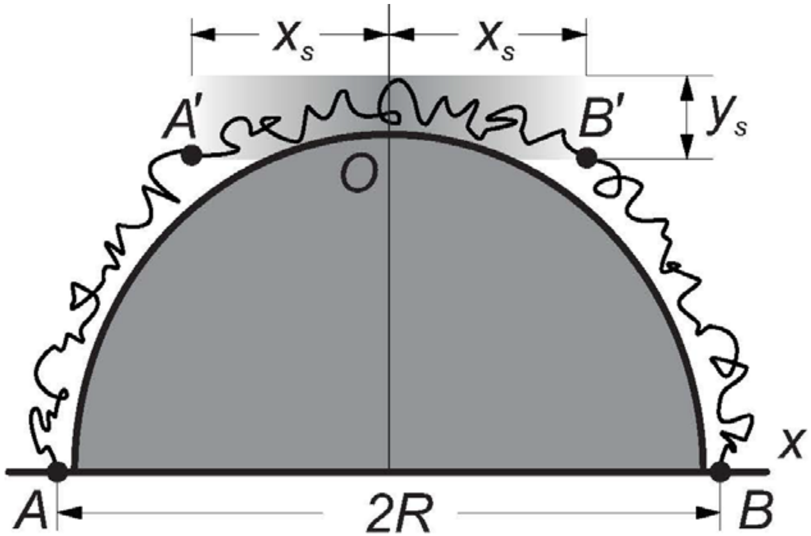

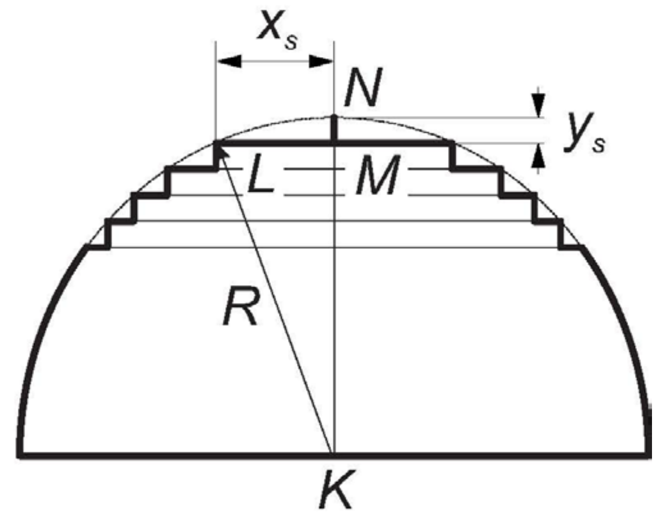

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Above the flat line one has a random walk $y \sim \sqrt{x}$

thus $x \sim \sqrt{R\sqrt{x}}$, and finally, $x \sim R^{2/3}$, $y \sim R^{1/3}$

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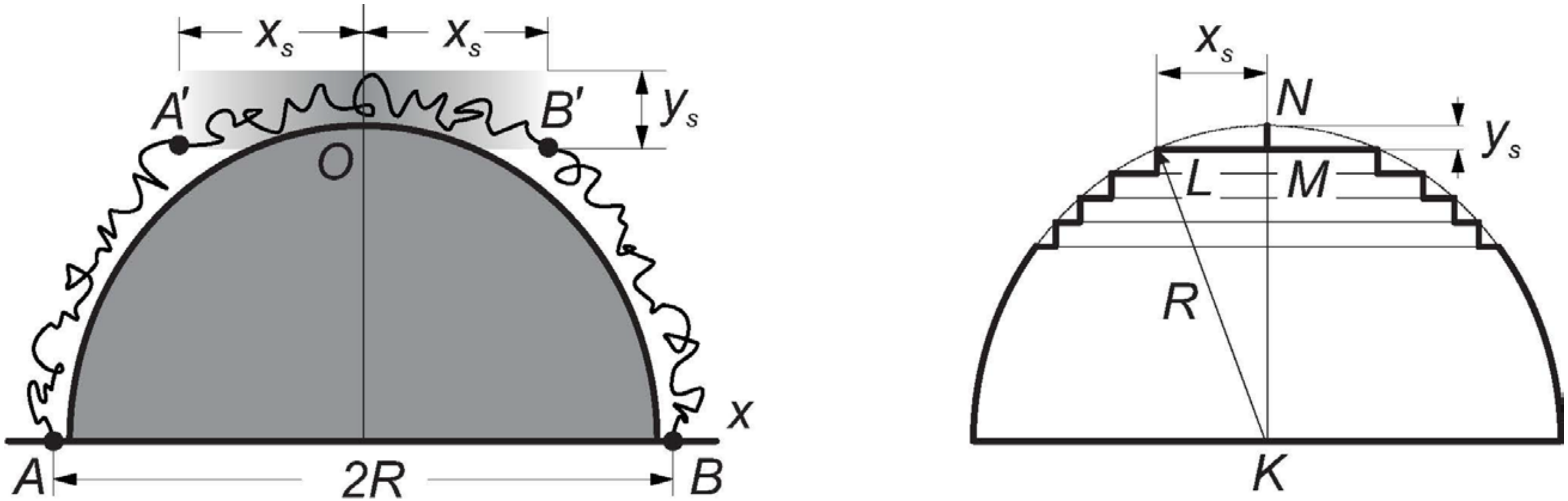

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Stretching above algebraic curve $\frac{y}{R} \approx \left(\frac{x}{R}\right)^{\eta}$ provides generic scaling

$$y_G(R) \sim R^{\gamma}; \qquad \gamma = \frac{\eta - 1}{2\eta - 1}$$

Exponent $\gamma = 1/3$ emerges for uniformly curved surface

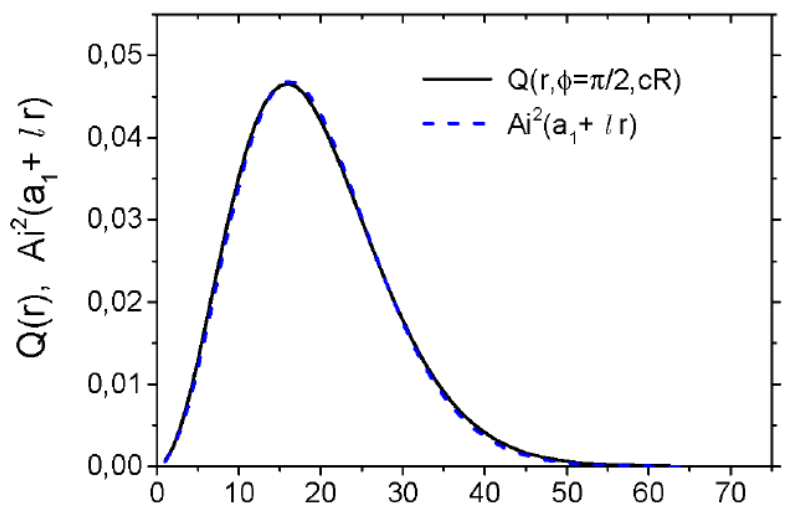
$$
\begin{cases}
\dfrac{\partial P(\rho,\phi,t)}{\partial t} = D\left[\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial P(\rho,\phi,t)}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 P(\rho,\phi,t)}{\partial\phi^2}\right] \\
P(\rho = R,\phi,t) = P(\rho\to\infty,\phi,t) = P(\rho,\phi=0,t) = P(\rho,\phi=\pi,t) = 0 \\
P(\rho,\phi,0) = \delta(\rho-\rho_0)\delta(\phi-\phi_0)
\end{cases}
$$

$$
P(\rho,\phi,t) = \sum_{k=1}^{\infty}\frac{2\rho_0}{\pi}\sin(k\phi_0)\sin(k\phi)\int_0^\infty e^{-\lambda^2 Dt} Z_k(\lambda\rho,\lambda R) Z_k(\lambda\rho_0,\lambda R)\lambda d\lambda
$$

$$
Z_k(\lambda\rho,\lambda R) = \frac{-J_k(\lambda\rho)N_k(\lambda R) + J_k(\lambda R)N_k(\lambda\rho)}{\sqrt{J_k^2(\lambda R)+N_k^2(\lambda R)}} \qquad \lambda = \frac{\mu}{R}, \qquad \rho = R + r
$$

$$
Q\left(r,\phi=\frac{\pi}{2},t\right) = \frac{1}{\mathcal{N}}P^2\left(r,\phi=\frac{\pi}{2},t=cR\right)
$$

$$
\mathcal{N} = \int_0^\infty P^2\left(r,\phi=\frac{\pi}{2},t\right)dr
$$



1D random walk trapping in a Poissonian field: Lifshitz tail of 1D Anderson localization *vs* KPZ

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Free energy of stretched paths: scaling approach

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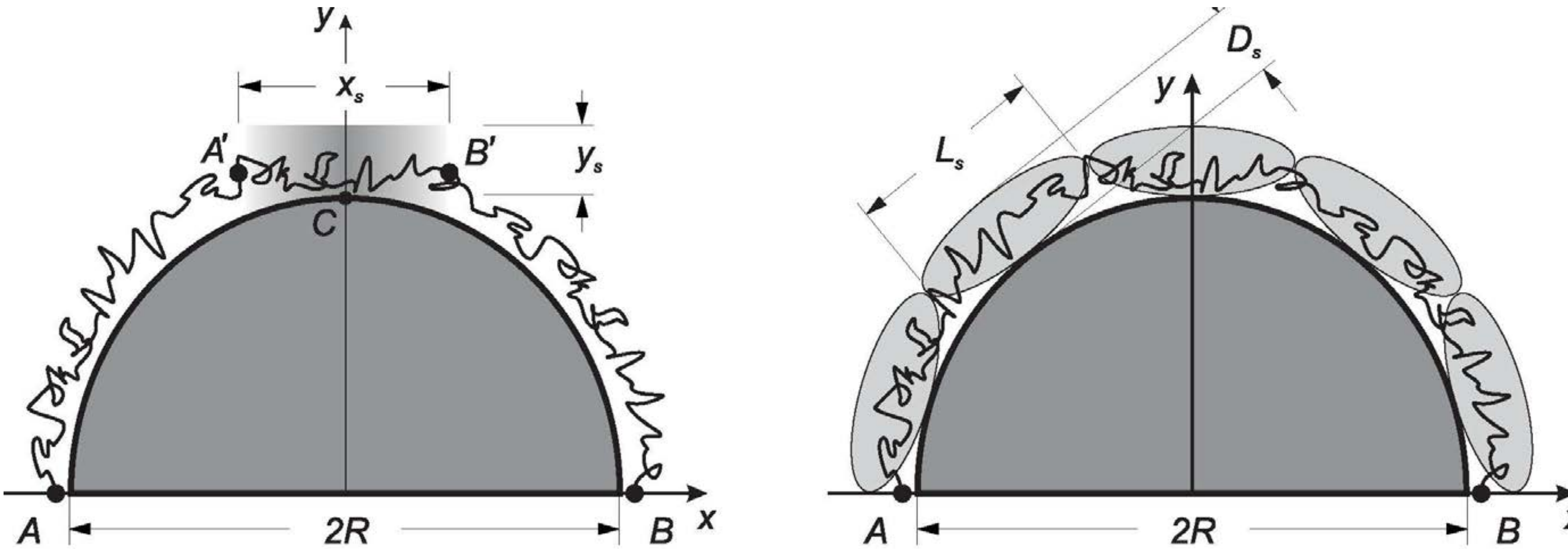

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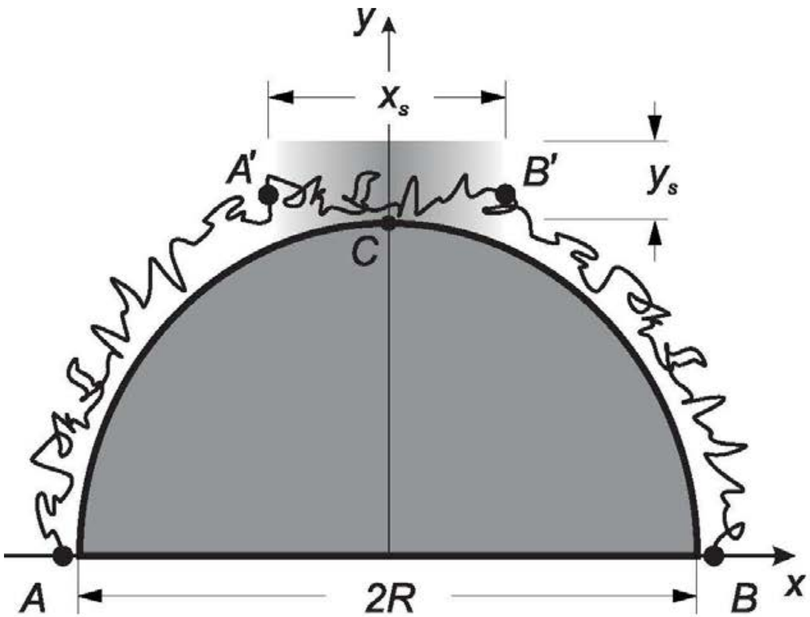

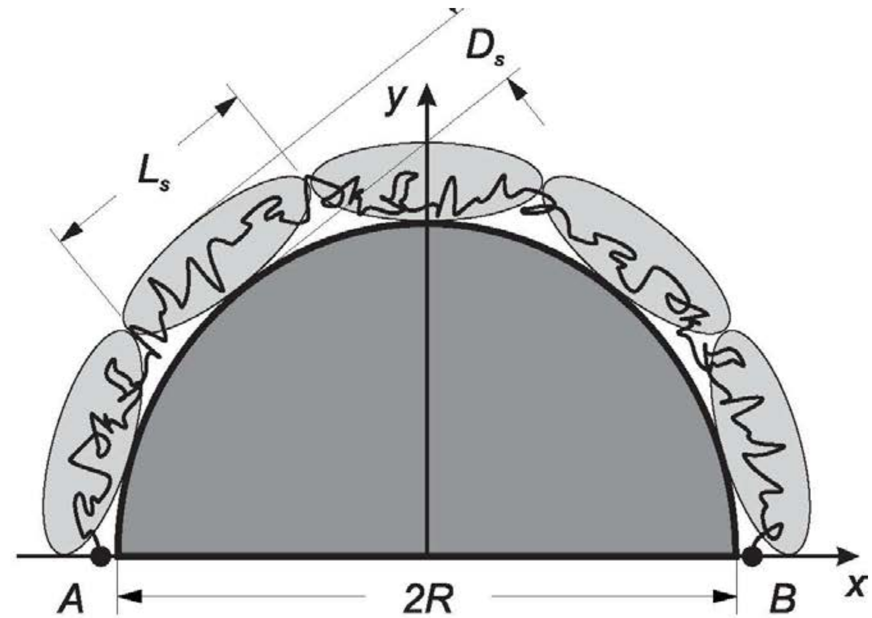

Blob's width $D_s \sim R^{1/3}$, blob's length $L_s \sim R^{2/3}$

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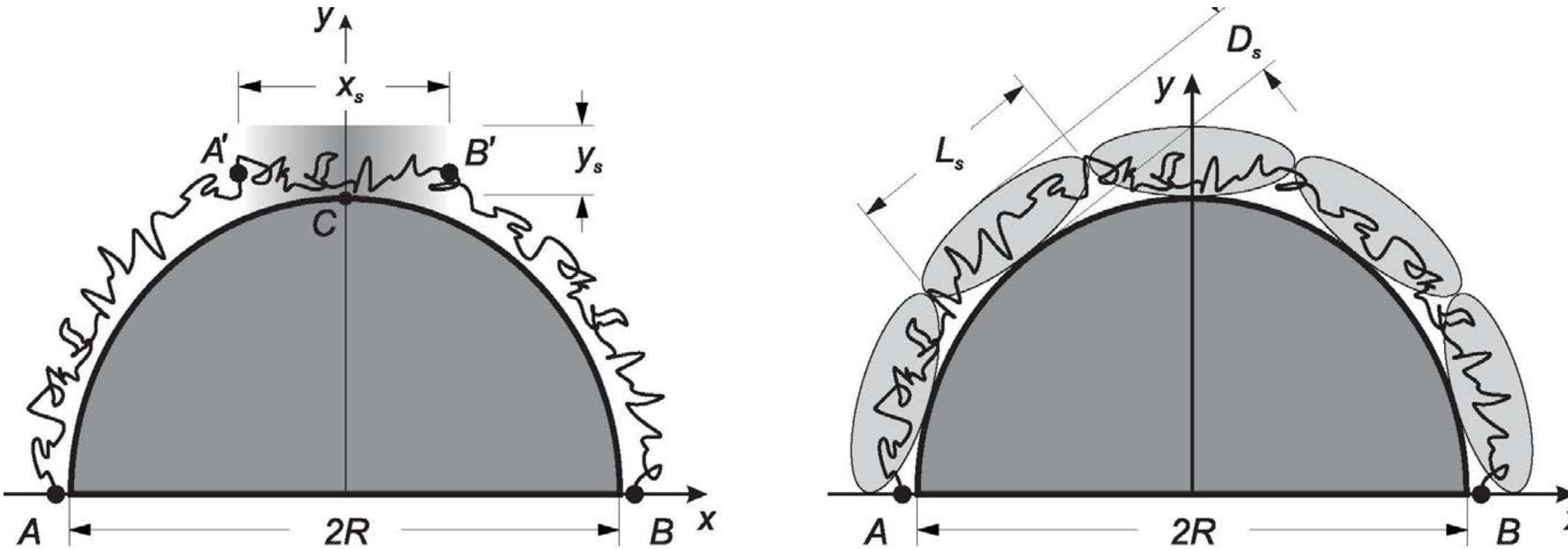

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Free energy of a chain stretched above semicircle

$$F \sim \frac{N}{L_s} \sim \frac{R}{R^{2/3}} \sim R^{1/3}$$

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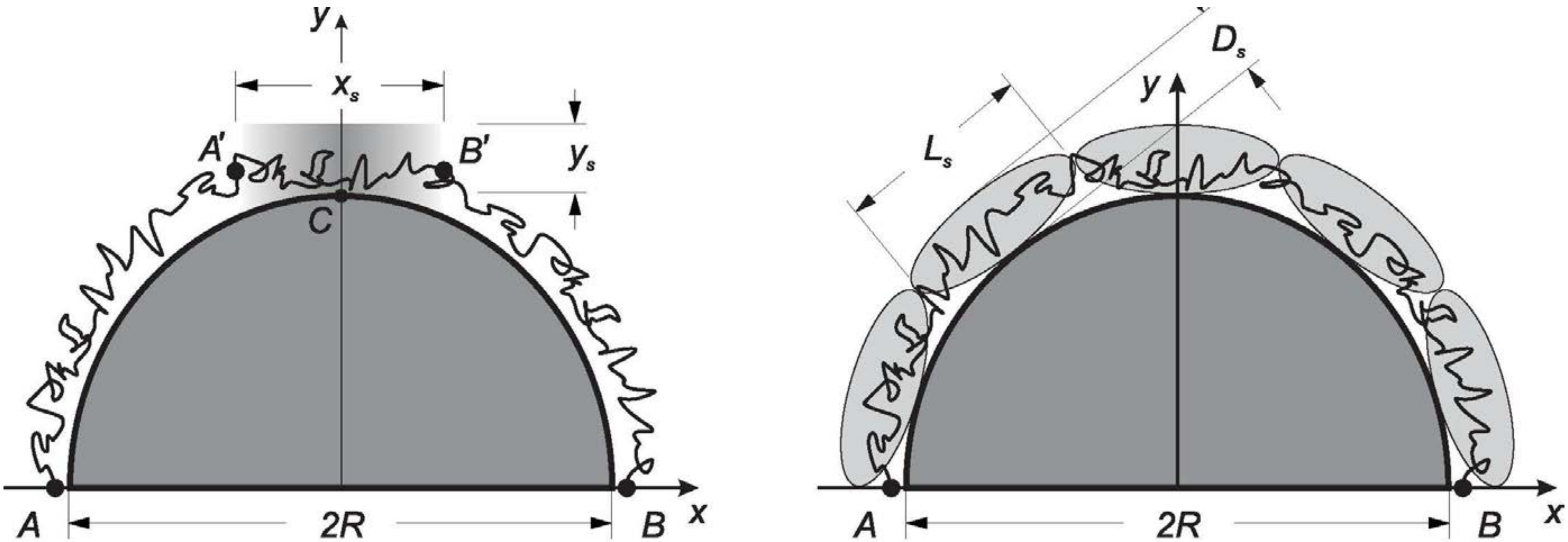

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Free energy of a chain stretched above semicircle

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Gibbs measure $W(R) = e^{-F(R)} \sim e^{-\alpha R^{1/3}}$

Stretched KPZ exponent *vs* Lifshitz tail:
Optimal fluctuation for survival probability

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Gibbs measure of stretched path in curved channel

Survival probability in 1D trapping in Poissonian field

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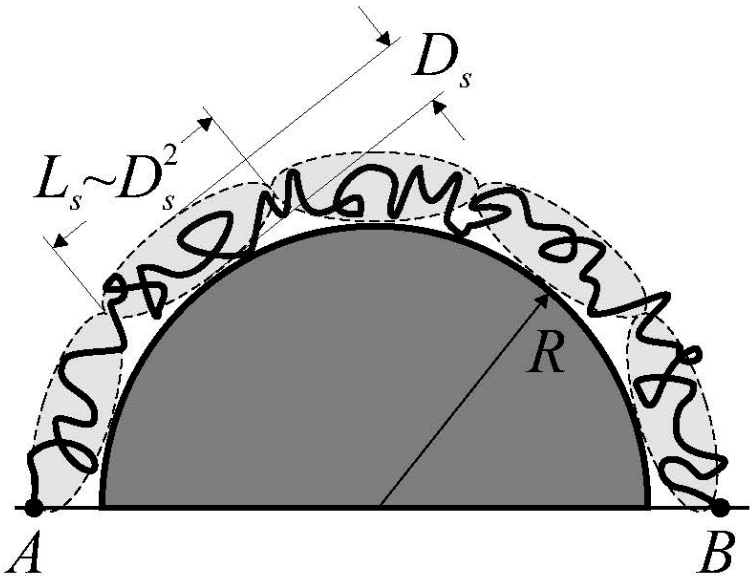

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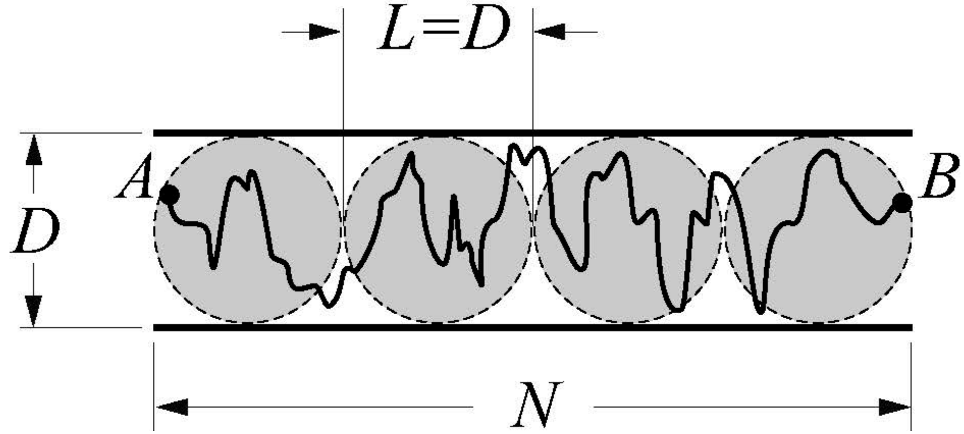

Stretched KPZ exponent *vs* Lifshitz tail: Optimal fluctuation for survival probability

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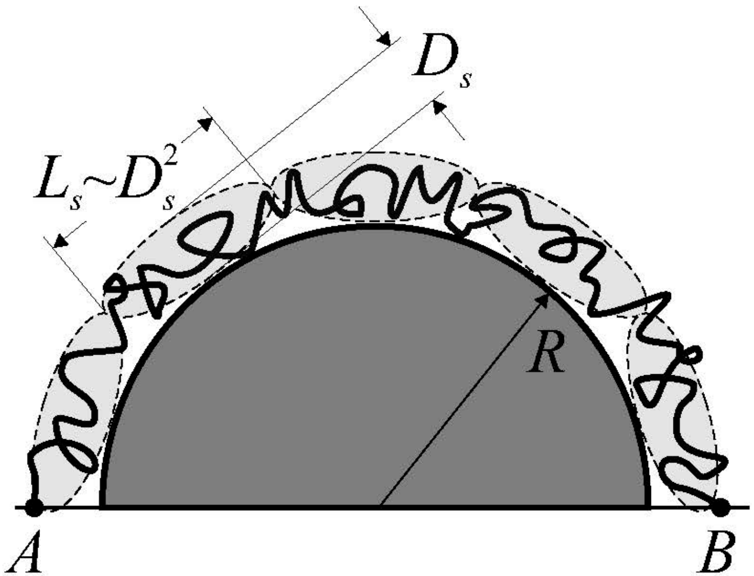

Since $N = c\,R$, one gets

$$W(R \sim N) \sim e^{-\alpha N^{1/3}}$$

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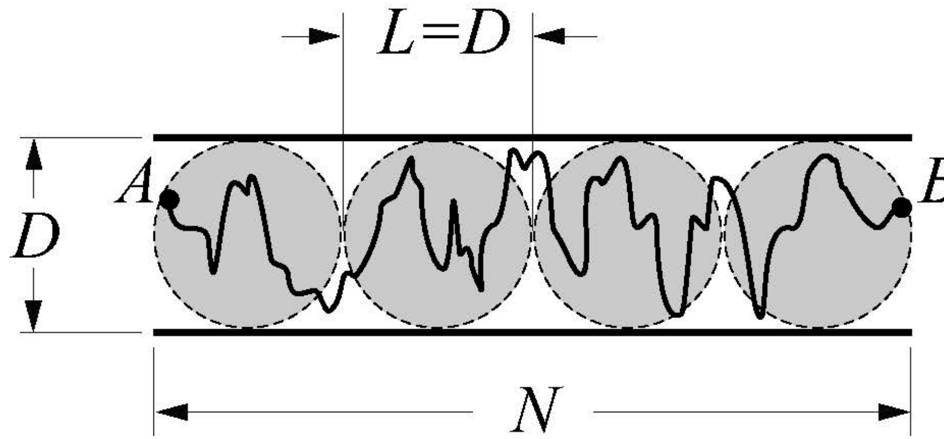

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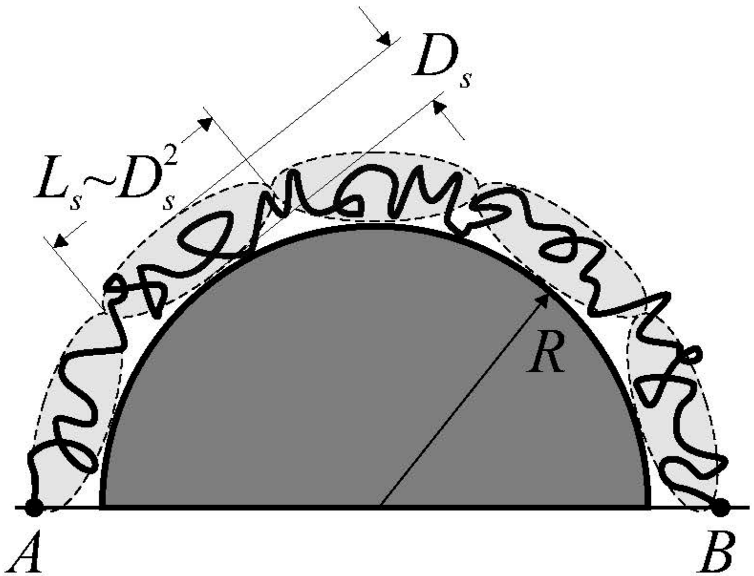

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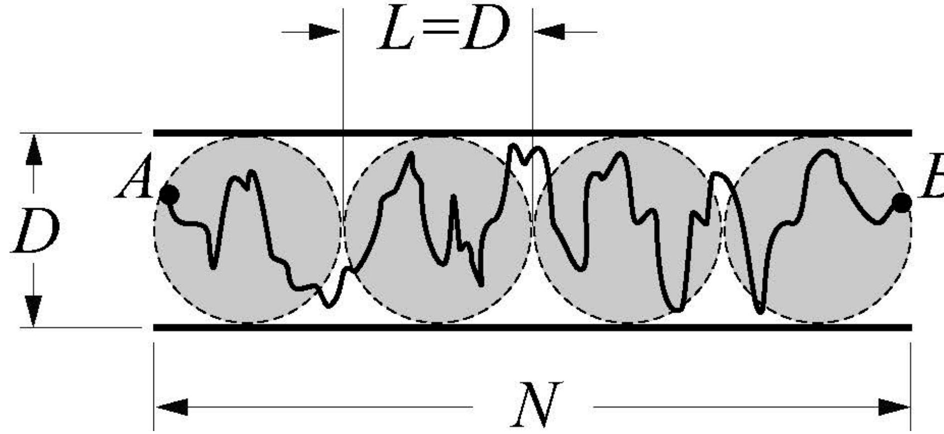

Free energy to be minimized over D

$$F(N,D) = F_e(N) - \ln Q(D) = \frac{N}{D^2} + \beta D$$

one gets $\bar{D}(N) \sim N^{1/3}$ and survival probability is (Balagurov, Vaks, 1974)

$$P(N) = e^{-F(N,\bar{D})} \sim e^{-c\beta^{2/3}N^{1/3}}$$

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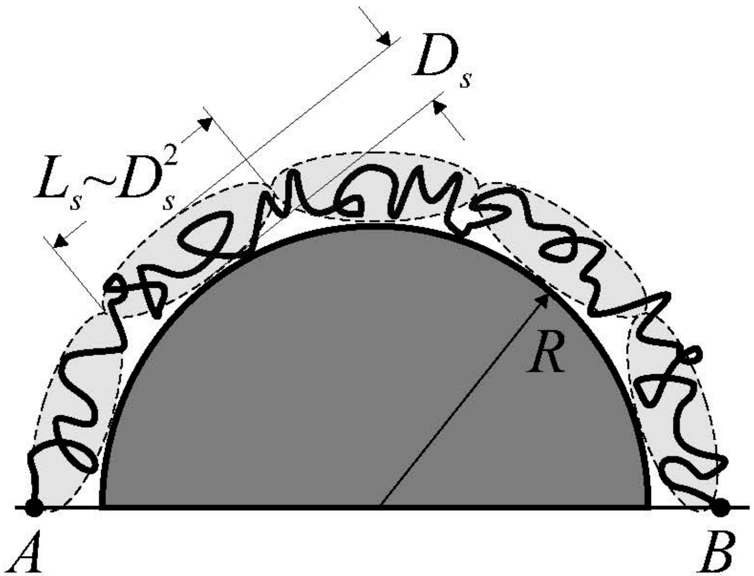
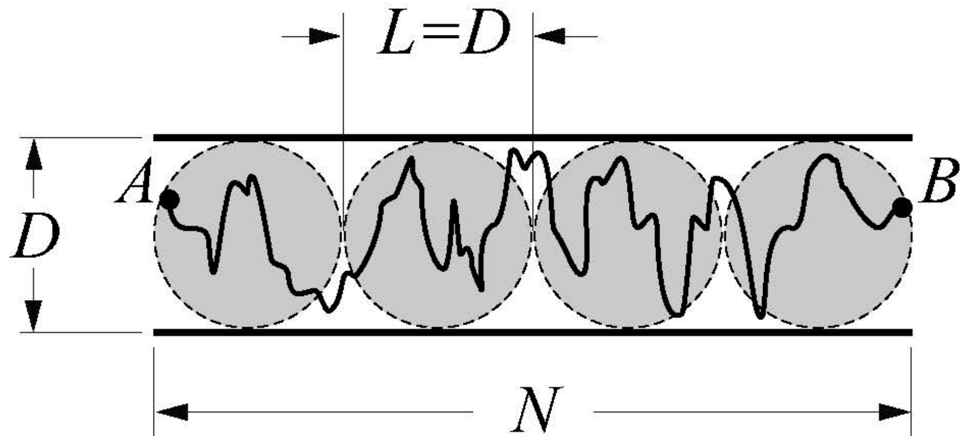


Since $N = c\,R$, one gets

$$W(R \sim N) \sim e^{-\alpha N^{1/3}}$$

$$\int_0^\infty W(R)\, e^{-sR} dR \sim e^{-\frac{a^{2/3}}{\sqrt{s}}}$$

Survival probability in 1D trapping in Poissonian field



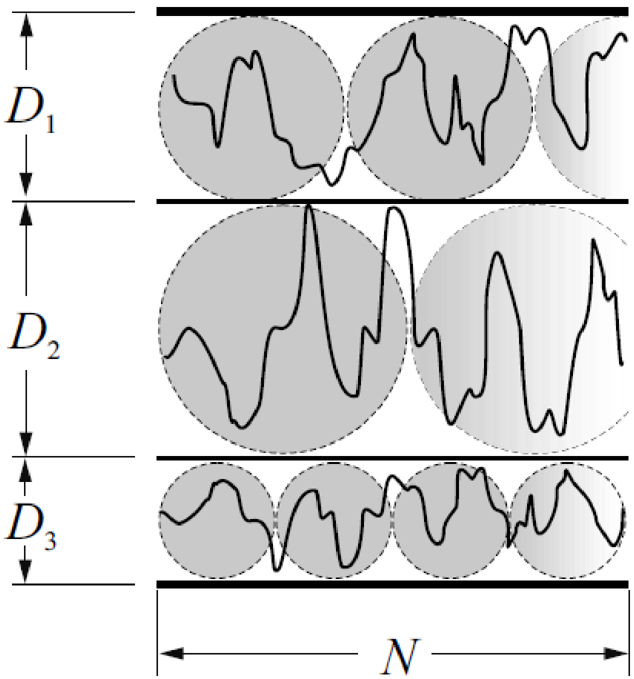
Free energy to be minimized over D

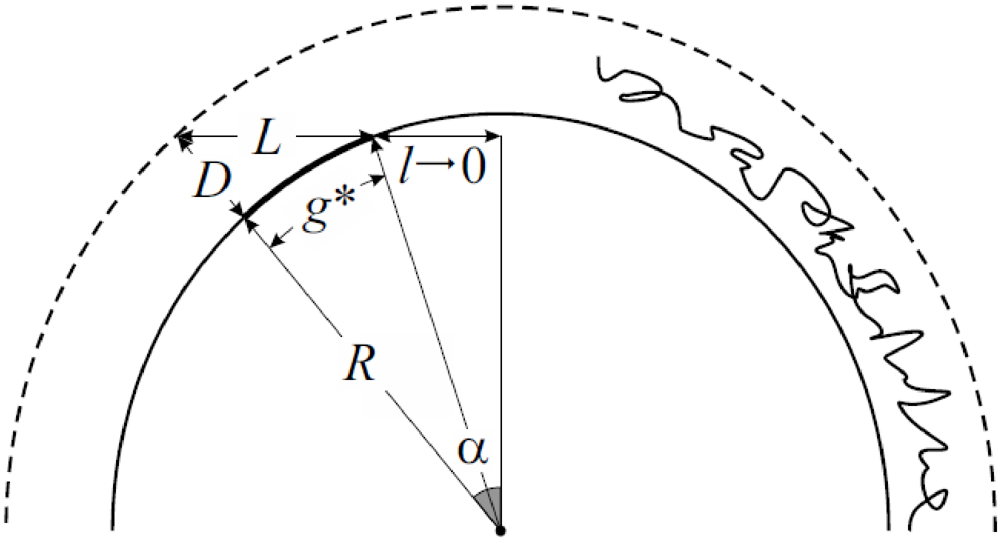
$$F(N, D) = F_e(N) - \ln Q(D) = \frac{N}{D^2} + \beta D$$

one gets $\bar{D}(N) \sim N^{1/3}$ and survival probability is (Balagurov, Vaks, 1974)

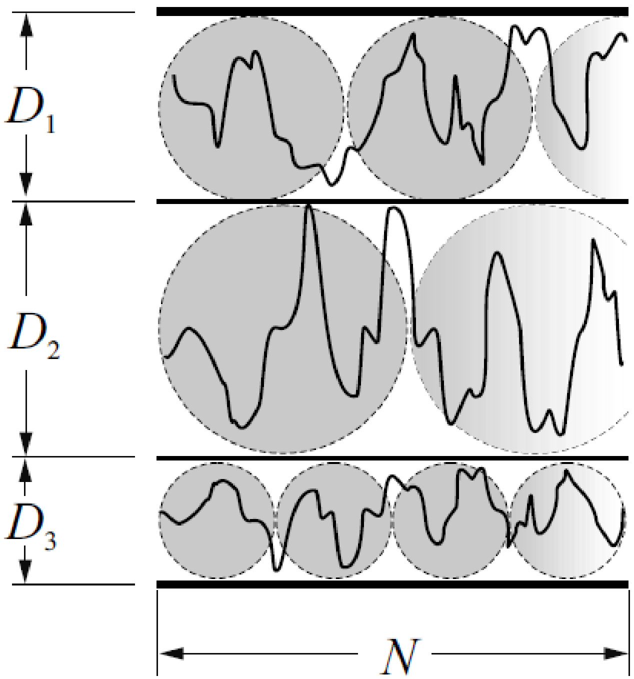
$$P(N) = e^{-F(N, \bar{D})} \sim e^{-c\beta^{2/3} N^{1/3}}$$

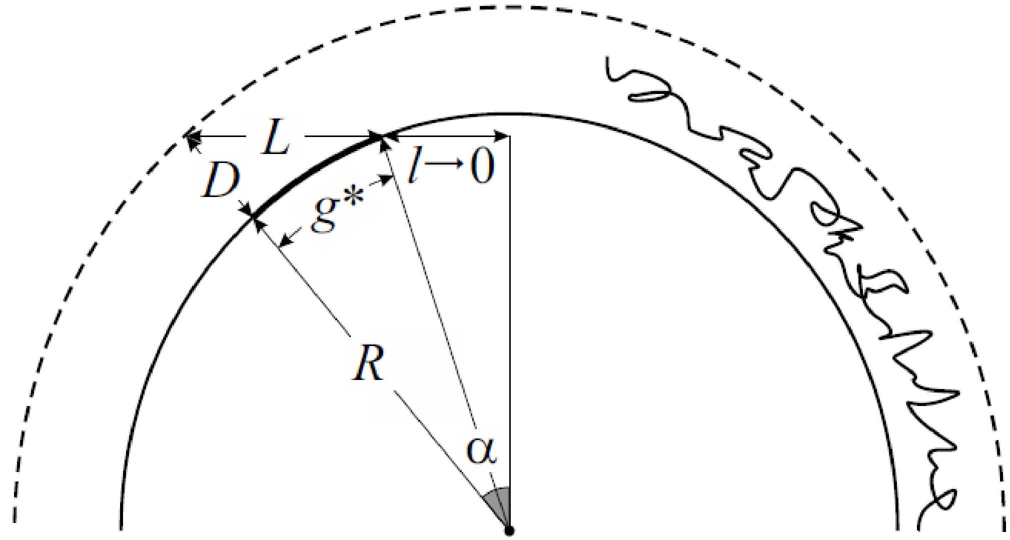
Stretched KPZ exponent *vs* Lifshitz tail: Optimal fluctuation for survival probability





Stretched KPZ exponent *vs* Lifshitz tail: Optimal fluctuation for survival probability





Free energy $F(D,R)$ of the stretched ($N = cR$) paths confined in a curved slit of size D, can be estimated as

$$F(D,R) \sim \frac{R}{D^2} - \frac{R}{g^*(R)}$$

g^* - average number of monomers in a blob.

Stretched KPZ exponent *vs* Lifshitz tail: Optimal fluctuation for survival probability

One can estimate g^* as follows: $g^* \sim R\,\alpha_{\min}$, where $\alpha_{\min} \sim \sqrt{\dfrac{D}{R}}$ which gives the following expression for $F(D,R)$:

$$F(D,R) \sim \frac{R}{D^2} - \sqrt{\frac{D}{R}}$$

Minimizing $F(D,R)$ with respect to D, we get equilibrium width of a slit D for stretched paths evading the semicircle: $\bar{D}(N) \sim N^{1/3}$

Stretched KPZ exponent *vs* Lifshitz tail: Optimal fluctuation for survival probability

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Conclusion / conjecture:

A **disorder-free** model of stretched paths above a semicircle with KPZ scaling, has in a *grand canonical formulation* a Gibbs measure with a Lifshitz tail as for 1D random walk in a **Poissonian disorder**