Isoperimetric type inequalities for singular Schrödinger operators

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in collaboration with Sylwia Kondej and Vladimir Lotoreichik

A talk at Prieezhev Memorial Meeting

Dubna, September 10, 2019

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As an example, consider the optimization problem for a *Robin Laplacian* associated with the quadratic form

$$\psi \mapsto \int_\Omega |\nabla\psi(x)|^2 \mathrm{d}x + \alpha \int_{\partial\Omega} |\psi(s)|^2 \mathrm{d}s$$

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$$\psi \mapsto \int_\Omega |\nabla\psi(x)|^2 \mathrm{d}x + \alpha \int_{\partial\Omega} |\psi(s)|^2 \mathrm{d}s$$

on $H^1(\Omega)$. As long as $\alpha > 0$ the result is similar to Faber-Krahn: the principal eigenvalue $\lambda_1^\alpha(\Omega)$ is *uniquely minimized* among the sets of *the same volume* by $\lambda_1^\alpha(\mathcal{B})$ where $\mathcal{B}$ is the *ball*

Attractive Robin boundary

The situation changes if $\alpha < 0$. In this case it was conjectured that $\lambda_1^\alpha(\mathcal{B})$ is *maximal* among ground states for sets of the same volume.

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This is true for local deformations of a ball, but *fails globally*: as an example, $\lambda_1^\alpha(\Omega) > \lambda_1^\alpha(\mathcal{B})$ may hold if Ω is a *spherical shell*.

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D. Krejčiřík, V. Lotoreichik: Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, *J. Convex Anal.* **25** (2018), 319–337.

D. Krejčiřík, V. Lotoreichik: Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, II: non-convex domains and higher dimensions, Potential Anal. (2019), to appear; `arXiv:1707.02269 [math.SP]`.

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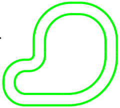
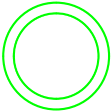
ground state of  $<$ ground state of 

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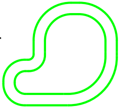
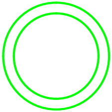
ground state of  $<$ ground state of 

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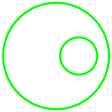
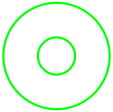
Similarly, for a *circular obstacle in circular cavity* we have

ground state of  $<$ ground state of 

whenever the obstacle is off center; the minimum is reached when it is touching the boundary.

E.M. Harrell, P. Kröger, K. Kurata: On the placement of an obstacle or a well so as to optimize the fundamental eigenvalue, *SIAM J. Math. Anal.* **33** (2001), 240–259.

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The δ-interaction supported by a manifold

A natural tool to define the corresponding singular Schrödinger operator is to employ the appropriate quadratic form, namely

$$q_{\delta,\alpha}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \alpha\|\psi|_\Gamma\|^2_{L^2(\Gamma)}$$

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We denote the associated unique self-adjoint operator $H_{\alpha,\Gamma}$. If Γ is a *smooth manifold* with $\operatorname{codim}\Gamma = 1$ one can equivalently characterized it by boundary conditions: it acts as $-\Delta$ on functions from $H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

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Alternatively, one sometimes uses the symbol $-\Delta_{\delta,\alpha}$ for this operator.

Planar loops

Let Γ be a *loop* in $\mathbb{R}^d$, $d \geq 2$, parametrized by its arc length, i.e. a piecewise differentiable function $\Gamma : [0, L] \to \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot{\Gamma}(s)| = 1$ for all but finitely many $s \in [0, L]$

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Theorem

Let $d = 2$. For any $\alpha > 0$ and $L > 0$ we have $\lambda_1(\alpha, \Gamma) \leq \lambda_1(\alpha, \mathcal{C})$, where $\mathcal{C}$ is a circle of perimeter L, the inequality being sharp unless Γ is congruent with $\mathcal{C}$.

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Proof idea: One employs a generalized *Birman-Schwinger principle* by which there is one-to-one correspondence between eigenvalues $-\kappa^2$ of $H_{\alpha,\Gamma}$ and solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \text{where } \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s) - \Gamma(s')|)$$

on $L^2([0, L])$, where K_0 is the Macdonald function

Proof idea, continued

We employ *inequalities on mean values of chords* denoted as $C_L^p(u)$:

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Remark: The (reverse) inequalities hold also for $p\in[-2,0)$ showing, e.g., that a *charged loop in the absence of gravity takes a circular form*

A discrete analogue: polymer loops

Consider the same loop as above with *point interactions* placed at the arc distances $\frac{jL}{N}$, $j = 0, \dots, N_1$, in other words, the formal Hamiltonian

$$H^N_{\alpha,\Gamma} = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta\left(x - \Gamma\left(\frac{jL}{N}\right)\right)$$

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Introduce the *generalized boundary values* as the coefficients in the expansion of H^*_Y where H_Y is the Laplacian restricted to functions vanishing at the vicinity of the points of Y

Point interactions 'necklaces'

A reminder: fixing the points $y_j \in Y$ the said expansions look as

$$\psi(x) = -\frac{1}{2\pi} \log|x-y_j|\, L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x-y_j|), \quad d=2,$$

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$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0\,, \quad \alpha \in \mathbb{R};$$

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Theorem

The ground state of $H^N_{\alpha,\Gamma}$ is uniquely maximized by a N-regular polygon

P.E.: Necklaces with interacting beads: isoperimetric problems, in Proceedings of the "International Conference on Differential Equations and Mathematical Physics" (Birmingham 2006), AMS *Contemporary Mathematics* Series, vol. 412, Providence, R.I., 2006; pp. 141-149.

New effects in three dimensions

In three dimensions the discrete spectrum of $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$ *may be empty* is α is small enough. As an example, for Γ being a *sphere of radius R* bound states are known to exist *iff* $\alpha R > 1$

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Remarks: (a) The results *fails to hold globally*: if a *surface-preserving* deformation of a critical surface is *elongated enough*, the discrete spectrum is *empty*.

(b) In contrast, deformation of a critical surface *always produces a nonvoid discrete spectrum* if it is *capacity preserving*.

More singular interactions in 2D

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For any $\beta > 0$ we have $\max_{|\Gamma|=L} \lambda_1^\beta(\Gamma) = \lambda_1^\beta(\mathcal{C})$, where $\mathcal{C}$ is a circle of perimeter $L > 0$ and the maximum is taken over all C^2 smooth loops.

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The Birman-Schwinger method does not work in this case, one has to use instead *locally orthogonal coordinates* in a way similar to those employed in [Krejčiřík-Lotoreichik'18, loc.cit.] to treat exterior of a Robin obstacle

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The C^2-smooth *cone* $\Sigma_R(\mathcal{T}) \subset \mathbb{R}^3$ of radius $R \in (0, \infty]$ with a C^2-smooth loop $\mathcal{T} \subset \mathbb{S}^2$ as its *cross-section* is

$$\Sigma_R(\mathcal{T}) := \left\{ r\mathcal{T} \in \mathbb{R}^3 \colon r \in [0, R) \right\};$$

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The cone $\Sigma_R(\mathcal{T})$ is called *circular* if its cross-section $\mathcal{T}$ is a circle and *non-circular* otherwise. An infinite circular cone with the cross-section length 2π is, in fact, a plane.

Results in the finite case

If $R < \infty$ it is not difficult to check that $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [0, \infty)$; we are interested in the principal eigenvalue $\lambda_1(H_{\alpha,\Gamma})$. We have

Theorem

Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $L := |\mathcal{C}| = |\mathcal{T}| \in (0, 2\pi]$. Let $\Gamma_R := \Sigma_R(\mathcal{C})$ and $\Lambda_R := \Sigma_R(\mathcal{T})$ be finite cones of radius $R > 0$ with the cross-sections $\mathcal{C}$ and $\mathcal{T}$, respectively; then

- $\#\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma_R}) \geq 1$ *if and only if* $\alpha > \alpha_{\mathrm{crit}}$ *for a certain value* $\alpha_{\mathrm{crit}} = \alpha_{\mathrm{crit}}(L, R) > 0$.

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- $\#\sigma_{\mathrm{disc}}(H_{\alpha,\Lambda_R}) \geq 1$ *for all* $\alpha \geq \alpha_{\mathrm{crit}}$ *(the borderline case $\alpha = \alpha_{\mathrm{crit}}$ is included) and the spectral isoperimetric inequality*

$$\lambda_1(H_{\alpha,\Lambda_R}) < \lambda_1(H_{\alpha,\Gamma_R})$$

is satisfied for all $\alpha > \alpha_{\mathrm{crit}}$.

P.E., V. Lotoreichik: A spectral isoperimetric inequality for cones, *Lett. Math. Phys.* **107** (2017), 717–732.

Cones, continued

We see the effect we have encountered before with spheres:

Corollary

Any (fixed-radius, smooth, conical) deformation of a critical circular cone gives rise to a non-void discrete spectrum of the corresponding $H_{\alpha,\Gamma}$

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Moreover, we even know its *accumulation rate*: for circular cones we have

$$\mathcal{N}_{-\frac{1}{4}\alpha^2 - E}(-\Delta_{\delta,\alpha}) \sim \frac{\cot\theta}{4\pi}\,|\ln E|\,, \quad E \to 0+\,,$$

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and similar results also holds in the non-circular case

T. Ourmières-Bonafos, K. Pankrashkin: Discrete spectrum of interactions concentrated near conical surfaces, *Appl. Anal.* **97** (2018) 1628–1649.

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Proof sketch in the finite case

The strategy is to employ the *generalized Birman-Schwinger principle* in combination with a minimization result about the energy of knots, cf. [E-Harrell-Loss'06, loc.cit.] and the following earlier paper

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$$\sigma\colon [0,R) \times [0,L] \to \mathbb{R}^3, \quad \sigma(r,s) := r\tau(s)\,;$$

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Proposition

Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\Gamma_R := \Sigma_R(\mathcal{C})$. Then the eigenfunction corresponding to the largest eigenvalue of the BS operator $S_{\Gamma_R}(\kappa)$ is rotationally invariant, i.e. it depends on the distance from the tip of the cone only.

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Let $f \in C([0,\infty);\mathbb{R})$ be convex and decreasing. Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $|\mathcal{T}| = |\mathcal{C}| = L$ for some $L \in (0, 2\pi]$. Then the following isoperimetric inequality holds

$$\Phi_f[\mathcal{C}] < \Phi_f[\mathcal{T}].$$

G. Lűko: On the mean length of the chords of a closed curve, *Israel J. Math.* **4** (1966), 23–32.

J. O'Hara: *Energy of knots and conformal geometry*, World Scientific 2003.

Proof sketch, concluded

In particular, the above proposition holds with the function

$$f(x) := \frac{e^{-a\sqrt{bx+c}}}{\sqrt{bx+c}}\,,$$

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we have to exclude situations where $r=0$, $r'=0$ or $r=r'$, but this is a zero measure set. With a bit of work, this yields finally the result. $\square$

Another object of interest: stars

Let us return to planar leaky graphs and consider *star-shaped* ones:

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Given $\alpha > 0$, we ask again about the *spectral threshold* of the operator H_{α,Σ_N} corresponding to the formal expression $-\Delta - \alpha\delta(x - \Sigma_n)$

An illustration

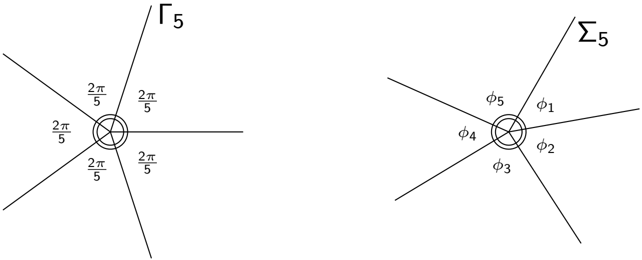

Figure: The star graphs Γ_5 and Σ_5 with $N = 5$ and $L < \infty$.

Star optimization

It is easy to see that $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [0,\infty)$ if $L<\infty$ and with the set $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [-\frac14\alpha^2,\infty)$ if $L=\infty$.

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For the lowest eigenvalue we have

Theorem

For any $\alpha>0$ we have the relation

$$\max_{\Sigma_N(L)} \lambda_1^\alpha(\Sigma_N(L)) = \lambda_1^\alpha(\Gamma_N(L)),$$

where the maximum is taken over all star graphs with $N\geq 2$ edges of a given length $L\in(0,\infty]$. If $L<\infty$ the equality is achieved iff Σ_N and Γ_N are congruent.

P. Exner, V. Lotoreichik: Optimization of the lowest eigenvalue for leaky star graphs, in Proceedings of the conference "Mathematical Results in Quantum Physics" (QMath13, Atlanta 2016; F. Bonetto, D. Borthwick, E. Harrell, M. Loss, eds.), Contemporary Math., vol 717, AMS, Providence, R.I., 2018; pp. 187–196.

Star optimization, continued

For *infinite stars* the condition $\Sigma_N(\infty) \cong \Gamma_N(\infty)$ is apparently also necessary and sufficient, just the method used in the proof of the theorem needs to be amended

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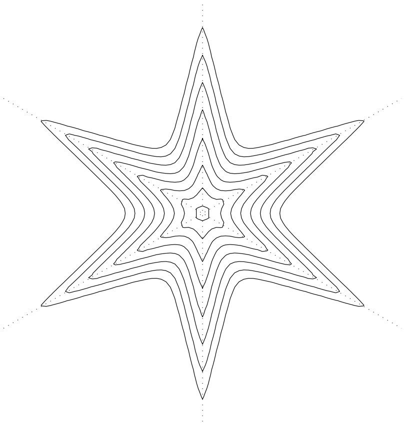
Moreover, the ground state has then some *esthetic quality*:



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To establish the relation for $L = \infty$ one uses the *strong resolvent convergence* which gives, in particular,

$$\lim_{L\to\infty} \lambda_1^\alpha(\Sigma_N(L)) = \lambda_1^\alpha(\Sigma_N(\infty))$$

and the analogous relation for symmetric stars

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Without being too technical, one takes the Laplacian defined on functions that are H^2 outside Γ and imposed the *generalized boundary conditions* defining 2D point interaction in the cross planes to the edges of Γ.

P.E., S. Kondej: Curvature-induced bound states for a δ interaction supported by for a curve in $\mathbb{R}^3$, *Ann. H. Poincaré* **3** (2002), 967–981.

Recall some related problems

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Universal optimality by Cohen and Kumar

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Remark: The remaining Platonic solids, *cube* and *dodekahedron*, do not qualify for universality having m=3 and 4, respectively. Note that they *do not represent Thomson problem solutions!*

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Lemma

Consider an N-arm star with edges of length $L \in (0, \infty]$ determined by unit vectors $\{\bar{\gamma}_i\}_{i=1}^N$, and let $\{\bar{\sigma}_i\}_{i=1}^N$ corresponds to a sharp-configuration star. Then we have

$$\sum_{i,j\; i\neq j} T_{\kappa;s,t}(|\bar{\gamma}_i - \bar{\gamma}_j|^2) \geq \sum_{i,j\; i\neq j} T_{\kappa;s,t}(|\bar{\sigma}_i - \bar{\sigma}_j|^2)$$

for any $s, t \in [0, L]$ and the inequality is sharp unless the two stars are congruent. Here $T_{\kappa;s,t}(x) := \frac{\mathrm{e}^{-\kappa\sqrt{a+bx}}}{4\pi\sqrt{a+bx}}$ with $a = (s-t)^2$ and $b = st$

Application to star leaky graphs, continued

Next we use the fact that the largest eigenvalue of the Birman-Schwinger operator corresponding to a sharp-configuration star has the *maximum symmetry*, $\tilde{f}_\sigma = (f_\sigma, \dots, f_\sigma) \in \bigoplus_1^N L^2([0, L])$.

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Theorem

Assume that $N \in \{2, 3, 4, 6, 12\}$, then the ground state energy of the N-arm leaky star assumes the unique maximum for $\gamma = \sigma$, where σ is the corresponds to the appropriate sharp configuration listed above.

P.E., S. Kondej: Ground state optimization for leaky star graphs in dimension three, arXiv:1906.00390

Open questions

Ignoring various technical ones which appeared in the course of the presentation, there are deeper and more interesting questions, for instance

- for 1D point interaction on an *interval with periodic b.c.*, is the equidistant distribution a maximizer *for all values of α?*

P.E.: An optimization problem for finite point interaction families, *J. Phys. A: Math. Theor.* **52** (2019), to appear; `arXiv:1906.01229`

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Спасибо за внимание!