Higher twist nucleon distribution amplitudes in Wandzura-Wilczek approximation

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Preamble

We start with definitions of the twist (in a nutshell):

- geometrical twist: $\tau = d - j$ defined for local quark-gluon operators.
- collinear twist-t for non-local quark-gluon operators associated with the behaviour on the light-cone or within the infinite momentum frame ($\psi_\pm = \mathscr{P}_\pm \psi$ with $\mathscr{P}_\pm = (\gamma_0 \mp \gamma_3)(\gamma_0 \pm \gamma_3)/4$):

$$
\begin{aligned}
t = 2 &\Rightarrow (\bar{\psi}_+ \, \psi_+)\,, \\
t = 3 &\Rightarrow (\bar{\psi}_+ \, \psi_-)\,, \quad (\psi_+ \, \psi_+ \, \psi_+)\,, \\
t = 4 &\Rightarrow (\bar{\psi}_- \, \psi_-)\,, \quad (\psi_+ \, \psi_+ \, \psi_-) \quad \text{etc.}
\end{aligned}
$$

- the light-cone basis: $n^\mu = (1/2, \mathbf{0}_T, -1/2)$, $\tilde{n}^\mu = (1/2, \mathbf{0}_T, 1/2)$ with $n^2 = \tilde{n}^2 = 0$ and $n \cdot \tilde{n} = 1/2$.

Matching:

$$\text{leading twist-}t \Longleftrightarrow \text{leading twist-}\tau$$
$$\text{next-to-leading twist-}t \Longleftrightarrow \tau \leq \text{next-to-leading twist-}t$$

That is, any "amplitude" can be presented as

$$(\text{L-twist-}t\text{ operator}) \oplus (\text{NL-twist-}t\text{ operator}) \oplus \ldots .$$

where

$$(\text{NL-twist-}t\text{ operator}) \ni (\text{L-twist-}\tau\text{ operator})\,,$$

or

$$\mathscr{O}^{nl-tw} = (\partial \mathscr{O}^{l-tw}) + \text{"genuine higher twist"}$$
$$\mathscr{O}^{nl-tw}_{WW} \stackrel{def}{=} (\partial \mathscr{O}^{l-tw})\,.$$

see, for example, Balitsky-Braun'88, Ball-Braun'96

- Geometrical twist 2:

$$
\left[\bar{\psi}(x)\gamma_\mu[x,-x]\psi(-x)\right]^{\tau=2} =
$$

$$
\sum_k \frac{1}{k!} x_{\mu_1}\dots x_{\mu_k} \mathbf{S}'_{all}\,\bar{\psi}(0)\gamma_\mu \overleftrightarrow{D}_{\mu_1} \dots \overleftrightarrow{D}_{\mu_k} \psi(0) =
$$

$$
\int_0^1 du \frac{\partial}{\partial x_\mu}\left[\bar{\psi}(ux)\hat{x}\,[ux,-ux]\psi(-ux)\right]
$$

- Geometrical twist 3:

$$\left[\bar{\psi}(x)\gamma_\mu[x,-x]\psi(-x)\right]^{\tau=3} =$$

$$\sum_k \frac{1}{k!} x_{\mu_1}\dots x_{\mu_k} \mathbf{S}'_{\mu_1\dots\mu_k} \mathbf{A}_{\mu\,\mu_1} \bar{\psi}(0)\gamma_\mu \overleftrightarrow{D}_{\mu_1} \dots \overleftrightarrow{D}_{\mu_k} \psi(0) =$$

$$-i\varepsilon_{\mu\alpha\beta\sigma} \int\limits_0^1 du\, u\, x_\alpha \partial_\beta \left\{ \bar{\psi}(ux)\gamma_\sigma\gamma_5\,[ux,-ux]\psi(-ux) \right\} +$$

$$(\bar{\psi} G_{\mu\alpha} x_\alpha\, \hat{x}\psi) + (\bar{\psi}\tilde{G}_{\mu\alpha} x_\alpha\, \hat{x}\gamma_5\psi)$$

see Efremov-Teryaev'86, Anikin-Teryaev'01

An alternative way to derive the WW-relations (or to extract the WW-type contributions) based on the *n*-independence condition:

$$\frac{d}{dn_\beta}\Big(\text{factorized physical amplitude}\Big) = 0$$

and the QCD equation of motions.

Spinor/Twistor Formalism

see, for example, Braun-Manashov,'10

The Dirac bispinors read

$$q = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix} \equiv \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix}, \quad \bar{q} \equiv q^\dagger \gamma_0 = (\chi^\beta \, \bar{\psi}_{\dot{\alpha}})$$

where

$$(0, \frac{1}{2}) \Longrightarrow \varphi'_R \equiv \varphi_R(p) = e^{\sigma\cdot\phi/2}\varphi_R(0) \equiv M\varphi_R \,,$$
$$(\frac{1}{2}, 0) \Longrightarrow \varphi'_L \equiv \varphi_L(p) = e^{-\sigma\cdot\phi/2}\varphi_L(0) \equiv N\varphi_L \,,$$
$$N \neq \mathscr{S} M \mathscr{S}^{-1} \text{ but } N = \mathscr{C} M^* \mathscr{C}^{-1} \,,$$

with

$$\text{an arbitrary matrix } \mathscr{S} \in SL(2,\mathbb{C}), \quad \mathscr{C} = i\sigma_2$$

some definitions

- For any matrix $A = a_\mu \sigma^\mu \in SL(2,\mathbb{C})$ with

$$(\sigma^\mu)_{\alpha\dot\beta} = (1, \vec\sigma), \quad (\bar\sigma^\mu)^{\dot\alpha\beta} = (1, -\vec\sigma) = (\sigma^\mu)^{\beta\dot\alpha}$$

$$\text{and } a_\mu = \frac{1}{2}\mathrm{tr}_D(A\bar\sigma_\mu) = \frac{1}{2}\mathrm{tr}_D(\bar A\sigma_\mu);$$

- Using $\varepsilon_{12} = \varepsilon^{12} = -\varepsilon_{\dot1\dot2} = -\varepsilon^{\dot1\dot2} = 1$, we have

$$u^\alpha = \varepsilon^{\alpha\beta} u_\beta\,, \quad u_\alpha = u^\beta \varepsilon_{\beta\alpha}\,, \quad \bar u^{\dot\alpha} = \bar u_{\dot\beta}\varepsilon^{\dot\beta\dot\alpha}\,, \quad \bar u_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar u^{\dot\beta}\,;$$

- $(u)^* := \bar u\,, \quad (u\,v) := u^\alpha\, v_\alpha = -u_\alpha\, v^\alpha;$
- $(u\,v)^* = (\bar v\,\bar u)$ and $\bar u_{\dot\alpha}\bar v^{\dot\alpha} = -\bar u^{\dot\alpha}\bar v_{\dot\alpha}$

- The γ_μ matrices take the forms

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix}, \quad \hat{a} = \begin{pmatrix} 0 & A_{\alpha\dot\beta} \\ \bar{A}^{\dot\alpha\beta} & 0 \end{pmatrix}$$

and

$$\gamma_5 = \begin{pmatrix} -\delta_\alpha^\beta & 0 \\ 0 & \delta_{\dot\beta}^{\dot\alpha} \end{pmatrix}.$$

Let us introduce the so-called twistor basis:

$$\lambda_\alpha \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \lambda^\alpha \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mu_\alpha \sim \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \mu^\alpha \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $\lambda_\alpha \lambda^\alpha = \mu_\alpha \mu^\alpha = 0$ and $(\mu\lambda) = -(\lambda\mu) = 1$. Within this basic, we are able to expand the spinors as

$$\psi_\alpha = \lambda_\alpha \psi_- - \mu_\alpha \psi_+ \quad \bar{\chi}^{\dot{\alpha}} = \bar{\chi}^+ \bar{\lambda}^{\dot{\alpha}} + \bar{\chi}^- \bar{\mu}^{\dot{\alpha}}\,.$$

P.S.We also remind

$$\xi_\alpha^{+\frac{1}{2}}(\phi,\theta) = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}, \quad \xi_\alpha^{-\frac{1}{2}}(\phi,\theta) = \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$$

Within the twistor representation, the light-cone basis read

$$n_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}\,, \quad \tilde{n}_{\alpha\dot{\alpha}} \equiv p_{\alpha\dot{\alpha}} = \mu_\alpha \bar{\mu}_{\dot{\alpha}}\,.$$

The equation of motions for nucleon

$$m_N \hat{n} N^+(P) = 2(P.n) N^-(P) \quad \Big(\hat{p} N^-(P) = m_N N^+(P) \Big)$$

will take the following form

$$2(p.n)\psi_- = -m_N(\mu\lambda)\bar{\chi}_+$$

Conformal Group and its Collinear Subgroup

see, Braun,Korchemsky,Muller'03

Among the general co-ordinate transformations that conserve $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ there are the transformations that change only the scale:

$$g'_{\mu\nu} = \omega(x)\, g_{\mu\nu}$$

and conserve the angles and the light-cone invariant.
The full conformal algebra in $d = 4$ includes 15 generators:

- $\mathbb{P}_\mu \Longrightarrow$ 4 translations;
- $\mathbb{M}_{\mu\nu} \Longrightarrow$ 6 Lorentz rotations;
- $\mathbb{D} \Longrightarrow$ 1 dilatation;
- $\mathbb{K}_\mu = \mathbb{I}\mathbb{P}_\mu\mathbb{I} \Longrightarrow$ 4 special conformal transformations;

We remind the special conformal transformation (for an arbitrary a):

$$x'_\mu = \mathscr{I}\mathscr{P}\mathscr{I} x_\mu = \frac{x_\mu + a_\mu x^2}{1 + 2a\cdot x + a^2 x^2}$$

and the special case is

$$x'_- = \frac{x_-}{1 + 2a_+ \cdot x_-}\ , \text{where } x^2 = 0\,, a = a_+\,.$$

The collinear subgroup generates $(\Phi(z) = \Phi(zn))$:

$$z' = \frac{az+b}{cz+d}, \quad ad - bc = 1\,,$$

$$\Phi'(z) = (cz+d)^{-2j}\Phi\Big(\frac{az+b}{cz+d}\Big), \quad j = \frac{d+s}{2}\,.$$

Consider now the realization of conformal group. First, we introduce the raising and lowering operators:

$$\mathbb{L}_{\pm} = \mathbb{L}_1 \pm i\mathbb{L}_2 \,,$$
$$\mathbb{L}_0 = \frac{i}{2}(\mathbb{D} + \mathbb{M}_{-+}) \,, \quad \mathbb{E} = \frac{i}{2}(\mathbb{D} - \mathbb{M}_{-+}) \,,$$

where $\mathbb{L}_+ = -i\mathbf{P}_+$ and $\mathbb{L}_- = i/2\mathbf{K}_-$. Working with the function $\Phi(z) \equiv \Phi(zn)$, we have

$$[\mathbb{L}_+, \Phi(z)] = -\partial_z \Phi(z) \equiv \mathscr{L}_+ \Phi(z) \,,$$
$$[\mathbb{L}_-, \Phi(z)] = (z^2\partial_z + 2jz)\Phi(z) \equiv \mathscr{L}_- \Phi(z) \,,$$
$$[\mathbb{L}_0, \Phi(z)] = (z\partial_z + j)\Phi(z) \equiv \mathscr{L}_0 \Phi(z) \,.$$

Here, $\mathbb{L}_i$ act on the Hilbert space while $\mathscr{L}_i$ – on the field representation.

Conformal Tower

Consider the Taylor expansion of $\Phi(z)$:

$$\Phi(z) = \sum_k \frac{z^k}{\Gamma(k+1)} \partial^k \Phi(0) = \sum_k c_k z^k \mathscr{O}_k(0),$$

where

$$\mathscr{O}_k = [\mathbb{L}_+, ...[\mathbb{L}_+, \Phi(0)]...] = -(\partial)^k \Phi(z)\Big|_{z=0}.$$

can be referred to the conformal tower.

Notice that $\Phi(0)$ is a conformal operator, *i.e.*

$$[\mathbb{L}_-, \Phi(0)] = 0 \text{ and } [\mathbb{L}_0, \Phi(0)] = j\Phi(0)\,,$$
$$\sum_i [\mathbb{L}_i, [\mathbb{L}_i, \Phi(0)]] = j(j-1)\Phi(0) \equiv \mathscr{L}^2\Phi(0)\,.$$

Moreover, any local composite operator can be specified by a polynomial:

$$\mathscr{O}_k(0) = \mathscr{P}_k(\partial)\Phi(z)\Big|_{z=0} \text{ where } \mathscr{P}_k(u) = (-u)^k\,.$$

The $sl(2,\mathbb{R})$-algebra can be realized by different representations. Indeed, we are able to introduce the operators $\hat{L}_i$ acting on the polynomial space instead $\mathscr{L}_i$ acting on the field:

$$\mathscr{P}_k(\partial)\underline{\mathscr{L}_\pm \Phi(z)}\Big|_{z=0} = \underline{\hat{L}_\mp \mathscr{P}_k(\partial)}\Phi(z)\Big|_{z=0}.$$

That is, we deal with the adjoint representation:

$$\hat{L}_+ \mathscr{P}_k(u) = (u\partial^2 + 2j\partial)\mathscr{P}_k(u),$$
$$\hat{L}_- \mathscr{P}_k(u) = (-u)\mathscr{P}_k(u),$$
$$\hat{L}_0 \mathscr{P}_k(u) = (u\partial + j)\mathscr{P}_k(u).$$

However, the above-mentioned repres., in not convenient one owing to ∂^2. To avoid the problem, we use the following trick based on the observation that the derivatives of Φ map into the vector space spanned by $|n\rangle$ (n means the number of derivatives):

$$|n\rangle \equiv \mathbb{Z}^n = \frac{\partial^n \Phi(0)}{\Gamma(n+2j)}\,.$$

In other words, we can say that the polynomials z^n determine completely the structure of the local composite operator $\mathscr{O}_n$:

$$z^n (\text{or } \mathbb{Z}^n) \overset{map}{\Longleftrightarrow} \partial^n \Phi(0)\,.$$

Define the generator operators which act on the polynomials space:

$$[\mathbb{G}, \mathbb{Z}^n] = \frac{\partial^n}{\Gamma(n+2j)}[\mathbb{L}_+, \Phi(0)] = (n+2j)\mathbb{Z}^{n+1} = S_+\mathbb{Z}^n \,,$$

where $S_+ = 2j\mathbb{Z} + \mathbb{Z}^2\partial_{\mathbb{Z}}$.
Thus, to summarize we conclude that

$$\Phi(z) = \sum_n c_n\{S_+^N \mathbb{I}\}\{\partial^N \Phi(0)\} \,.$$

P.S. Going over to the most general case, the trivial local conformal operator $\Phi(0)$ should be replaced by the local conformal composite operator $\mathscr{O}_m(0)$ with $m-$ number of (covariant) derivatives:

$$[\mathbb{L}_-, \mathscr{O}_m(0)] = 0 \text{ and } [\mathbb{L}_0, \mathscr{O}_m(0)] = (J+m)\mathscr{O}_m(0) \,.$$

Nucleon DAs

The leading twist DA is defined as

$$\langle 0|\varepsilon^{ijk}\, u_+^{\downarrow i}(z_1 n) u_+^{\uparrow j}(z_2 n) d_+^{\downarrow k}(z_3 n)|P\rangle =$$
$$= -\frac{1}{2}(pn)\, N_+^{\downarrow} \int \mathscr{D}x\, e^{-i(pn)\sum x_i z_i}\, \Phi_3(x)\,.$$

The integration goes here over the simplex, i.e.

$$\mathscr{D}x = dx_1\, dx_2\, dx_3\, \delta(1 - x_1 - x_2 - x_3)\,.$$

The nucleon DAs of twist-4, Φ_4, Ψ_4, Ξ_4 are defined as

$$\langle 0|\varepsilon^{ijk} u_+^{\downarrow i}(z_1) u_+^{\uparrow j}(z_2) d_-^{\downarrow k}(z_3)|P\rangle = \\ = \frac{1}{4}(\mu\lambda) m_N N_+^{\uparrow} \int Dx\, e^{-i(pn)\sum z_k x_k}\, \Phi_4(x),$$

$$\langle 0|\varepsilon^{ijk} u_+^{\uparrow i}(z_1) u_-^{\downarrow j}(z_2) d_+^{\downarrow k}(z_3)|P\rangle = \\ = \frac{1}{4}(\mu\lambda) m_N N_+^{\uparrow} \int Dx\, e^{-i(pn)\sum z_k x_k}\, \Psi_4(x),$$

$$\langle 0|\varepsilon^{ijk} u_-^{\downarrow i}(z_1) u_+^{\downarrow j}(z_2) d_+^{\downarrow k}(z_3)|P\rangle = \\ = \frac{1}{4}(\mu\lambda) m_N N_+^{\downarrow} \int Dx\, e^{-i(pn)\sum z_k x_k}\, \Xi_4(x)$$

WW term for twist four

To the one loop accuracy , the expansion of nonlocal operators in terms of local multiplicatively renormalized operators reads

$$\mathbb{O}_3(z) = \sum_{N,k,q} a_{Nk}\, S_+^k \Phi_{Nq}(\vec{z})\, \partial_+^k\, \mathbb{O}_{Nq}^{t=3}\,,$$

where $\partial_+ = (n\partial)$ and $S_+ = S_{1,+} + S_{2,+} + S_{3,+}$ is the sum of one-particle generators.
The similar expansion can be written for the twist-4 operator:

$$\mathbb{O}_4(z) = \sum_{N,k,q} b_{Nk}\, S_+^k \Psi_{Nq}(\vec{z})\, \partial_+^k\, \mathbb{O}_{Nq}^{t=4}\,.$$

P.S. Here the coefficients are $a_{Nk} = \Gamma(2N+6)/(k!\Gamma(2N+6+k))$ and $b_{Nk} = \Gamma(2N+5)/(k!\Gamma(2N+5+k))$

Schematically, the local twist-4 operators can be formed by the following combinations:

$$\mathbb{O}_N^{t=4,(1)} \sim \psi_+ \, \psi_+ \, \psi_- \, ,$$
$$\mathbb{O}_N^{t=4,(2)} \sim \psi_+ \, \psi_+ \, (\partial^T \psi_+) \oplus \psi_+ \, \psi_+ \, (\partial_+ \psi_-) \, .$$

More exactly, we have

$$\begin{aligned}
\mathbb{O}_{Nq}^{t=4,(1)}(\mu) &= \frac{1}{N+2}(\mu\partial_\lambda)\,\mathbb{O}_{Nq}^{t=3} \, , \\
\mathbb{O}_{N+1,q}^{t=4,(2)}(\mu) &= \frac{1}{4(N+3)^2}\left(i\left[\mathbf{P}_{\mu\bar{\lambda}} \, , \mathbb{O}_{Nq}^{t=3}\right] \right. \\
&\quad \left. - \frac{N+2}{2N+5}\, i\left[\mathbf{P}_{\lambda\bar{\lambda}} \, , \mathbb{O}_{Nq}^{t=4,(1)}(\mu)\right]\right) .
\end{aligned}$$

Notice that the non-local twist-4 operator can be expressed as

$$\lambda\,\partial_\mu \mathbb{O}_4(\vec{z}) = \mathbb{O}_3(\vec{z})\,.$$

Therefore, we have the following relation:

$$\sum_{N,k,q} a_{Nk}\, S_+^k \Phi_{Nq}(\vec{z})\,\partial_+^k\,\mathbb{O}_{Nq}^{t=3} = \sum_{N,k,q} \Big\{ b_{Nk}\,\widetilde{S}_+^k \Psi_{Nq}^{(1)}(\vec{z})\,\partial_+^k\,\mathbb{O}_{Nq}^{t=3}$$

$$+\frac{b_{N+1k}}{2(N+3)(2N+5)}\widetilde{S}_+^k \Psi_{N+1q}^{(2)}(\vec{z})\,\partial_+^{k+1}\,\mathbb{O}_{Nq}^{t=3}\Big\},$$

where $\widetilde{S}_+ = S_+ - z_1$.

We now derive the recurrent relations:

- for $k = 0$,

$$\Psi_N^{(1)}(\vec{z}) = \Phi_N(\vec{z})\,;$$

- for $k = 1$,

$$\begin{aligned} \Psi_{N+1}^{(2)}(\vec{z}) &= \Big(a_{N\,1}\,S_+ - b_{N\,1}[S_+ - z_1]\Big)\Phi_N(\vec{z}) \\ &= -\Big[S_+^{(1\,1\,1)} - 2(N+3)z_1\Big]\Phi_N(\vec{z})\,. \end{aligned}$$

Thus, we derive the following expression for the contributions of the descendants of the twist−3 operators to the light-ray operators $\mathbb{O}_4(z)$:

$$\mathbb{O}_4^{WW}(z) = \sum_{N,k,q} b_{Nk}\, S_+^k \Big\{ \Psi^{(1)}_{Nq}(z)\, \partial_+^k\, \mathbb{O}^{t=4,(1)}_{Nq}(\mu) + \Psi^{(2)}_{Nq}(z)\, \partial_+^k\, \mathbb{O}^{t=4,(2)}_{Nq}(\mu) \Big\}.$$

In order to find $\Psi_4^{WW}(x)$, we take the nucleon matrix elements of both sides of the above-mentioned equation. By definition

$$\langle 0|\mathbb{O}_4^{WW}(z)|P\rangle = \frac{1}{4}(\mu\lambda)m_N N_+^{\uparrow}\int \mathscr{D}x\, e^{-i(pn)\sum z_k x_k}\Psi_4^{WW}(x_2,x_1,x_3).$$

In its turn, for the matrix elements of the operators $\mathbb{O}_{Nq}^{t=4,(1)}(\mu)$, $\mathbb{O}_{Nq}^{t=4,(2)}(\mu)$ one derives

$$\langle 0|\mathbb{O}_{Nq}^{t=4,(1)}|P\rangle = \frac{1}{4}(\mu\lambda)m_N N_+^{\uparrow}\frac{(-ipn)^N\phi_{Nq}}{N+2},$$

$$\langle 0|\mathbb{O}_{N+1q}^{t=4,(2)}|P\rangle = -\frac{1}{8}(\mu\lambda)m_N N_+^{\uparrow}\frac{(-ipn)^{N+1}\phi_{Nq}}{(N+3)^2(2N+5)}.$$

After some algebra, one can bring the matrix element to the form

$$\frac{1}{4}(\mu\lambda) m_N N^{\uparrow}_{+} \sum_{Nq} \Gamma(2N+5)\, \phi_{Nq} \int \mathscr{D}x\, x_2 x_3\, e^{-i(pn)\sum z_k x_k} \times \left\{ \frac{1}{N+2} P^{(1)}_{N,q}(x) - \frac{1}{N+3} P^{(2)}_{N+1,q}(x) \right\},$$

where the polynomials $P^{(1)}_{N,q}(x)$, $P^{(2)}_{N+1,q}(x)$ are given by the $sl(2)$ Fourier transform

$$P^{(k)}_{N}(x) = \langle \mathrm{e}^{\sum_{i=1}^{3} x_i z_i} | \Psi^{(k)}_{Nq} \rangle_{\frac{1}{2}11} \,.$$

Here $\langle *, * \rangle$ stands for the $sl(2)$ invariant scalar product.

We now have to express all polynomials in terms of P_{Nq} which enter the expansion for twist-3 nucleon DA, $\Phi_3(x)$. It follows that

$$c_{Nq} P_{Nq}(x) = \Gamma(2N+6) \langle e^{\sum_{i=1}^3 x_i z_i} | \Phi_{Nq} \rangle_{111} \,.$$

And, one gets for $P^{(1)}_{Nq}$,

$$P^{(1)}_{Nq}(x) = r_{Nq}\, \partial_{x_1} x_1 P_{Nq}(x) \,,$$

where $r_{Nq} = c_{Nq}/\Gamma(2N+6)$.
Since the generators S^j_+ and S^j_- (here j is multiindex $j = (j_1, j_2, j_3)$) are conjugated with respect to the corresponding scalar product, $\langle \Psi | S^j_+ \Phi \rangle_j = -\langle S_- \Phi | \Psi \rangle_j$, we obtain

$$P^{(2)}_{N+1,q}(x) = r_{Nq} \left[(2N+5) - x_{123} \partial_{x_1} \right] x_1 P_{Nq}(x) \,,$$

where $x_{123} = x_1 + x_2 + x_3$.

Finally, we derive the following expression

$$\Psi_4^{WW}(x) = -\sum_{Nq} \frac{c_{Nq}\phi_{Nq}}{(N+2)(N+3)} \times [N+2-\partial_{x_2}]\, x_1 x_2 x_3\, P_{Nq}(x_2, x_1, x_3)$$