JINR, Dubna, Bogoliubov Laboratory of Theoretical Physics, Seminar
Wednesday, 02 Oct 2013, 11:00h

Metric-Affine Gravity (MAG): General relativity and its gauge-theoretical extension

Friedrich W. Hehl, email: hehl@thp.uni-koeln.de

University of Cologne and University of Missouri-Columbia

• I'd like to thank Oleg Teryaev for the invitation to the seminar.

• Our program: We take the Minkowski space and read off the gravitational properties of matter by gauging the Poincaré group $T_4 \rtimes SO(1,3)$. Subsequently we generalize to the gauging of the affine group $T_4 \rtimes GL(4,R)$.

• See H., McCrea, Mielke, and Ne'eman, Physics Reports (1995) and Yuri Obukhov, Int.J.Geom.Meth.Mod.Phys. (2006), numerous results I talk about today were won in collaboration with Yuri Obukhov (Moscow).

- My Soviet/Russian connections as a student:
  - D. I. Blochinzew, *Grundlagen der Quantenmechanik,* 2nd corr. ed. (transl. from the Russian). Deutscher Verlag der Wissenschaften, Berlin (East) (1958)
  - D. Iwanenko and A. Sokolow, *Klassische Feldtheorie* (Classical Field Theory, transl. from the Russian). Akademie-Verlag, Berlin (East) (1953)
  - V. I. Rodichev, *Twisted space and nonlinear field equations,* Sov. Phys.–JETP **13**, 1029 (1961)
  - N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (transl. from the Russian). Interscience, London (1959)
  - D. I. Blokhintsev, *Space and Time in the Microworld* (transl. from the Russian). Reidel, Dordrecht (1973)

- First book on gauge theory of gravity: V. N. Ponomariev, A. O. Barvinsky, and Yu. N. Obukhov, *Geometrodynamical Methods and the Gauge Approach to the Theory of Gravitational Interactions* (Energoatomizdat, Moscow, 1985) (in Russian)

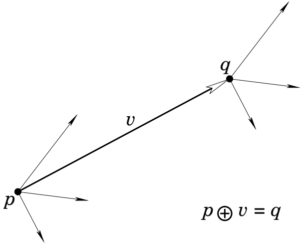
Contents

1. Minkowski space $\mathcal{M}(E, \oplus, V, g)$ as 4-dimensional flat affine space with a constant Lorentz metric

• Electrodynamics of moving bodies $\rightarrow$ SR and Minkowski spacetime $\mathcal{M}$. Equivalence of all inertial systems of reference. $\mathcal{M}$ is homogeneous in time $(t = x^0)$ and space (x^1, x^2, x^3) and isotropic in space.

• Each event is characterized by its coordinates x^i. E is the set of all events. An operation $\oplus$, called translation is defined that maps events into events. Let V be a 4d vector space. Then 4d affine space $(E, \oplus, V)$. V is attached to events, named $p, q, ... \in E$, vectors named $v, w, ... \in V$; see Kopczyński & Trautman 1992, Snapper & Troyer 1971.

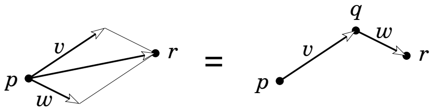
• Translating the event p into the event q by means of the translation vector v:



• Free transitive operations $\oplus$

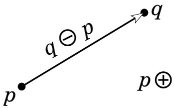
$$E \times V \rightarrow E\,, \quad (p, v) \longmapsto \underbrace{p}_{\text{event}} \oplus \underbrace{v}_{\text{vector}}$$

• Translating the event p first by the vector $(v + w)$ or, alternatively, first by v and subsequently by w. The result is the same, the event r:



$$p \oplus (v + w) = (p \oplus v) \oplus w$$

• The vector $q \ominus p$ is uniquely determined:



$$p \oplus (q \ominus p) = q$$

• In affine space the translation is a fundamental structure. No point (event) is distinguished in an affine space (in contrast to a vector space V). This is an expression of its homogeneity.

• Additionally, we need a Minkowski (aka Lorentz) metric for being able to measure distances and angles:

$$g(v,w) := g_{\alpha\beta}\, v^{\alpha} w^{\beta} = v^{\hat{0}} w^{\hat{0}} - v^{\hat{a}} w^{\hat{a}}\,,$$

frame indices $\alpha, \beta, ... = \hat{0}, \hat{1}, \hat{2}, \hat{3}$, frame $e_\alpha = e^i{}_\alpha\, \partial_i$. Minkowski *vector* space (V, g).

• Minkowski space $\mathcal{M}= (E, \oplus, V, g)$. Group of motion in $\mathcal{M}$ is the $4+6$ parameter

• Poincaré group $\mathcal{P} = \underbrace{T_4}_{\text{transl.}} \rtimes \underbrace{SO(1,3)}_{\text{Lorentz rot.}}$ (also called inhom. Lorentz group)

• Turn later to affine group $\mathcal{A} = \underbrace{T_4}_{\text{transl.}} \rtimes \underbrace{GL(4,R)}_{\text{linear gr.}}$, which will be gauged in the presence of a metric, yielding metric-affine gravity (MAG)

• Poincaré algebra: P_α generators of translations, $J_{\alpha\beta} = -J_{\beta\alpha}$ generators of Lorentz rotations with commutation rules

$$[P_\alpha, P_\beta] = 0\,, \quad [P_\alpha, J_{\beta\gamma}] = i\left(g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta\right)\,,$$

$$[J_{\alpha\beta}, J_{\gamma\delta}] = i\left(g_{\beta\gamma} J_{\alpha\delta} - g_{\alpha\gamma} J_{\beta\delta} + g_{\alpha\delta} J_{\beta\gamma} - g_{\beta\delta} J_{\alpha\gamma}\right)\,.$$

• Note the semi-direct product structure. Quadratic Casimir operators that commute with all P's and J's:

$$C_1 := P^\alpha P_\alpha\,, \quad C_2 := W^\alpha W_\alpha\,,$$

with the Pauli-Lubański operator (orbital angular momentum drops out)

$$W_\alpha := -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\, J^{\beta\gamma} P^\delta\,.$$

• Properties (mass-spin classification of elementary particles, Wigner 1939):

$$P^\alpha W_\alpha = 0\,, \quad C_1 \to \underbrace{m}_{\text{mass}}{}^2\,, \quad C_2 \to -m^2 \underbrace{s}_{\text{spin}}(s+1)$$

• Field theoretical correspondents: energy-momentum 3-form Σ_α and spin angular momentum 3-form $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$.

2. Heuristics: Read off gravitational properties from the behavior of matter in Minkowski space

• Study a classical Dirac field $\Psi(x)$ in Minkowski space $\mathcal{M}$ in non-inertial frames.

• Find the integrability conditions for the 4 coframe 1-forms $\vartheta^\alpha = e_i{}^\alpha dx^i$ and the 6 Lorentz connection 1-forms $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$ in $\mathcal{M}$.

• Relax these integrability conditions and arrive at the Einstein-Cartan theory (Sciama, Kibble 1961) or, more generally, at the Poincaré gauge theory, both formulated in a Riemann-Cartan space with gauge potentials $(\vartheta^\alpha, \Gamma^{\alpha\beta})$.

• Already in vacuum electrodynamics, because of its conformal invariance, the dil(at)ation current emerges. Such a current couples to the trace part of the connection $\Gamma_\gamma{}^\gamma$.

• Subsequently abandon the antisymmetry of the Lorentz connection, $SO(1,3) \Longrightarrow GR(4,R)$, study a *general linear* connection $\Gamma_\alpha{}^\beta$, its shear piece $\Gamma^{(\alpha\beta)} - \frac{1}{4} g^{\alpha\beta} \Gamma_\gamma{}^\gamma$ is related to the $SL(3,R)$, a dynamical group describing Regge trajectories (beyond the mass-spin classification!).

	Einstein's laboratory	Kibble's laboratory
elementary object in SR	mass point m	Dirac spinor $\Psi(x)$
inertial frame	Cart. coo. system x^i $ds^2 \stackrel{*}{=} o_{ij} dx^i dx^j$	holonomic orthon. frame $e_\alpha = \delta^i_\alpha \, \partial_i , \quad e_\alpha \cdot e_\beta = o_{\alpha\beta}$
force-free motion in IF	$\dot{u}^i \stackrel{*}{=} 0$	$(i\gamma^i \partial_i - m)\Psi \stackrel{*}{=} 0$
non-inertial frame	arbitrary curvilinear coord. system $x^{i'}$	anholon. orthon. frame $e_\alpha = e^i{}_\alpha \, \partial_i$ coframe $\vartheta^\alpha = e_i{}^\alpha dx^i$
force-free motion in NIF	$\dot{u}^{i'} + u^{j'} u^{k'} \left\{ {i' \atop j'k'} \right\} = 0$	$\left[i\gamma^\alpha e^i{}_\alpha (\partial_i + \Gamma_i) - m \right] \Psi = 0$ $\Gamma_i := \frac{1}{2} \Gamma_i{}^{\beta\gamma} \rho_{\beta\gamma} \quad \text{Lorentz}$
non-inertial object	$\left\{ {i' \atop j'k'} \right\}$ 40	$\vartheta^\alpha , \quad \Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ $16 \quad + \quad 24$
constraints in SR	$\tilde{R}(\partial\{\}, \{\}) = 0$ 20	$T(\partial e, e, \Gamma) = 0, R(\partial\Gamma, \Gamma) = 0$ $24 \quad + \quad 36$
global IF	$g_{ij} \stackrel{*}{=} o_{ij} \, , \; \left\{ {i \atop jk} \right\} \stackrel{*}{=} 0$	$\left(e_i{}^\alpha , \; \Gamma_i{}^{\alpha\beta} \right) \stackrel{*}{=} (\delta_i^\alpha , 0)$

Continued:

	Einstein's laboratory	Kibble's laboratory
non-inertial object	$\left\{ {i' \atop j'k'} \right\}$ 40	$\vartheta^\alpha, \quad \Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ $16 \quad + \quad 24$
constraints in SR	$\tilde{R}(\partial\{\},\{\}) = 0$ 20	$T(\partial e, e, \Gamma) = 0, R(\partial\Gamma, \Gamma) = 0$ $24 \quad + \quad 36$
global IF	$g_{ij} \overset{*}{=} o_{ij}\ , \ \left\{ {i \atop jk} \right\} \overset{*}{=} 0$	$\left(e_i{}^\alpha,\ \Gamma_i{}^{\alpha\beta}\right) \overset{*}{=} (\delta_i^\alpha, 0)$
switch on gravity	$\tilde{R} \neq 0$ Riemann	$T \neq 0, \quad R \neq 0$ Riemann-Cartan
local IF Einstein elev.	$g_{ij}\|_P \overset{*}{=} o_{ij}\ , \ \left\{ {i \atop jk} \right\}\|_P \overset{*}{=} 0$	$(e_i{}^\alpha,\ \Gamma_i^{\alpha\beta})\|_P \overset{*}{=} (\delta_i^\alpha, 0)$
field eqs.	$\tilde{R}ic - \frac{1}{2}tr(\tilde{R}ic) \sim mass$ **GR**	$Ric - \frac{1}{2}tr(Ric) \sim mass$ $Tor + 2\ tr(Tor) \sim spin$ **EC**

Thereby we arrived at the Einstein-Cartan theory of gravity (EC). We will now immediately generalize to the gauge of the affine group:

3. Weak and hypoth. strong gravity, metric-affine gauge th. of gravity

3.1 Geometry and coupling to matter

• The 'gravitational' potentials are

$g_{\alpha\beta}$	metric (weak, Newton-Einstein type gravity)
ϑ^{α}	coframe (weak, Newton-Einstein type gravity)
${\Gamma_\alpha}^\beta$	connection (strong, Yang-Mills type gravity)

• By differentiation, we find the field strengths

$$
\begin{aligned}
Q_{\alpha\beta} &= -Dg_{\alpha\beta} && \text{nonmetricity}\\
T_\alpha &= D\vartheta^\alpha && \text{torsion}\\
{R_\alpha}^\beta &= d{\Gamma_\alpha}^\beta - {\Gamma_\alpha}^\gamma \wedge {\Gamma_\gamma}^\beta && \text{curvature}
\end{aligned}
$$

• The material currents, coupled to the potentials $(g_{\alpha\beta}, \vartheta^\alpha, {\Gamma_\alpha}^\beta)$, are energy-momentum and hypermomentum $(\sigma_{\alpha\beta}, \Sigma_\alpha, {\Delta^\alpha}_\beta)$.

• The hypermomentum splits into spin current $\oplus$ dilation current $\oplus$ shear current (add. sources of gravity):

$$\Delta_{\alpha\beta} = \tau_{\alpha\beta} + \tfrac{1}{4}\, g_{\alpha\beta}\, {\Delta^\gamma}_\gamma + \overset{\frown}{\not\Delta}_{\alpha\beta}\,, \qquad \tau_{\alpha\beta} = -\tau_{\beta\alpha}$$

• The 3 potentials span the geometry of spacetime: It is the metric-affine space (L_4, g). The corr. first order Lagrangian gauge field theory is called MAG. It is a framework for gravitational gauge field theories. We developed mainly the *bosonic*, Yuval Ne'man, together with Šijački, its *fermionic* version.

3.2 Field equations of metric-affine gravity (MAG)

- Lagrangian:

$$L_{\text{total}} = V(g_{\alpha\beta}\,, \vartheta^{\alpha}\,, Q_{\alpha\beta}\,, T^{\alpha}\,, R_{\alpha}{}^{\beta}) + L_{\text{matter}}(g_{\alpha\beta}\,, \vartheta^{\alpha}\,, \Psi\,, \overset{\Gamma}{D}\,\Psi)\,.$$

- Define the excitations (field momenta):

$$M^{\alpha\beta} = -2\frac{\partial V}{\partial Q_{\alpha\beta}}\,, \quad H_{\alpha} = -\frac{\partial V}{\partial T^{\alpha}}\,, \quad H^{\alpha}{}_{\beta} = -\frac{\partial V}{\partial R_{\alpha}{}^{\beta}}\,.$$

- Then the field equations of MAG read (Einstein sector),

$$\begin{aligned}
DM^{\alpha\beta} - m^{\alpha\beta} &= \sigma^{\alpha\beta} && (\delta/\delta g_{\alpha\beta}\text{: 0th field eq.})\,,\\
DH_{\alpha} - E_{\alpha} &= \Sigma_{\alpha} && (\delta/\delta\vartheta^{\alpha}\text{: 1st field eq.})\,,\\
DH^{\alpha}{}_{\beta} - E^{\alpha}{}_{\beta} &= \Delta^{\alpha}{}_{\beta} && (\delta/\delta\Gamma_{\alpha}{}^{\beta}\text{: 2nd field eq.})\,,\\
\frac{\delta L}{\delta\Psi} &= 0 && (\delta/\delta\Psi\text{: matter eq.})\,,
\end{aligned}$$

with the energy-momentum and the hypermomentum of the gauge fields as

$$\begin{aligned}
m^{\alpha}{}_{\beta} &:= \vartheta^{\alpha}\wedge E_{\beta} + Q_{\beta\gamma}\wedge M^{\alpha\gamma} - T^{\alpha}\wedge H_{\beta} - R_{\gamma}{}^{\alpha}\wedge H^{\gamma}{}_{\beta} + R_{\beta}{}^{\gamma}\wedge H^{\alpha}{}_{\gamma},\\
E_{\alpha} &:= e_{\alpha}\rfloor V + (e_{\alpha}\rfloor T^{\beta})\wedge H_{\beta} + (e_{\alpha}\rfloor R_{\beta}{}^{\gamma})\wedge H^{\beta}{}_{\gamma} + \frac{1}{2}(e_{\alpha}\rfloor Q_{\beta\gamma})\,M^{\beta\gamma},\\
E^{\alpha}{}_{\beta} &:= -\vartheta^{\alpha}\wedge H_{\beta} - g_{\beta\gamma}\,M^{\alpha\gamma}.
\end{aligned}$$

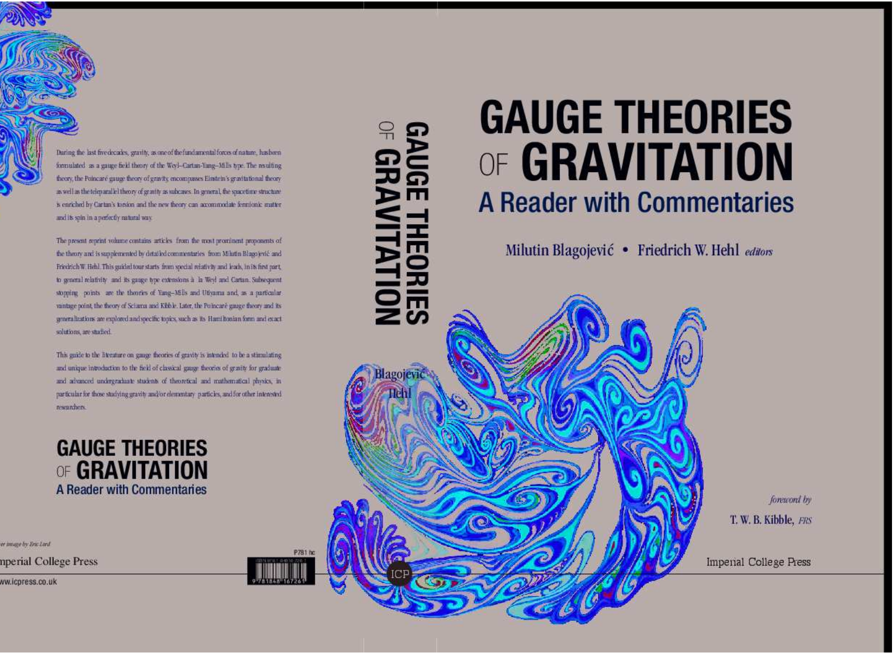
- We have the potentials $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta)$. The field strength of $\Gamma_\alpha{}^\beta$ is the curvature with $R_{\alpha\beta} := g_{\beta\gamma} R_\alpha{}^\gamma$. It can be decomposed into 3 pieces: $Z_{\alpha\beta} := R_{(\alpha\beta)}$, $W_{\alpha\beta} := R_{[\alpha\beta]}$, $Z := Z_\alpha{}^\alpha$:

$$R_{\alpha\beta} = \underbrace{Z_{\alpha\beta}}_{\text{strain}} + \underbrace{W_{\alpha\beta}}_{\text{rotation}} = \underbrace{\not{Z}_{\alpha\beta}}_{\text{shear}} + \underbrace{Z g_{\alpha\beta}/4}_{\text{dilation}\sim\text{gen.Weyl}} + \underbrace{W_{\alpha\beta}}_{\text{rotation}} .$$

- Quadratic gravitational YM-type master Lagrangian [Yang (1974) $\rightarrow W^2$]:

$$V_{\text{MAG}} \sim \underbrace{\frac{1}{\kappa}}_{\text{grav.const.}} \left(R + \lambda + T^2 + Q^2 + QT\right) + \underbrace{\frac{1}{\rho}}_{\text{dim.less}} \left(W^2 + Z^2 + WZ\right) ,$$

see M.Blagojević & FWH (eds.), Gauge Theories of Gravitation, a reader with commentaries, Imperial College Press, London (2013).


During the last five decades, gravity, as one of the fundamental forces of nature, has been formulated as a gauge field theory of the Weyl–Cartan-Yang–Mills type. The resulting theory, the Poincaré gauge theory of gravity, encompasses Einstein's gravitational theory as well as the teleparallel theory of gravity as subcases. In general, the spacetime structure is enriched by Cartan's torsion and the new theory can accommodate fermionic matter and its spin in a perfectly natural way.
The present reprint volume contains articles from the most prominent proponents of the theory and is supplemented by detailed commentaries from Milutin Blagojević and Friedrich W. Hehl. This guided tour starts from special relativity and leads, in its first part, to general relativity and its gauge type extensions à la Weyl and Cartan. Subsequent stopping points are the theories of Yang–Mills and Utiyama and, as a particular vantage point, the theory of Sciama and Kibble. Later, the Poincaré gauge theory and its generalizations are explored and specific topics, such as its Hamiltonian form and exact solutions, are studied.
This guide to the literature on gauge theories of gravity is intended to be a stimulating and unique introduction to the field of classical gauge theories of gravity for graduate and advanced undergraduate students of theoretical and mathematical physics, in particular for those studying gravity and/or elementary particles, and for other interested researchers.
GAUGE THEORIES OF GRAVITATION
A Reader with Commentaries
Imperial College Press
www.icpress.co.uk
P781 hc
GAUGE THEORIES OF GRAVITATION
Blagojević
Hehl
ICP
GAUGE THEORIES OF GRAVITATION
A Reader with Commentaries
Milutin Blagojević • Friedrich W. Hehl editors
foreword by
T. W. B. Kibble, FRS
Imperial College Press

3.3 An exact solution of metric-affine gravity (MAG)

OVETH-solution (Obukhov, Vlachynsky, Esser, Tresguerres, H., 1996)

- Spherically symmetric field configuration

Spherical polar coordinates (t, r, θ, ϕ), coframe of Schwarzschild type

$$\vartheta^{\hat{0}} = f\, dt\,, \quad \vartheta^{\hat{1}} = \frac{1}{f}\, dr\,, \quad \vartheta^{\hat{2}} = r\, d\theta\,, \quad \vartheta^{\hat{3}} = r\, \sin\theta\, d\phi\,,$$

with unknown function $f(r)$. Coframe assumed to be *orthonormal*, local Minkowski metric $o_{\alpha\beta} :=$ diag$(-1,1,1,1) = o^{\alpha\beta}$:

$$ds^2 = o_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta = -f^2\, dt^2 + \frac{dr^2}{f^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,.$$

For torsion and nonmetricity the *triplet ansatz* with 3 covector pieces. Nonmetricity = only shear $\oplus$ dilation ($\Lambda := \vartheta^\alpha e^\beta \rfloor \not{Q}_{\alpha\beta}$):

$${}^{(3)}Q_{\alpha\beta} = \frac{4}{9}\left(\vartheta_{(\alpha} e_{\beta)} \rfloor \Lambda - \frac{1}{4} g_{\alpha\beta} \Lambda\right)\,, \qquad {}^{(4)}Q_{\alpha\beta} = Q\, g_{\alpha\beta}\,.$$

Torsion with only the vector piece:

$$T^\alpha = {}^{(2)}T^\alpha = \frac{1}{3}\, \vartheta^\alpha \wedge T\,, \qquad \text{with} \qquad T := e_\alpha \rfloor T^\alpha\,.$$

Thus we are left with the three non-trivial 1-forms Q, Λ, and T. Thus,

$$Q = u(r)\, \vartheta^{\hat{0}}\,, \qquad \Lambda = v(r)\, \vartheta^{\hat{0}}\,, \qquad T = \tau(r)\, \vartheta^{\hat{0}}\,.$$

• Use computer algebra packages Reduce-Excalc and GRG. Simplify Lagrangian and come up with

$$\begin{aligned} V &= \frac{1}{2\kappa}\left[-a_0\, R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\lambda\,\eta + T^{\alpha}\wedge{}^{*}\left(\sum_{I=1}^{3} a_I\,{}^{(I)}T_{\alpha}\right)\right. \\ &+ \left. 2\left(\sum_{I=2}^{4} c_I\,{}^{(I)}Q_{\alpha\beta}\right)\wedge\vartheta^{\alpha}\wedge{}^{*}T^{\beta} + Q_{\alpha\beta}\wedge{}^{*}\left(\sum_{I=1}^{4} b_I\,{}^{(I)}Q^{\alpha\beta}\right)\right] \\ &- \frac{z_4}{2}\,R^{\alpha\beta}\wedge{}^{*(4)}Z_{\alpha\beta} \qquad \Longleftarrow \quad \text{strong gravity}\,. \end{aligned}$$

• The new solution with four types of charge:
By substitution into the field equations we find an exact solution with

$$f = \sqrt{1 - \frac{2\kappa M}{r} - \frac{\lambda\, r^2}{3a_0} + z_4\frac{\kappa(k_0 N)^2}{2a_0\, r^2}}$$

and

$$u = \frac{k_0 N}{f r}\,, \qquad v = \frac{k_1 N}{f r}\,, \qquad \tau = \frac{k_2 N}{f r}\,.$$

Here M and N are arbitrary *integration constants*, and the coefficients k_0, k_1, k_2 are constructed in terms of the dimensionless coupling constants.

Collect our results, orthonormal coframe, nonmetricity, and torsion are

$$\vartheta^{\hat{0}} = f\,dt\,, \quad \vartheta^{\hat{1}} = \frac{1}{f}\,dr\,, \quad \vartheta^{\hat{2}} = r\,d\theta\,, \quad \vartheta^{\hat{3}} = r\,\sin\theta\,d\phi\,,$$

$$Q^{\alpha\beta} = \frac{1}{fr}\left[k_0N\,o^{\alpha\beta} + \frac{4}{9}\,k_1N\left(\vartheta^{(\alpha}e^{\beta)}\rfloor - \frac{1}{4}\,o^{\alpha\beta}\right)\right]\vartheta^{\hat{0}}\,,$$

$$T^{\alpha} = \frac{k_2N}{3\,fr}\,\vartheta^{\alpha}\wedge\vartheta^{\hat{0}}\,.$$

Besides mass, this solution carries dilation, shear, and spin charges, each of them of the (co)vectorial type. We have the following assignments:

$M \longrightarrow$ mass of Schwarzschild type ,

$k_0N \longrightarrow$ dilation ('Weyl') charge of type CONOM or ${}^{(4)}Q^{\alpha\beta}$,

$k_1N \longrightarrow$ shear charge of type VECNOM or ${}^{(3)}Q^{\alpha\beta}$,

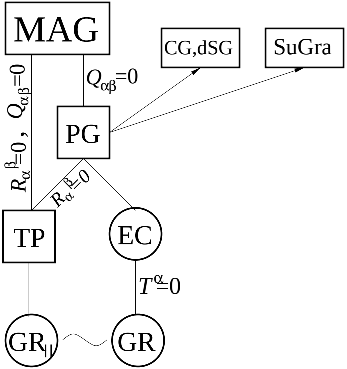
$k_2N \longrightarrow$ spin charge of type TRATOR or ${}^{(2)}T^{\alpha}$.

For $N = 0$ and $a_0 = 1$, recover Schwarzschild-deSitter solution of GR.
Two typical curvature pieces (of rotational and shear type) as examples ($q^2 := z_4(k_0N)^2/(2a_0)$):

$${}^{(1)}W^{\hat{0}\hat{1}} = -2\kappa\,\frac{Mr - q^2}{r^4}\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}, \qquad {}^{(4)}Z^{\alpha\beta} = o^{\alpha\beta}\,\frac{k_0N}{2r^2}\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}\,.$$

• Review of exact solutions, H. & Macias, IJMPD (1999).

4. Models within the framework of MAG (nonm. $Q_{\alpha\beta}$, tors. T^{α}, curv. $R_{\alpha}{}^{\beta}$)

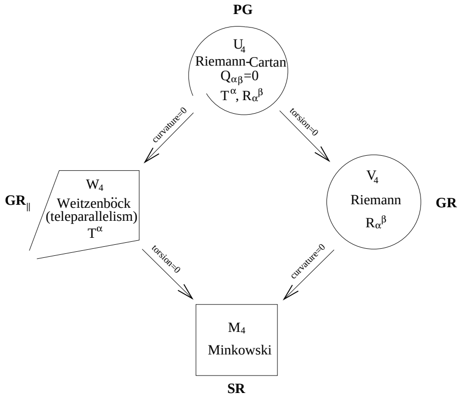


Rectangle $\square \rightarrow$ class of theories; circle $\circ \rightarrow$ definite viable theories

PG = Poincaré gauge theory, **EC** = Einstein–Cartan theory, **GR** = general relativity, **TG** = translation gauge theory aka teleparallel theory, $\mathbf{GR}_{||}$ = a specific TG known as teleparallel equivalent of GR (spoken “GR teleparallel”), **MAG** = metric-affine gauge theory, **CG** = conformal gauge theory, **dSG** = (anti-)de Sitter gauge theory, **SuGra** = supergravity

4.1 Viable models

All in Riemann-Cartan spacetime: Einstein-Cartan theory (EC), general relativity (GR), teleparallel equivalent of GR ($GR_{||}$). The corresponding geometries:



Nonmetricity=0 (Lorentz invariance): A space with a metric and a metric compatible connection is called a Riemann-Cartan space U_4. It can either become a Weitzenböck space W_4, if its curvature vanishes, or a Riemann space V_4, if the torsion happens to vanish.

• *Einstein-Cartan(-Sciama-Kibble) theory (EC)*:

Curv. scalar of Riemann–Cartan spacetime as simplest PG Lagrangian, $V_{\rm EC} \sim \frac{1}{\kappa} R_{\rm sc} \sim \frac{1}{\kappa}{}^\star(\vartheta^\alpha \wedge \vartheta^\beta) \wedge R_{\alpha\beta}(\Gamma^{\gamma\delta})$ (κ = Einstein's grav. const.):

$$\begin{aligned} \mathrm{Ric}_{ij} - \tfrac{1}{2} g_{ij} \mathrm{Ric}_k{}^k + \Lambda g_{ij} &= \kappa\, \Sigma_{ij}\,, \\ T_{ij}{}^k - \delta_i^k T_{j\ell}{}^\ell + \delta_j^k T_{i\ell}{}^\ell &= \kappa\, \tau_{ij}{}^k\,. \end{aligned}$$

Formulated in terms of exterior calculus:

$$\begin{aligned} \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} - \Lambda \eta_\alpha &= \kappa\, \Sigma_\alpha &= \kappa \frac{\delta L_{\rm mat}}{\delta \vartheta^\alpha}\,, \\ \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge T^\gamma &= \kappa\, \tau_{\alpha\beta} &= \kappa \frac{\delta L_{\rm mat}}{\delta \Gamma^{\alpha\beta}}\,. \end{aligned}$$

$\alpha, \beta, ...$ are (anholonomic) frame indices,
$\eta_\alpha = {}^\star\vartheta_a,\ \eta_{\alpha\beta} = {}^\star(\vartheta_a \wedge \vartheta_\beta),\ \eta_{\alpha\beta\gamma} = {}^\star(\vartheta_a \wedge \vartheta_\beta \wedge \vartheta_\gamma),\ {}^\star$ = Hodge star

GR plus an additional spin contact interaction. If spin $\tau\ \rightarrow 0$, then EC $\rightarrow$ GR, and RC-spacetime $\rightarrow$ Riem. spacetime; thus, GR is included.
With $\tau \neq 0$, modified source of Einstein's equation: $\rho \rightarrow \rho + \kappa\tau^2\ \Rightarrow$ at suff. high densities $\kappa\tau^2 \sim \rho \Rightarrow \rho_{\rm EC} \sim m/\left(\lambda_{\rm Co}\ell_{\rm Pl}^2\right)$ and $r_{\rm EC} \sim \left(\lambda_{\rm Co}\ell_{{\rm P}\ell}^2\right)^{1/3}$, more than $10^{52} g/cm^3$ or $10^{24}\ K$ for electrons, see H., v.d.Heyde, Kerlick, Nester, RMP 1976. This is valid up to $10^{-34}\ s$ after the big bang ($10^{-43}\ s$ corr. to Planck era). For Dirac spins, the contact interaction is repulsive (O'Connell). EC is a viable gravitational theory.— See Magueijo, Zlosnik, Kibble, "Cosmology with a spin" (2012); Khriplovich, Rudenko, "Gravitational four-fermion interaction and dynamics of the early Universe" (2013).

• *Teleparallel equivalent $GR_{||}$ of general relativity GR:*

$$V_{\|} = \frac{1}{\kappa}V_{\mathrm{T}^2} + R_\alpha{}^\beta \wedge \lambda^\alpha{}_\beta \,,$$

$$V_{\mathrm{T}^2} := -\frac{1}{2}T^\alpha \wedge {}^\star\Big(-\underbrace{{}^{(1)}T_\alpha}_{\text{tensor}} + 2\underbrace{{}^{(2)}T_\alpha}_{\text{vector}} + \frac{1}{2}\underbrace{{}^{(3)}T_\alpha}_{\text{axial vector}}\Big)\,.$$

Viable set! Yields local Lorentz invariance ⇒ Einstein's GR.
$GR_{||}$ in gauge $\Gamma \stackrel{*}{=} 0$, W_4 spacetime, field eq. is Maxwell-like:

$$D_k\, T^{ki}{}_\alpha + \text{nonlin. terms} \sim \kappa \times \Sigma_\alpha{}^i$$
$$\Box\, e^i{}_\alpha + \text{nonlin. terms} \sim \kappa \times \Sigma_\alpha{}^i \quad \text{(in Hilbert gauge)}$$

Compare Einstein's equation ($g_{ij} = g_{ji}$):

$$\Box\, g_{ij} + \text{nonlin. terms} \sim \kappa \times \sigma_{ij} \quad \text{(in Hilbert gauge)}$$

For scalar and for Maxwell matter, that is, for $\Sigma_{ij} = \sigma_{ij}$, it can be shown that $GR_{||}$ and GR are equivalent.

Distinguish teleparallelism as a unified field theory (Einstein 1929) from dualistic teleparallel gravity (Moeller 1958... Cho 1976... H.Meyer 1982... Gronwald 1997... Itin 2004..., see books of Blagojević 2002, Ortín 2004, and Aldrovandi & Pereira 2012 and review of Maluf 2013).

4.2 Poincaré gauge theory (PG)

- Field eqs. of PG: $L_{\rm tot} = V(g_{\alpha\beta}\,,\vartheta^\alpha\,,T^\alpha\,,R^{\alpha\beta}) + L_{\rm mat}(g_{\alpha\beta}\,,\vartheta^\alpha\,,\Psi\,,\overset{\Gamma}{D}\,\Psi)\,.$

$$\begin{aligned} DH_\alpha - E_\alpha &= \Sigma_\alpha\,,\\ DH_{\alpha\beta} - E_{\alpha\beta} &= \tau_{\alpha\beta}\,,\\ \frac{\delta L}{\delta\Psi} &= 0\,. \end{aligned}$$

Energy-momentum $E_\alpha := \partial V/\partial\vartheta^\alpha$ and spin $E_{\alpha\beta} := \partial V/\partial\Gamma^{\alpha\beta}$ of gravitational gauge fields:

$$E_\alpha = e_\alpha\rfloor V + (e_\alpha\rfloor T^\beta)\wedge H_\beta + (e_\alpha\rfloor R^{\beta\gamma})\wedge H_{\beta\gamma}\,,\qquad E_{\alpha\beta} = -\vartheta_{[\alpha}\wedge H_{\beta]}\,.$$

E_α and $E_{\alpha\beta}$ are gauge covariant 3-forms, they are gravitationally 'charged'...

- Irreducible decomposition of the torsion:

$$T^\alpha = \underbrace{{}^{(1)}T^\alpha}_{\text{tentor 16}} + \underbrace{{}^{(2)}T^\alpha}_{\text{trator 4}} + \underbrace{{}^{(3)}T^\alpha}_{\text{axitor 4}} = {}^{(1)}T^\alpha - \tfrac{1}{3}\,\mathcal{V}\wedge\vartheta^\alpha + \tfrac{1}{3}\,{}^\star(\mathcal{A}\wedge\,\vartheta^\alpha)\;.$$

with 1-forms for vector and axial vector: $\mathcal{V} := e_\beta\rfloor T^\beta$ and $\mathcal{A} := {}^\star(\vartheta_\alpha\wedge T^\alpha)$.

- Irreducible decomposition of the curvature:

$$R_{\alpha\beta} \;=\; \underbrace{{}^{(1)}R_{\alpha\beta}}_{\text{weyl 10}} + \underbrace{{}^{(2)}R_{\alpha\beta}}_{\text{paircom 9}} + \underbrace{{}^{(3)}R_{\alpha\beta}}_{\text{pscalar 1}} + \underbrace{{}^{(4)}R_{\alpha\beta}}_{\text{ricsymf 9}} + \underbrace{{}^{(5)}R_{\alpha\beta}}_{\text{ricanti 6}} + \underbrace{{}^{(6)}R_{\alpha\beta}}_{\text{scalar 1}}\;.$$

R (curv. scalar) and $X = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}R^{[\alpha\beta\gamma\delta]}$ (curv. pseudoscalar)

Gauge field theories of Abelian and non-Abelian groups compared

• Maxwell–Dirac theory as a $U(1)$ gauge field theory is Abelian. Recall: Excitation $H = (\mathcal{D}, \mathcal{H})$, field strength $F = (E, B)$.

• The group $SU(2)$ is noncommutative, that is non-Abelian, the Yang–Mills theory is a nonlinear theory and its gauge current, $\overset{A}{I} = -A \wedge H$, even though it is not gauge covariant, carries its own isospin, note $\overset{A}{D}\, H = dH - A \wedge H$.

• The Poincaré group is also non-Abelian. Hence PG shares numerous properties with the Yang–Mills theory. The gauge currents carry their own charge and the theories become nonlinear.

Maxw:	$dH = J,$	$dF = 0,$	$H = Y_0\,{}^\star F.$
YM:	$\overset{A}{D}\, H = I,$	$\overset{A}{D}\, F = 0,$	$H = \alpha_0\,{}^\star F.$
PG1:	$\overset{\Gamma}{D}\, H_\alpha - E_\alpha = \Sigma_\alpha,$	$\overset{\Gamma}{D}\, T^\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta,$	$H_\alpha = H_\alpha(T^\gamma, R^{\delta\varepsilon});$
PG2:	$\overset{\Gamma}{D}\, H_{\alpha\beta} + \vartheta_{[\alpha} \wedge H_{\beta]} = \tau_{\alpha\beta},$	$\overset{\Gamma}{D}\, R^{\alpha\beta} = 0,$	$H_{\alpha\beta} = H_{\alpha\beta}(T^\gamma, R^{\delta\varepsilon}).$

PG1 refers to the field equation related to translations and PG2 to the Lorentz rotations, but there are mixing terms between both potentials. Universality!

• Quadratic master Lagrangian of PG including parity violating pieces:

$$V_{\rm qPG} = \frac{1}{2\kappa}\Big[\,(\,a_0 R - 2\Lambda_0 + b_0 X\,)\,\eta + \frac{a_2}{3}\mathcal{V}\wedge{}^{\star}\mathcal{V} - \frac{a_3}{3}\mathcal{A}\wedge{}^{\star}\mathcal{A} - \frac{2\sigma_2}{3}\mathcal{V}\wedge{}^{\star}\mathcal{A} + a_1{}^{(1)}T^{\alpha}\wedge{}^{\star(1)}T_{\alpha}\Big]$$
$$-\frac{1}{2\varrho}\Big[\big(\frac{w_6}{12}R^2 - \frac{w_3}{12}X^2 + \frac{\mu_3}{12}RX\big)\,\eta + w_4{}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta}$$
$$+{}^{(2)}R^{\alpha\beta}\wedge\big(w_2{}^{\star(2)}R_{\alpha\beta} + \mu_2{}^{(4)}R_{\alpha\beta}\big) + {}^{(5)}R^{\alpha\beta}\wedge\big(w_5{}^{\star(5)}R_{\alpha\beta} + \mu_4{}^{(5)}R_{\alpha\beta}\big)\Big]$$

[Obukhov, Ponomarev, Zhytnikov: Gen.Rel.Grav. 21, 1107 (1989), Baekler, H., Nester: PRD 83, 024001 (2011), Diakonov, Tumanov, Vladimirov: PRD 84, 124042 (2011), Baekler, H.: CQG 28, 215017 (2011)]

• Cosmol. models with acc. expan. by Shie, Nester, Yo (without parity viol.):

$$V_{\rm SNY} = \underbrace{\frac{1}{2\kappa}\left(a_0\,{}^{\star}R + \frac{1}{3}a_2\mathcal{V}\wedge{}^{\star}\mathcal{V}\right)}_{\text{weak Newt.-Einst. grav.}} - \underbrace{\frac{1}{24\varrho}w_6 R^2\eta}_{\text{strong YM-type grav.}}\quad .$$

• Cosmol. models with acc. expan. by Baekler, H., Nester (with parity viol.):

$$V_{\rm BHN} = \frac{1}{2\kappa}(a_0{}^{\star}R + b_0{}^{\star}X - 2\lambda_0\eta) + \frac{1}{6\kappa}\left(a_2\mathcal{V}\wedge{}^{\star}\mathcal{V} - a_3\mathcal{A}\wedge{}^{\star}\mathcal{A} - 2\sigma_2\mathcal{V}\wedge{}^{\star}\mathcal{A}\right)$$
$$-\frac{1}{24\varrho}\left(w_6 R\,{}^{\star}R - w_3 X\,{}^{\star}X + \mu_3 R\,{}^{\star}X\right)\,.$$

Cosmological model of Shie-Nester-Yo as example, see last slide [PRD 78, 023522 (2008)], more gen.models by Minkevich et al. CQG 2007 and references therein. In the SNY parametrization, we have

$$V_{\rm SNY} \sim \frac{1}{\kappa}\left(a_0 R_{\rm sc} + a_1 V_{\mathsf{T}^2}\right) + b\, R_{\rm sc}^2\,,$$

$a_1 > 0,\ b > 0$. In an Hamiltonian analysis, they found, for $a_0 \neq a_1$, a massive torsion mode of spin 0^+ with effective mass $\mu := a_1 - a_0$, together with 2 conventional graviton helicities.

1st field eq. (vector torsion $T_i := T_{ik}{}^k$) and 2nd vacuum field eq. (matter spin negligible):

$$\frac{2a_1}{3}\left(e^i{}_\beta D_\alpha T^\beta - e^i{}_\alpha \widetilde{D}_j T^j\right) - e^i{}_\alpha\left(-\frac{a_0}{2}R + \frac{b}{24}R^2 - \frac{a_1}{3}T_j T^j\right)$$
$$+R_\alpha{}^i\left(\frac{b}{6}R - a_0\right) = \kappa\Sigma_\alpha{}^i\,,$$

$$D_\alpha R = -\frac{2}{3}\left(R + \frac{6\mu}{b}\right)T_\alpha\,,\quad T_{ij}{}^\alpha = \frac{2}{3}T_{[i}e_{j]}{}^\alpha$$

Both are YM-type eqs. for $T_{ij}{}^\alpha$ and R, respectively. 1st field eq. can be rewritten in a quasi-Einsteinian form.

• The authors investigate a Friedmann model and eventually conclude: "... we show that for suitable ranges of the parameters the dynamic scalar torsion model can display features similar to those of the presently observed accelerating universe."

• Confirmed by X. -C. Ao and X. -Z. Li, "Torsion Cosmology of Poincaré gauge theory and the constraints of its parameters via SNeIa data," JCAP 1202, 003 (2012)

• See, however, C. -Q. Geng, C. -C. Lee, and H. -H. Tseng, "Asymptotic cosmological behavior of scalar-torsion mode in Poincaré gauge theory," PRD 87, 027301 (2013)

4.3 Additional shear transformations, Ne'eman's world spinors

• Independent connection yields a new type of shear current, corresponds to the quotient $SL(4,R)/SO(1,3)$. Recall splitting of hypermomentum current:

$$\begin{aligned} \Delta_{\alpha\beta} &= \tau_{\alpha\beta} + \tfrac{1}{4}\, g_{\alpha\beta}\, \Delta + \widehat{\not\Delta}_{\alpha\beta} \\ &\sim \text{spin current} \oplus \text{dilation current} \oplus \text{shear current}\,, \end{aligned}$$

$\tau_{\alpha\beta} := \Delta_{[\alpha\beta]}$, $\Delta := \Delta_\gamma{}^\gamma$, and $\widehat{\not\Delta}_{\alpha\beta} := \Delta_{(\alpha\beta)} - \frac{1}{4}\, g_{\alpha\beta}\, \Delta$. New Noether law:

$$^\dagger D \widehat{\not\Delta}_{\alpha\beta} + {}^\dagger Q_{\mu(\alpha} \wedge \not\Delta^\mu{}_{\beta)} + \vartheta_{(\alpha} \wedge \widehat{\Sigma}_{\beta)} - \not\sigma_{\alpha\beta} = 0\,.$$

"Hypermom. in hadron dynamics and in grav.," PRD 1978, Lord, Ne'eman, H.

• $SL(3,R)$ current originally proposed by Dothan, Gell-Mann, Ne'eman (1965) as dynamical group for classification of sequences of hadrons (Regge trajectories). Lie algebra of $SO(3)$ extended by five operators of the time derivatives of the quadrupole moments of the 'hadronic' energy-momentum current. Spin 2 excitations. Later generalized to $SL(4,R)$ and to $GL(4,R)$. [Test: For Dirac field, we can directly relate the time derivatives of the quadrupole exitations to the (orbital) shear current, $\frac{d}{dt}\int d^3x\, x^\alpha x^\beta\, \Sigma^{0\kappa}$ $= 2\int d^3x\, x^{(\alpha}\Sigma^{\beta)\kappa}$.] Are shear currents conceivable in a quark-gluon-plasma?

• Ne'eman 1977 generalized Dirac spinors to world spinors, he showed existence of double-valued linear infinite spinorial representations of diffeomorphism group, see Kirsch and Šijački 2002.

• "Fermionic" hyperfluid of Obukhov and Tresguerres 1993.

• Incorporation into MAG of Jacobsson's et al. Einstein-aether model (Heinicke, Baekler, H. 2005); incorporation into MAG of Fronsdal-Vasiliev massless spin 3 (Baekler, Boulanger, H. 2006).

• Numerous exact solutions in the context of MAG have been found. But MAG, from a physical point of view, is still wide open...

5. Operational significance of torsion and nonmetricity

Eqs. of motion of test matter in PG and MAG has been investig. since 1970's.

PG: see H., Obukhov, Puetzfeld 2013: *On Poincaré gauge theory of gravity, its equations of motion, and Gravity Probe B.* GPB cannot sense torsion!

MAG: see the recent review of Puetzfeld & Obukhov 2013: *Unraveling gravity beyond Einstein with extended test bodies.* If minimally coupled, microscopic matter is required for measuring torsion and nonmetricity!

5.1 Torsion

• Precession of elementary particle spin (of electron or neutron, e.g.) in torsion field. Rumpf (1979) (polarisation vector $\mathbf{w}$ of the spin) found

$$\dot{\mathbf{w}} = 3\mathbf{t} \times \mathbf{w}\,, \quad t^{\alpha} := -\epsilon^{\alpha\beta\gamma\delta}\, T_{\beta\gamma\delta}/3!$$

Independent of particular model. On the Earth, using $\text{GR}_{||}$, find only $|\mathbf{T}| \sim 10^{-15}\,\frac{1}{s}$, see Lämmerzahl (1997). He determined experimental limits of admissible torsion by using Hughes & Drever tpye experiments.

• Gravity Probe B, instead of the quartz balls, spin polarized balls: Spin gyroscope (see Ni `gr-qc/0407113`, Vasilakis et al. PRL 2009).

• Papini et al. (2004): measuring a spin flip of a neutrino induced by torsion, Lambiase calculated cross section for corr. process.

5.2 Nonmetricity

• Eq. of motion of a matter field in MAG: Apply the Bianchi identities for MAD (tilde denotes the Riemannian part) (Obukhov '96, Ne'eman & H '96),

$$\widetilde{D}\left[\Sigma_\alpha + \Delta^{\beta\gamma}\left(e_\alpha \rfloor \diamondsuit\Gamma_{\beta\gamma}\right)\right] + \Delta^{\beta\gamma}\wedge\left(\pounds_{e_\alpha}\diamondsuit\Gamma_{\beta\gamma}\right) = \tau^{\beta\gamma}\wedge\left(e_\alpha\rfloor\widetilde{R}_{\gamma\beta}\right)\,,$$

where $\diamondsuit\Gamma_{\beta\gamma} := \Gamma_{\beta\gamma} - \widetilde{\Gamma}_{\beta\gamma}$ is post-Riemannian part of connection. On the r.h.s., Mathisson-Papapetrou force density of GR for matter with spin $\tau^{\beta\gamma} := \Delta^{[\beta\gamma]}$.

• For $\Delta^{\beta\gamma} = 0$, we have $\widetilde{D}\Sigma_\alpha = 0$. Without dilation, shear, and spin "charges" the particle follows a *Riem. geodesic*, irresp. of the form of $V_{\rm MAG}$.

• Thus, test matter for nonmetricity has to carry dilation or shear charges. Ne'eman's world spinors do carry shear (they kind of represent Regge trajectories). They are the appropriate test matter for nonmetricity.

• Detailed discussions show that nonmetricity induces pulsations (mass quadrupole excitations) on suitable test matter. For the first time, one has a good idea about the interpretation of the nonmetricity.

6. Discussion

• The gauging of the Poincaré group led to PG with the master Lagrangian quadratic in torsion and curvature, including parity violating pieces.

• $GR_{||}$, EC, and GR are viable theories within the framework of PG.

• Within MAG, the dilation and shear currents as well as the nonmetricity can be understood consistently. However, MAG, at the present time, should be considered as speculative.