Historical Introduction into Relativistic Mean Field Theory

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1949 y. **Nuclear Shell Model**

- Magic numbers
- Nuclear Mean Filed: independent motion of nucleons in an appropriately chosen potential. The long mean free path of nucleons in a nucleus.
- This theory is differ significantly from the Hartree model in atomic physics. In a nucleus nucleons themselves completely determine the nuclear potential.
- Strong spin-orbit interaction

$$\lambda \frac{1}{r} \frac{dV}{dr} \vec{l} \cdot \vec{s}$$

- The long mean free path of nucleons in a nucleus can not be easily reconciled with the cross section for collision between free nucleons.

1952 y. **Generalized (Collective) nuclear model**

- Nuclear potential, nuclear density are almost constant inside nucleus
- Low-lying collective excitations of nuclei are related to surface oscillations

$$R(\theta,\varphi) = R_0\left(1 + \sum_{\lambda=2,\ldots}\sum_{\mu}\alpha *_{\lambda\mu} Y_{\lambda\mu}(\theta,\varphi)\right)$$

Nobel Prize for Physics for 1949: Prof. H.Yukawa

Prof.H.Yukawa is best known for his theory of nuclear forces which, in 1935, first postulated the existance of a particle a few hundred times heavier than the electron. The nuclear forces would then be in the same relation to the possible emission and absorbtion of such a particle as the electromagnetic forces to the emission and absorbtion of light. The discovery of the μ-meson in cosmic rays appeared to be a confirmation of Yukawas prediction, but the study of its properties gradually led to the conviction that it could not be identical with the particle required for Yukawas theory. It was not until 1947 that Powell and his collaborators demonstrated the existance of a second short-lived particle, the π-meson, which is strongly linked to protons and neutrons.

1955 y. **Classical field theory of nuclear forces**

M.M.Johnson and E.Teller, Phys. Rev. **98**, 783 (1955)

- Nuclear interactions are strong which has the consequence that at high energies the multiple production of nuclear quanta-meson is the rule, where the multiple production of gamma-quanta is a rare event. Consequently in nucleon-nucleon collisions several mesons may be expected in virtual states.
- For heavy nucleus in which the expectation value of mesons present is considerably larger than one, the mesons obeying the Bose statistics will tend to occupy the same quantum states. The wave function of this quantum state will correspond to the classical potential of nuclear forces.

Interaction can take form:

$$\sum_l \Psi * O_l \Psi \cdot \Phi_l(\Phi) \quad (1)$$

where Φ is the amplitude of the meson field

- Φ_l must be a simple scalar and isotopic singlet
- If $\Psi * O_l \Psi$ is an isotopic triplet it will have components in which the nucleon changes charge
- A pseudoscalar Φ_l can be ruled out because nuclear mean filed conserves parity

Thus meson is scalar and neutral (isoscalar). This meson need not be an elementary particle. It may be a virtual state composed of other mesons. It can be even a superposition of such virtual states. It may decay into π-mesons so quickly that it cannot be observed (resonance with a large width).

The simplest Hamiltonian valid for the interior of nuclei is

$$H_1 = \int d\tau \left\{ \frac{\hbar^2}{2m} \sum_j |\nabla \Psi_j|^2 + \mu^2 c^4 \Phi^2 - \hbar c g \Phi \sum_j |\Psi_j|^2 \right\} \quad (2)$$

At the moment it does not include kinetic energy of meson field.

This Hamiltonian has two shortcomings

- it does not explain saturation
- it predicts too large neutron excess in heavy nuclei

$$\frac{\delta H_1}{\delta \Phi(r)} = 2\mu^2 c^4 \Phi - \hbar c g \sum_j |\Psi_j|^2 \equiv 2\mu^2 c^4 \Phi - \hbar c g \rho = 0$$

at equilibrium. Then

$$H_1 = \int d\tau \left\{ \frac{\hbar^2}{2m} \sum_j |\nabla \Psi_j|^2 + \frac{\mu^2 c^4 \hbar^2 c^2 g^2}{4\mu^4 c^8} \rho^2 - (\hbar c g)^2 \frac{\rho^2}{2\mu^2 c^4} \right\}$$

$$= \int d\tau \left\{ \frac{\hbar^2}{2m} \sum_j |\nabla \Psi_j|^2 - \frac{1}{4} \frac{(\hbar c g)^2}{2\mu^2 c^4} \rho^2 \right\}$$

Nucleon kinetic energy $\sim \rho^{5/3}$. Therefore, there is no saturation.

The Coulomb energy should be added to H_1. The energy minimum occur when the kinetic energy at the top of the proton Fermi distribution differs from the energy at the top of the neutron Fermi distribution by the Coulomb potential. This minimum occurs in heavy nuclei for two large neutron excess.

Both difficulties can be removed by adding to H_1 a velocity-dependent term:

$$H_2 = H_1 + \int d\tau \left\{ \hbar^3 \mu^{-2} c^{-1} f \Phi \sum_j |\nabla \Psi_j|^2 \right\} \tag{3}$$

- – The velocity-dependent term added in H_2 is positive and increases with ρ as $\rho^{8/3}$. The nuclear potential has a minimum at a finite ρ
- – Since $m_{eff} < m$ nuclear velocity and kinetic energies will be increased. Therefore, a smaller neutron excess produces the difference in kinetic energies at top of the Fermi distributions necessary to balance the Coulomb potential

$$m_{eff} = \frac{m}{1 + 2m\hbar f \Phi \mu^{-2} c^{-1}} \approx 0.5$$

The kinetic energy of a neutron at the top of the Fermi distribution is

$$\left(\frac{p_{max}^2}{2m}\right)_n = \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2}{2mR^2} N^{2/3}$$

and for protons

$$\left(\frac{p_{max}^2}{2m}\right)_p = \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2}{2mR^2} Z^{2/3}$$

For ^{238}U $\left(\frac{p_{max}^2}{2m}\right)_n - \left(\frac{p_{max}^2}{2m}\right)_p = 10.7$ MeV. For equilibrium this quantity must be equal to Coulomb potential, i.e. 21 MeV. Equilibrium is achieved for $m_{eff} = 0.51m$

A Hamiltonian capable to describe a surface energy must contain kinetic energy of the meson field:

$$H_3 = H_2 + \int d\tau \hbar^2 c^2 |\nabla\Phi|^2$$

The presence of $|\nabla\Phi|^2$ in the integral prevents Φ from dropping sharply to zero at the surface.
Independent on the kinetic energy the part of the integrand of H_3 is

$$\mu^2 c^4 \Phi^2 - \hbar c g \Phi \rho$$

The first term is the potential of the meson field, the second term is the potential for nucleons times ρ. Since Φ diffused at the surface V_{nucl} obtain some diffusion which creates a surface effect.

Since Φ is diffused meson field contributes to the surface energy. Let Φ is constant inside nucleus and goes to zero with a constant slope in a layer of thickness $L(L << R)$
Then the surface energy is equal:

$$\int_{R}^{R+L} d\tau \mu^2 c^4 \Phi^2 = \frac{L}{R}\frac{4\pi}{3}R^3(\mu c^2 \Phi)^2$$

Contribution of the kinetic energy of mesons to the surface energy is

$$\int d\tau \hbar^2 c^2 |\nabla(\Phi)|^2 = \frac{3L}{R}(\hbar \mu^{-1} c^{-1} L^{-1})^2 \cdot \frac{4\pi}{3}R^3(\mu c^2 \Phi)^2$$

The sum of two terms is

$$S = (L + \frac{3}{L}(\hbar \mu^{-1} c^{-1})^2) \cdot \frac{4\pi}{3}R^2(\mu c^2 \Phi)^2$$

$$\frac{dS}{dL} = 0 \to L_{min} = \sqrt{3}\hbar \mu^{-1} c^{-1}$$

and $S_{min} = 116\hbar \mu^{-1} c^{-1} \rho^{1/2} A^{2/3}$ MeV. The experimental value of $S_{exp} = 18A^{2/3}$ MeV and we obtain $\mu c^2 = 660$ MeV

Spin-orbit coupling estimated by analogy to atomic physics have a correct sign within nuclei. However, its value is too small.

$$V_{so} = \frac{\hbar}{2m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\vec{\sigma} \cdot \vec{l}$$

With m_{eff} instead m the spin-orbit coupling approach experimental value.

Summary of the ideas of the paper by Johnson and Teller

- The nuclear model is proposed in which it is assumed that the scalar nuclear potential arises from a linear coupling of the scalar neutral (isoscalar) meson field. The idea of the scalar meson condensate is introduced.
- Effective nucleon mass $m_{eff} \approx 0.5m$
- An additional linear coupling to the scalar meson field is introduced which is proportional to the kinetic energy of the nucleons. This additional term is repulsive. This term increases with density. As a result a saturation properties are correctly given.
- The kinetic energy dependent coupling has the effect of decreasing the mass of a nucleon within a nucleus
- Scalar meson which can be a resonance with large width and a mass ~ 600 MeV

H.-P.Dürr, Phys.Rev. **103**, 469 (1956)

Let us try to find a relativistic formulation of an interaction which in the relativistic limit will lead to a velocity dependence as it is proposed by Johnson and Teller. We also introduce vector meson field in addition to scalar meson field. Dirac equation ($\hbar = c = 1$)

$$i\gamma_\mu p_\mu + m = O_j$$

In general,

$$\begin{aligned} O_1 &= V_s - \quad \text{scalar} \\ O_2 &= i\gamma_\nu A_\nu,\ A_\nu = (\vec{A}, iA_0) - \quad \text{vector} \\ O_3 &= i\gamma_5 V_{ps} - \quad \text{pseudoscalar} \end{aligned}$$

We restrict consideration to the interactions which have a non-vanishing linear averages. In the time-independent problem this amounts to selfconsistent field treatment. On the basis of the transformation properties of the fields only the scalar interaction and the fourth component of the vector vector field will contribute to nonrelativistic Hamiltonian. Because of the parity conservation pion field contribution vanishes.

Dirac Hamiltonian

$$H = \vec{\alpha} \cdot \vec{p} + \beta m - \beta V + A_0$$

$\vec{\alpha} \cdot \vec{p}$ mixes large and small component of a Dirac spinor. Because of this, a weight of a small component will increase with $\vec{p}$. The nuclear potential $-\beta V + A_0$ is attractive for small $|\vec{p}|$ $(V > A_0)$ and becomes repulsive for higher $|\vec{p}|$.

Let $V = am\Phi$, $A_0 = bm\Phi_0$

It can be shown that the interaction with uncharged scalar field lead always to attraction. The interaction with the fourth component of an uncharged vector field always gives repulsion.

$$L_{int} = g_s \bar{\Psi}\Psi\Phi \to V^s_{eff}(r) = -\frac{g_s^2}{4\pi}\frac{e^{-m_s r}}{r}$$

For vector meson field we get

$$V^v_{eff}(r) = \frac{g_v^2}{4\pi}\frac{e^{-m_v r}}{r}$$

Foldy-Wouthuysen transformation

$$H' = e^{iS} H e^{-iS} = H + [iS, H] + \frac{1}{2}[iS, [iS, H]] + \dots$$

with $iS = -\frac{1}{2}\vec{\alpha} \cdot \vec{p}\beta\frac{1}{m(1-a\Phi)}$ and assuming that Φ and Φ_0 are constant inside the nucleus we get

$$H' = \frac{\vec{p}^2}{2m(1-a\Phi)}\beta + \beta m(1-a\Phi) + bm\Phi_0$$

We omit here ∇ and $\vec{\sigma}$ dependent terms.
For the gross component of Dirac spinor

$$H' = \frac{\vec{p}^2}{2m} - (1 - \frac{1}{1-a\Phi}\frac{\vec{p}^2}{2m^2})am\Phi + bm\Phi_0$$

The total Hamiltonian

$$
\begin{aligned}
H' &= H_{nucl} + H^s_{int} + H^v_{int} + H^s_m + H^v_m \\
H_{nucl} &= \int d\tau E_k \rho \\
H^s_{int} &= -\int d\tau \left(1 - \frac{1}{1 - a\Phi}\frac{E_k}{m}\right) am\Phi\rho \\
H^v_{int} &= \int d\tau bm\Phi_0\rho \\
H^s_m &= \frac{1}{2}\int d\tau \left(|\nabla\Phi|^2 + \mu_1^2\Phi^2\right) \\
H^v_m &= -\frac{1}{2}\int d\tau \left(|\nabla\Phi_0|^2 + \mu_2^2\Phi_0^2\right)
\end{aligned}
$$

By variation of H' by Φ and Φ_0 we obtain

$$\begin{aligned} -\nabla^2\Phi + \mu_1^2\Phi &= \left(1 - \frac{1}{1-a\Phi}\frac{E_k}{m}\right) am\rho \\ -\nabla^2\Phi_0 + \mu_2^2\Phi_0 &= bm\rho \end{aligned}$$

Assuming a constancy of the nuclear density inside a nucleus we get

$$\begin{aligned} \mu_1^2\Phi &= \left(1 - \frac{1}{1-a\Phi}\frac{E_k}{m}\right) am\rho \\ \mu_2^2\Phi_0 &= bm\rho \end{aligned}$$

From H' the volume energy per nucleon $E_V = \frac{H'}{A}$ is

$$E_V = E_k + \frac{1}{1-a\Phi}\frac{E_k}{m}am\Phi - am\Phi + bm\Phi_0 + \frac{\mu_1^2}{2}\frac{\Phi^2}{\rho} - \frac{\mu_2^2}{2}\frac{\Phi_0^2}{\rho}$$

Minimizing E_V over ρ we obtain

$$\rho\frac{\partial E_V}{\partial \rho} = \frac{2}{3}\frac{1}{1-a\Phi}E_k - \frac{1}{2}\mu_1^2\frac{\Phi^2}{\rho} + \frac{1}{2}\mu_2^2\frac{\Phi_0^2}{\rho}$$

After some algebraic manipulations we get

$$E_V = -\left(\frac{1}{1-a\Phi} - \frac{4}{3}\right)\frac{E_k}{1-a\Phi}$$

The empirical Bethe-Weizsäcker value of E_V is -15.75 MeV. With $r_0 = 1.22 \cdot 10^{-13}$ cm we get $E_k = 19.25$ MeV and from the experimental value of the volume energy we get $a\Phi = 0.44$. This means $m_{eff} = 0.56m$.

If we take into account the electrostatic repulsion experienced by protons and treat neutrons and protons as separate Fermi gases of densities ρ_n and ρ_p, respectively, we obtain the following additional term to the total kinetic energy per nucleon

$$E'_k - E_k = \frac{5}{9} E_k \Delta^2, \qquad \Delta = \frac{N-Z}{N+Z}$$

Minimization with respect to the total density leads to the additional term to the volume binding energy per nucleon

$$\frac{5}{9}\left(\frac{10}{3} - \frac{1}{1-a\Phi}\right)\frac{E_k}{1-a\Phi}\Delta^2$$

which is the symmetry energy. Numerically, we get for this term: $29.6\Delta^2$ MeV. The experimental value is $23.42\Delta^2$ MeV.

In this approach, the nucleons are the source of two kinds of mesons: scalar and vector. The exchange by scalar meson leads to attraction, the exchange by vector mesons - to repulsion. However, the source strength of the scalar meson is not a constant but has the form

$$a_{eff} = a\left[1 - \frac{1}{1 - a\Phi}\frac{\vec{p}^2}{2m^2}\right]$$

It decrease with increasing momentum $\vec{p}$ of a nucleon and decrease with increasing scalar field amplitude. This can be interpreted as a saturation of the scalar meson interaction.

Appearance of the spin-orbit interaction.

Let charge particle moves with a velocity $\vec{V}$ in the field of a nucleus located at the center of a coordinate system. Nucleus create a Coulomb field of the strength:

$$e\vec{E} = -\frac{\vec{r}}{r}\frac{\partial V}{\partial r}.$$

In the coordinate system of a charged particle a moving nucleus creates electric field $\vec{E}$ and magnetic field

$$\vec{H} = -\frac{1}{c}\vec{V} \times \vec{E} = -\frac{1}{mc}\vec{p} \times \vec{E}$$

The corresponding Hamiltonian is

$$H_{so} = \frac{e\hbar}{2m^2c^2}\vec{\sigma} \cdot (\vec{p} \times \vec{E}).$$

Because of the Thomas precession which is a kinematical effect related to the variation of an orientation of vector (coupled to the noninertial system) with respect to the lab.frame. This decreases spin orbit interaction by factor 2.

In the approach based on Dirac equation we have a superposition of a strong attractive scalar and a strong repulsive vector potentials. The vector field precession will add to the scalar field precession constructively and will lead to extremely strong spin-orbit coupling!

This coupling is in addition enhanced by the effective mass in the interior of a nucleus.

This result follows from the nonrelativistic limit of Dirac equation.

$$H' = e^{iS} H e^{-iS}$$

$$S = -\frac{i}{4m}\beta\vec{\alpha}\left(\frac{1}{(1-a\Phi)}\vec{p} + \vec{p}\frac{1}{(1-a\Phi)}\right)$$

$$\begin{aligned} H' = \beta m &+ \beta\frac{1}{8m}\left(\vec{p}\frac{2}{(1-a\Phi)}\vec{p} + \vec{p}^2\frac{1}{(1-a\Phi)} + \frac{1}{(1-a\Phi)}\vec{p}^2\right) \\ &- \beta a m\Phi + b m\Phi_0 + \frac{1}{8m^2}\nabla\left(\frac{\nabla b\Phi_0}{(1-a\Phi)^2}\right) \\ &+ \frac{1}{4m^2(1-a\Phi)^2}\vec{\sigma}\cdot(\nabla(\beta a\Phi + b\Phi_0)\times\vec{p}) \end{aligned}$$

For nucleus in the nonrelativistic limit we take the upper rows of Dirac equation, i.e. $\beta \to 1$. We obtain very strong spin-orbit coupling

$$H_{so} = \frac{1}{4m^2(1-a\Phi)^2}\vec{\sigma}\cdot(\nabla(a\Phi + b\Phi_0)\times\vec{p})$$

$$= \frac{1}{4m^2(1-a\Phi)^2}\frac{1}{r}\frac{\partial}{\partial r}(a\Phi + b\Phi_0)\vec{\sigma}\cdot\vec{l}$$

In the shell-model

$$H_{so} = \lambda\frac{1}{4m^2}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)\vec{\sigma}\cdot\vec{l}$$

With our results for $a\Phi$ and $b\Phi_0$ we get $\lambda \approx 33$ which is in reasonable agreement with the phenomenology.

Let us calculate the potential which acts on a nucleons of a kinetic energy $E' = E - m \ (c = 1)$ impinging on a heavy nucleus at rest.

For constant Φ and Φ_0 the Hamiltonian operator

$$H = \vec{\alpha} \cdot \vec{p} + \beta m - \beta a m \Phi + b m \Phi_0$$

reduces inside the nucleus to

$$H = \sqrt{m^2(1 - a\Phi)^2 + p^2} + b m \Phi_0 = E$$

We consider this as resulting from an effective potential which is defined by

$$\sqrt{m^2 + p^2} + V_{eff} = E$$

by eliminating p^2 from both equations we obtain

$$V_{eff} = E - \sqrt{(E - bm\phi_0)^2 + m^2 a\phi(2 - a\phi)} \quad (c = 1).$$

With the values of $a\phi$=0.44 and $b\phi_0$=0.36 defined above we obtain for very low kinetic energy

$$V_{eff} = -0.04m = -43.8MeV.$$

The effective potential is negative up to $E' = E - m$=121 MeV and then becomes repulsive for higher particle energies.

In the case of the antinucleon - nucleus potential the vector meson potential becomes attractive. On the other hand, the spin-orbit coupling will be only of the order of the spin-orbit coupling in atoms. Antinucleon-nucleus potential is always strongly attractive.

- Velocity-dependent meson-nucleon coupling is not introduced into consideration.
- Effective mass is produced by the coupling to the scalar meson.
- The empirical value of the nuclear size and the symmetry energy are reproduced.
- Appearance of the strong spin-orbit interaction is explained.
- Saturation is explained as relativistic effect (small component of a Dirac spinor, decrease of the coupling to the scalar meson with momentum $\vec{p}$ increases).

Present formulation of the Relativistic Mean Field theory (RMF)

RMF theory is a phenomenological description of the nuclear many-body problem. This theory is fully Lorentz invariant and nucleons are treated as point-like particles. They are described by Dirac equation, although we know that nucleons are composite objects. The nucleons interact between themselves only by exchange of effective point-like particles - mesons. In a phenomenological theory the number of these mesons, their quantum numbers and the values of their masses and coupling constants are determined to reproduce the experimental data.

Only as few mesons as possible are included. It is accepted that the parameters within the nuclear medium do not necessary have the same values as in the free space. The lightest and therefore the most important meson is the pion. Its quantum numbers: J=0, T=1 and $P = -1$. However, since it carries negative parity, the corresponding mean field breaks parity on the Hartree level. This is certainly not the case in the real nuclei where mean field is parity conserving to a very high degree of accuracy.

Two and any even number of pions, however, contribute to positive parity fields. In the RMF theory a phenomenological σ-meson is introduced with the quantum numbers J=0, T=0 and P=+1. This can be understood as an approximation of a more complicated object with the same quantum numbers formed from quark-antiquark pairs and gluons. As a consequence of the relativistic structure of the theory the exchange of such scalar mesons leads to an attractive forces between nucleons.

The repulsive part of the interaction is determined by the exchange of vector mesons. The most important one is the ω-meson with the quantum numbers J=1, T=0. A meson of this kind is experimentally known. Its mass is $m_\omega \approx$783 MeV. On the mean field level it produces a vector field $\omega^\mu(x)$, whose time-like component is strongly repulsive, in close analogy to the electromagnetic field of photons which also carries spin J=1. It is presented by the vector potential $A^\mu(x)$ and its time-like component represents the Coulomb repulsion.

In addition we know, that nuclear forces depends on the isospin. The isospin dependence is caused by the exchange of ρ-mesons with quantum numbers J=1, T=1. In principle, there are many more mesons which play a role in a quantitative description of the bare nucleon-nucleon forces by meson exchange. The δ-meson, for instance, would lead to scalar nuclear potential different for protons and neutrons. In order to simplify equations in many nuclear models only the most important fields $\sigma(x)$, $\omega^\mu(x)$, $\vec{\rho}^\mu(x)$ and the photon $A^\mu(x)$ are taken into account.

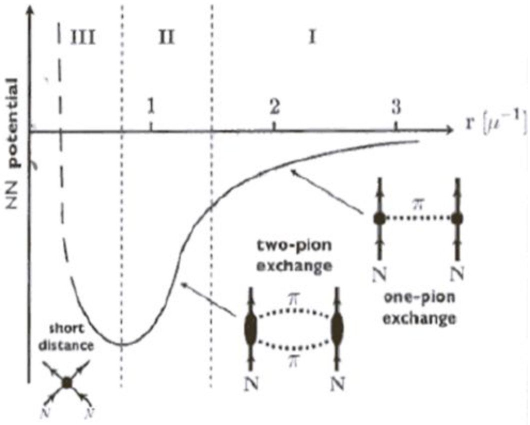

Figure : Hierarchy of scales governing the nucleon-nucleon interaction (adapted from Taketani [5]). The distance r is given in units of the pion Compton wavelength. $\mu^{-1} \approx 1.4$ fm.

$$L = L_N + L_M + L_{int},$$

$$L_N = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi,$$

$$L_\sigma = \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2\right),$$

$$L_\omega = -\frac{1}{2}\left(\Omega_{\mu\nu}\Omega^{\mu\nu} - \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu\right),$$

$$L_\rho = -\frac{1}{2}\left(\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} - \frac{1}{2} m_\rho^2 \vec{\rho}_\mu \vec{\rho}^\mu\right).$$

$$L_A = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu}.$$

$$\Omega^{\mu\nu} = \partial^\mu \omega^\nu - \partial^\nu \omega^\mu,$$

$$\vec{R}^{\mu\nu} = \partial^\mu \vec{\rho}^\nu - \partial^\nu \vec{\rho}^\mu,$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

$$L_{int} = -g_\sigma \bar{\psi} \sigma \psi - g_\omega \bar{\psi} \gamma_\mu \omega^\mu \psi - g_\rho \bar{\psi} \gamma_\mu \vec{\tau} \vec{\rho}^\mu \psi - e \bar{\psi} \gamma_\mu A^\mu \psi.$$

The pion does not contribute to the mean field on the Hartree level. However, in the case of pairing or in time dependent calculations it contributes.
It was recognized that essential nuclear properties such as compressibility or surface properties can not be reproduced quantitatively by the ansatz above.
In particular, linear parameter sets are not able to reproduce the nuclear deformations. For this reason the model was extended to include a nonlinear self-coupling amongst the σ-mesons

$$g_3 \sigma^3 + g_4 \sigma^4$$

A classical spinor field $\psi_i (i = 1...A)$ is introduced for each nucleon and the classical equations of motion are derived.
Dirac equation for the spinor fields:

$$\left\{\gamma_\mu(i\partial^\mu + V^\mu) + m + S\right\}\psi_i = 0$$

with the relativistic fields:

$$S(x) = g_\sigma \sigma(x),$$
$$V^\mu(x) = g_\omega \omega^\mu(x) + g_\rho \vec{\tau}\vec{\rho}^\mu(x) + eA^\mu(x).$$

For the meson fields:

$$(\Box + m_\sigma)\sigma = -g_\sigma \rho_s,$$
$$(\Box + m_\omega)\omega^\mu = g_\omega j^\mu,$$
$$(\Box + m_\rho)\vec{\rho}^\mu = g_\rho \vec{j}^\mu,$$
$$\Box A^\mu = e j_c^\mu.$$

The scalar density and currents are:

$$\rho_s(x) = \sum_{i=1}^{A} \bar{\psi}_i(x)\psi_i(x),$$
$$j^\mu = \sum_{i=1}^{A} \bar{\psi}_i(x)\gamma^\mu\psi_i(x),$$

$$\vec{j}^{\mu} = \sum_{i=1}^{A} \bar{\psi}_i(x)\gamma^{\mu}\vec{\tau}\psi_i(x),$$

$$j_c^{\mu}(x) = \sum_{i=1}^{A} \bar{\psi}_i(x)\frac{1}{2}(1+\tau_3)\gamma^{\mu}\psi_i(x).$$

The four-current $j^{\mu} = (\rho_v, \vec{j})$ contains the usual 3-dimensional current of the nucleons $\vec{j}$) and the normal density of the nucleons ρ_v. The index v indicates that it is the time-like component of a Lorentz vector. ρ_v is different from the scalar density ρ_s.

In the static approximation we assume time-independence for the meson fields and a time-dependent phase $\exp(i\varepsilon_i t)$ for the spinors ψ_i. With time reversal invariants and good parity space like component of all currents $\vec{j}, j$ and j_c vanish and we come to the stationary RMF equations:

$$(-i\vec{\alpha}\cdot\nabla + \beta(m+S) + V)\,\psi_i = \varepsilon_i\psi_i$$

$$\begin{aligned}(-\Delta + m_\sigma)\sigma &= -g_\sigma\rho_\sigma \\ (-\Delta + m_\omega)\omega^0 &= g_\omega\rho_v \\ (-\Delta + m_\rho)\rho_3^0 &= g_3\rho_3 \\ -\Delta\rho_c^0 &= e\rho_c\end{aligned}$$

where

$$
\begin{aligned}
\rho_s &= \sum_i^A \bar{\psi}_i \psi_i, \\
\rho_v &= \sum_i^A \psi_i^+ \psi_i, \\
\rho_3 &= \sum_i^A \psi_i^+ \tau_3 \psi_i, \\
\rho_c &= \sum_i^A \psi_i^+ \frac{1}{2}(1+\tau_3)\psi_i, \\
V(\vec{r}) &= g_\omega \omega^0(\vec{r}) + g_\rho \tau_3 \rho_3^0(\vec{r}) + eA^0(\vec{r}), \\
S(\vec{r}) &= g_\sigma \sigma(\vec{r}), \quad m^* = m + S(\vec{r}).
\end{aligned}
$$

Resulting total energy

$$
\begin{aligned}
E &= \int d^3r \mathcal{H}(\vec{r}) = \sum_i^A \int d^3r \psi_i^+(-i\vec{\alpha}\cdot\nabla + \beta m)\psi_i \\
&+ \frac{1}{2}\int d^3r \left\{(\nabla\sigma)^2 + m_\sigma^2\sigma^2\right\} \\
&- \frac{1}{2}\int d^3r \left\{(\nabla\omega^0)^2 + m_\omega^2(\omega^0)^2\right\} \\
&- \frac{1}{2}\int d^3r \left\{(\nabla\rho^0)^2 + m_\rho^2(\rho_3^0)^2\right\} - \frac{1}{2}\int d^3r(\nabla A^0)^2 \\
&+ \int d^3r \left\{g_\sigma\rho_s\sigma + g_\omega\rho_v\omega^0 + g_\rho\rho_3\rho_3^0 + e\rho_c A^0\right\}
\end{aligned}
$$

For the selfconsistent solution we obtain

$$E = \sum_{i}^{A} \varepsilon_i - \frac{1}{2} \int d^3r \left\{ g_\sigma \rho_s \sigma + g_\omega \rho_v \omega^0 + g_\rho \rho_3 \rho_3^0 + e \rho_c A^0 \right\}.$$

The masses of the σ and ω mesons are quite large and for a qualitative discussion we can neglect the Laplace operator Δ in the field equations. Then the fields σ and ω are proportional to the scalar and vector densities. In finite nuclei these fields take more or less the Saxon-Woods shape.

$$\psi = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\begin{pmatrix} m + S + V & -i\vec{\sigma} \cdot \nabla \\ -i\vec{\sigma} \cdot \nabla & -m - S + V \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = (m + \varepsilon) \begin{pmatrix} f \\ g \end{pmatrix}$$

$$g = \frac{-i}{2m + \varepsilon + S - V} \vec{\sigma} \cdot \nabla f,$$

$$\left\{ -\vec{\sigma} \cdot \nabla \frac{1}{2m + \varepsilon + S - V} \vec{\sigma} \cdot \nabla + V + S \right\} f = \varepsilon f$$

Using the quantity $\varepsilon/(2m+S-V)$ as a small one and do an expansion we obtain

$$\left\{-\nabla\frac{1}{2m_{eff}}\nabla+V_{pot}+\frac{1}{2m^2}\nabla V_{ls}\cdot(\vec{p}\times\vec{s})\right\}f=\varepsilon f,$$

$$m_{eff}=m-\frac{1}{2}(V-S),$$

$$V_{pot}=V+S,\quad V\approx 350MeV,\quad S\approx -400MeV,$$

$$V_{ls}=\frac{m}{m_{eff}}(V-S).$$

In the case of spherical symmetry we obtain for the spin-orbit interaction

$$\frac{1}{2m^2}\left(\frac{1}{r}\frac{\partial V_{ls}(r)}{\partial r}\right)\vec{l}\cdot\vec{s}.$$

In the RMF theory collapse is prevented by a specific relativistic effect

$$\rho_s(\vec{r}) = \sum_{i=1}^{A} \left(|f_i(\vec{r})|^2 - |g_i(\vec{r})|^2 \right),$$

$$\rho_v(\vec{r}) = \sum_{i=1}^{A} \left(|f_i(\vec{r})|^2 + |g_i(\vec{r})|^2 \right)$$

Here ρ_v is the normal baryon density and is normalized to the particle number. ρ_s decreases whenever the small components become important, for instance, in the case of a possible collapse. This mechanism automatically reduces attraction and stabilizes the nucleus.

We can get

$$\rho_s = \rho_v - 2\sum_{i=1}^{A} g_i^2 \approx \rho_v - \frac{1}{m_{eff}}\sum_{i=1}^{A} |\nabla f_i|^2 \approx \rho_v - 2\tau_{kin}.$$

For large densities the kinetic energy increases.

Density dependent Relativistic Mean Field Theory

Traditional RMF models have been following closely to description of the interaction of nucleons and mesons in a pure phenomenological way. A more elaborate but more fundamental approach is based on derivation in-medium interaction microscopically. A fully covariant consistent field theory requires to treat interaction vertices as a functional of the field operators.

$$
\begin{aligned}
L_{int} = \bar{\Psi}\hat{\Gamma}_c(\hat{\rho})\Psi\phi_\sigma - \bar{\Psi}\hat{\Gamma}_\omega(\hat{\rho})\gamma_\mu\Psi A^{(\omega)\mu}, \\
+\bar{\Psi}\hat{\Gamma}_\delta(\hat{\rho})\vec{\tau}\Psi\phi_\delta - \bar{\Psi}\hat{\Gamma}_\delta(\hat{\rho})\gamma_\mu\vec{\tau}\Psi A^{(\rho)\mu}, \\
-c\bar{\Psi}\hat{Q}\gamma_\mu\Psi A^{(\gamma)\mu}.
\end{aligned}
$$

In the mean field approximation for mesons, with only nucleons treated as a quantum field we obtain

$$
\begin{aligned}
\left\{\gamma_\mu \left(i\partial^\mu - \Sigma_b^\mu(\rho)\right) - \left(M - \Sigma_b^s(\rho)\right)\right\}\psi_b = 0, \\
\Sigma_b^s(\rho) = \Gamma_\sigma(\rho)\phi_\sigma + \tau_b \Gamma_\delta(\rho)\phi_\delta, \\
\Sigma_b^0(\rho) = \Sigma_b^{0(0)}(\rho) + \Sigma_b^{0(r)}(\rho), \\
\Sigma_b^{0(0)}(\rho) = \Gamma_\omega(\rho) A_0^{(\omega)} + \tau_b \Gamma_\rho(\rho) A_0^{(\rho)} + e\frac{1-\tau_b}{2} A_0^{(\gamma)}, \\
\Sigma_b^{0(r)}(\rho) = \frac{\partial \Gamma_\omega}{\partial \rho} A_0^{(\omega)} \rho + \frac{\partial \Gamma_\rho}{\partial \rho} A_0^{(\rho)} \rho_3, \\
-\frac{\partial \Gamma_\sigma}{\partial \rho}\phi_\sigma \rho^s - \frac{\partial \Gamma_\delta}{\partial \rho}\phi_\delta \rho_3^s.
\end{aligned}
$$

The effective baryon-meson field theory is introduced to describe finite nuclei.

The model includes the spin properties in a natural way and distinguishes in a nucleus a large attractive scalar field and a large repulsive vector field.