Quark-hadron matter models for compact stars

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1. **Functional Integral approach to the partition function**
   Walecka model – String-flip model - Phase transition
2. **Special problems of dense quark-hadron matter**
   Quark Pauli-blocking – Mott dissociation – Color superconductivity
3. **Application: hybrid stars w. quark core (→ D. Alvarez-Castillo)**
   M-R relationships for hybrid stars – High-mass “twin” stars

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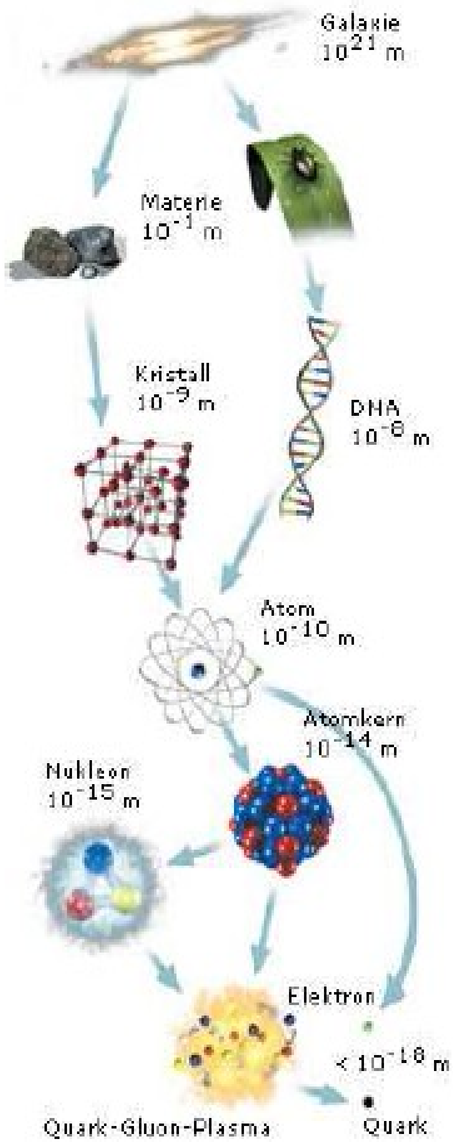
Many Particle Systems & Quantum Field Theory



Elements	Bound states	System
humans, animals	couples, groups, parties	society
molecules, crystals	(bio)polymers	animals, plants
atoms	molecules, clusters, crystals	solids, liquids, ...
ions, electrons	atoms	plasmas
nucleons, mesons	nuclei	nuclear matter
quarks, anti-quarks	nucleons, mesons	quark matter

Highly Compressed Matter ⇔ Pauli Principle

Partition function: $Z = \mathrm{Tr}\left\{ e^{-\beta(H-\mu_i Q_i)} \right\}$

Partition function for Quantum Chromodynamics (QCD)

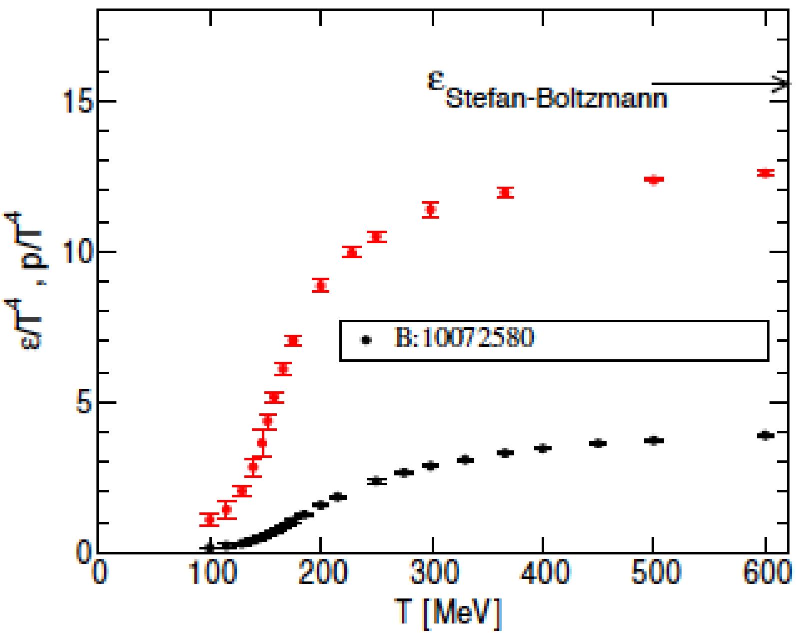
- Partition function as a Path Integral (imaginary time $\tau = i\,t$, $0 \le \tau \le \beta = 1/T$)

$$Z[T,V,\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \exp\left\{-\int_0^\beta d\tau \int_V d^3x\, \mathcal{L}_{QCD}(\psi,\bar{\psi},A)\right\}$$

- QCD Lagrangian, non-Abelian gluon field strength: $F^a_{\mu\nu}(A) = \partial_\mu A^a\nu - \partial_\nu A^a_\mu + g\, f^{abc}[A^b_\mu, A^c_\nu]$

$$\mathcal{L}_{QCD}(\psi,\bar{\psi},A) = \bar{\psi}[i\gamma^\mu(\partial_\mu - igA_\mu) - m - \gamma^0\mu]\psi - \frac{1}{4}F^a_{\mu\nu}(A)F^{a,\mu\nu}(A)$$

- Numerical evaluation: Lattice gauge theory simulations (hotQCD, Wuppertal-Budapest)



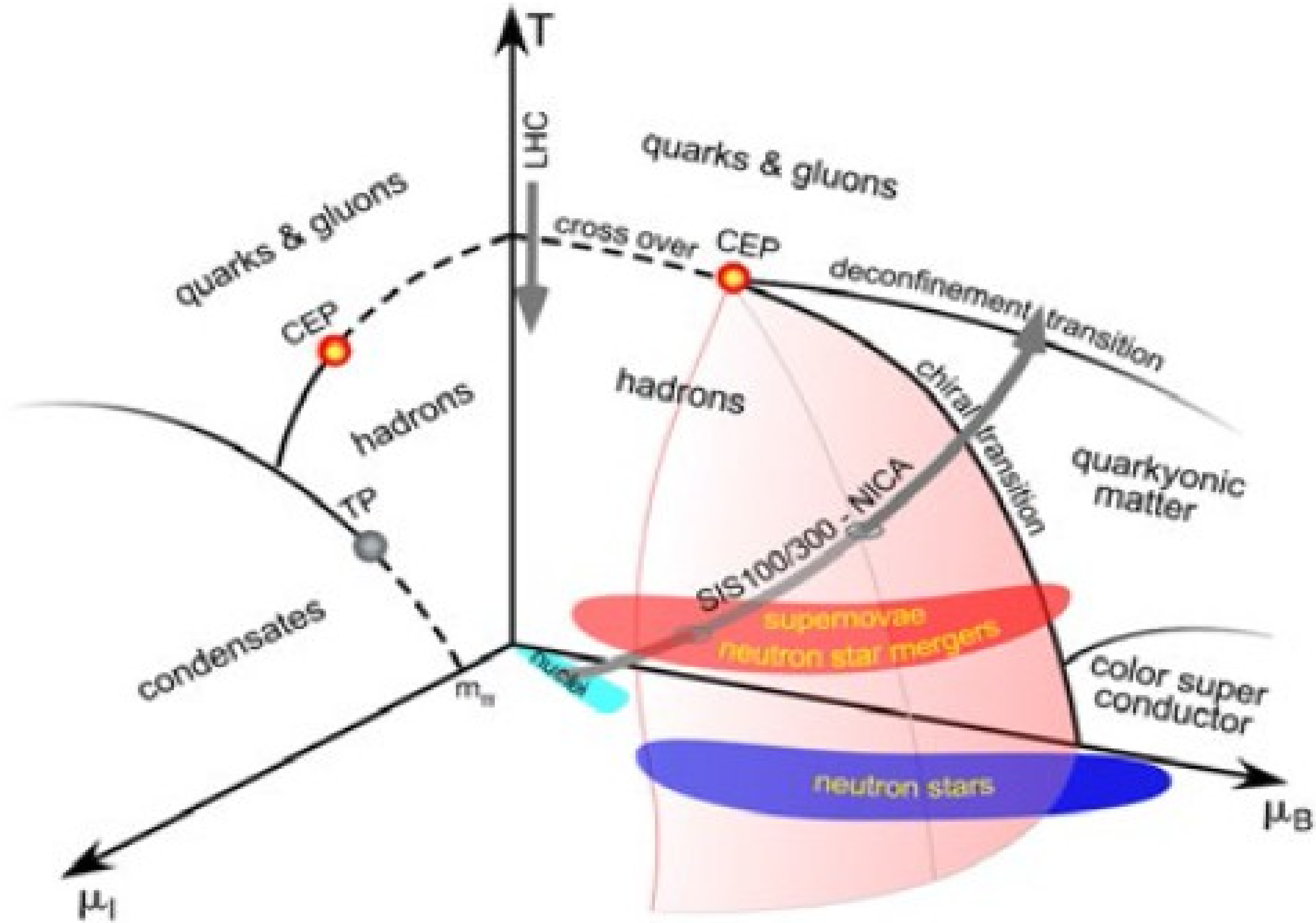
- Equation of state: $\varepsilon(T) = -\partial \ln Z[T,V,\mu]/\partial\beta$ (1/V)
- Phase transition at $T_c = 155$ MeV
- Problem: Interpretation ?

$\varepsilon/T^4 = \frac{\pi^2}{30}N_\pi \sim 1$ (ideal pion gas)

$\varepsilon/T^4 = \frac{\pi^2}{30}(N_G + \frac{7}{8}N_Q) \sim 15.6$ (quarks and gluons)

- Hadron resonance gas

CEP in the QCD phase diagram: HIC vs. Astrophysics



NuPECC Long Range Plan 2017; http://www.nupecc.org

A. Andronic, D. Blaschke, et al., “Hadron production ...”, Nucl. Phys. A 837 (2010) 65 - 86

Ensembles and Partition Function

- *microcanonical ensemble:* isolated system, fixed energy E, particle number N, volume V
- *canonical ensemble:* system with a heat reservoir at temperature T; fixed: T, N, and V
- *grand canonical ensemble:* system can exchange particles and energy with a reservoir. fixed variables: T, V, and the chemical potential μ

System: Hamiltonian $\hat{H}$ and conserved number operators $\hat{N}_i$ (hermitean, commute with $\hat{H}$)
In relativistic QED: $N_e = N_{e-} - N_{e+}$ is conserved, not N_{e-} or N_{e+} separately.

- statistical density matrix is: $\hat{\rho} = \exp\left[-\beta(\hat{H} - \mu_i \hat{N}_i)\right]$,
- ensemble average of an operator $\hat{A}$ is given by: $A = \mathrm{Tr}\hat{\rho}\hat{A}/\mathrm{Tr}\hat{\rho}$.
- grand canonical partition function is: $Z = \mathrm{Tr}\hat{\rho}$.

The partition function $Z = Z(T, V, \mu_1, \mu_2, \ldots)$ is the single most important function !
From it all other standard thermodynamic properties may be determined:
$P = T\partial \ln Z/\partial V$, (pressure) $N_i = T\partial \ln Z/\partial \mu_i$, (particle numbers)
$S = \partial(T \ln Z)/\partial T$, (entropy) $E = -PV + TS + \mu_i N_i$, (energy)

Note: Extension to the nonequilibrium situation possible; generalized Gibbs ensemble
Nonequilibrium characterized by further observables, e.g., currents or reaction variables.
relevant statistical operator $\Longrightarrow$ Zubarev formalism.

PATH INTEGRAL APPROACH TO PARTITION FUNCTION

Partition function quantum statistics: sum over all (eigen-)states.

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta(\hat{H}-\mu_i \hat{N}_i)} = \int d\phi_a \langle \phi_a | \mathrm{e}^{-\beta(\hat{H}-\mu_i \hat{N}_i)} | \phi_a \rangle \ ,$$

Similar to the transition amplitude (time evolution operator) in Quantum Field Theory,
Introduce *imaginary time* variable $\tau = \mathrm{i}\, t$ with integration limited to $0 < \tau < \beta$.
For system with conserved charges $\mathcal{N}(\pi, \phi)$, make the replacement

$$\mathcal{H}(\pi, \phi) \to \mathcal{K}(\pi, \phi) = \mathcal{H}(\pi, \phi) - \mu \mathcal{N}(\pi, \phi) \ ,$$

Representation of the partition function Z as a functional integral

$$Z = \int \mathcal{D}\pi \int_{\mathrm{periodic}} \mathcal{D}\phi \exp\left\{ \int_0^\beta \int d^3x \left(i\, \pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi, \phi) + \mu \mathcal{N}(\pi, \phi) \right) \right\} \ .$$

"Periodic": field integration constrained, so that $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$

Key lesson:
Quantization: Path integral over all admissible (constraints!) *classical field* configurations;
Statistical operator quantization on the other hand requires introduction of *field operators*.

Partition Function as a Path Integral - Equivalence

Be $\hat{\phi}(\vec{x}, 0)$ a field operator in the Schrödinger picture at time $t = 0$ and $\hat{\pi}(\vec{x}, 0)$ the corresponding canonically conjugated field momentum operator. For eigenstates $\mid \phi\rangle$ of the field holds the eigenvalue equation

$$\hat{\phi}(\vec{x}, 0) \mid \phi\rangle = \phi(\vec{x}) \mid \phi\rangle \ ,$$

where $\phi(\vec{x})$ is the "eigenvalue" corresponding to the field operator. For the eigenstates of the fields completeness and orthonormality shall hold

$$\int d\phi(\vec{x}) \mid \phi\rangle\langle\phi \mid \ = \ 1 \ ; \quad \langle\phi_a \mid \phi_b\rangle = \delta\left[\phi_a(\vec{x}) - \phi_b(\vec{x})\right] \ .$$

For the field momentum operator and its eigenstates $\mid \pi\rangle$ holds analogously

$$\hat{\pi}(\vec{x}, 0) \mid \pi\rangle \ = \ \pi(\vec{x}) \mid \pi\rangle$$

$$\int \frac{d\pi(\vec{x})}{2\pi} \mid \pi\rangle\langle\pi \mid \ = \ 1 \ ; \quad \langle\pi_a \mid \pi_b\rangle = \delta\left[\pi_a(\vec{x}) - \pi_b(\vec{x})\right] \ .$$

The transition amplitude between coordinates and momenta eigenstates $\langle x \mid p\rangle = \exp(ipx)$ is generalized to the quantum field theory case by

$$\langle\phi \mid \pi\rangle = \exp\left[i \int d^3x \, \pi(\vec{x})\phi(\vec{x})\right] \ .$$

Partition Function as a Path Integral - Equivalence

For a dynamical description of the system we require the Hamiltonian operator

$$\hat{H} = \int d^3x \mathcal{H}\left(\hat{\pi}, \hat{\phi}\right) .$$

Consider the state $| \phi_a\rangle$ at $t = 0$, which at a later time t_f has evolved to $e^{-i\hat{H}t_f} | \phi_a\rangle$. For the quantum statistical partition function, the system returns at $t = t_f$ to the initial state at $t = 0$. The time interval $(0, t_f)$ is decomposed into equidistant parts $\triangle t = t_f/N$. At each time step we introduce a complete set of field and field-momentum states

$$\begin{aligned}\langle\phi_a | e^{-iHt_f} | \phi_a\rangle &= \lim_{N\to\infty}\int\left(\prod_{i=1}^{N}\frac{d\pi_i d\phi_i}{2\pi}\right)\langle\phi_a | \pi_N\rangle\langle\pi_N | e^{-iH\Delta t} | \phi_N\rangle\langle\phi_N | \pi_{N-1}\rangle \\ &\quad\times\langle\pi_{N-1} | e^{-iH\Delta t} | \phi_{N-1}\rangle \times \ldots \times \langle\phi_2 | \pi_1\rangle\langle\pi_1 | e^{-iH\Delta t} | \phi_1\rangle\langle\phi_1 | \phi_a\rangle\end{aligned}$$

We make use of the following expressions

$$\langle\phi_1 | \phi_a\rangle = \delta\left(\phi_1 - \phi_a\right) \; ; \; \langle\phi_{i+1} | \pi_i\rangle = \exp\left[i\int d^3x\, \pi_i(\vec{x})\phi_{i+1}(\vec{x})\right] .$$

For $\triangle t \to 0$ the exponential function can be expanded with $H_i = \int d^3x \mathcal{H}\left(\pi_i(\vec{x}), \phi_i(\vec{x})\right)$

$$\langle\pi_i | e^{-i\hat{H}\Delta t} | \phi_i\rangle \simeq \langle\pi_i | (1 - \hat{H}\Delta t) | \phi_i\rangle = \langle\pi_i | \phi_i\rangle(1 - H_i\Delta t) = (1 - H_i\Delta t)\exp\left[i\int d^3x\, \pi_i(\vec{x})\phi_i(\vec{x})\right]$$

PARTITION FUNCTION AS A PATH INTEGRAL - EQUIVALENCE

Taken all expressions together yields

$$\langle \phi_a \mid e^{-i\hat{H}t_f} \mid \phi_a \rangle = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \frac{d\pi_i d\phi_i}{2\pi} \right) \delta\left(\phi_1 - \phi_a\right) \exp\{-i\Delta t \sum_{j=1}^{N} \int d^3x [\mathcal{H}(\pi_j, \phi_j) - \pi_j \frac{\phi_{j+1} - \phi_j}{\Delta t}]\}$$

Here holds $\phi_{N+1} = \phi_a = \phi_1$. In the continuum limit we obtain

$$\langle \phi_a \mid e^{-i\hat{H}t_f} \mid \phi_a \rangle = \int \mathcal{D}\pi \int_{\phi(\vec{x},0)=\phi_a}^{\phi(\vec{x},t_f)=\pm\phi_a} \mathcal{D}\phi \exp \left[i \int_0^{t_f} dt \int d^3\vec{x} \underbrace{\left(\pi \frac{\partial \phi}{\partial t} - \mathcal{H}(\phi, \pi) \right)}_{\mathcal{L}(\phi,\pi)} \right]$$

$\mathcal{D}\pi$ and $\mathcal{D}\phi$ stand for the Functional Integration over fields and their conjugate momenta. The result for the partition function reads

$$Z = \int [d\pi] \int_{\pm} [d\phi] \exp \left(\int_0^{\beta} d\tau \int d^3x \left(i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi, \phi) + \mu_i \mathcal{N}_i(\pi, \phi) \right) \right) \quad (1)$$

The index $\pm$ stands for the symmetry (antisymmetry) of the Bose (Fermi) fields at the borders of the imaginary time interval: $\phi(\vec{x}, 0) = \pm\phi(\vec{x}, \beta)$.

Example: Neutral Scalar Field

The most general renormalizable Lagrangian for a neutral scalar field is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - U(\phi) \ , \quad \text{with the potential} \ \ U(\phi) = g\phi^3 + \lambda\phi^4 \ ,$$

and $\lambda \geq 0$ for stability of the vacuum. The momentum conjugate to the field is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \frac{\partial\phi}{\partial t} \ ,$$

and the Hamiltonian is

$$\mathcal{H} = \pi\frac{\partial\phi}{\partial t} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi) \ .$$

There is no conserved charge.
The first step in evaluating the partition function is to return to the discretized version

$$\begin{aligned} Z \ &= \lim_{N\to\infty}\left(\Pi_{i=1}^{N}\int_{-\infty}^{\infty}\frac{d\pi_i}{2\pi}\int_{\text{periodic}} d\phi_i\right) \\ &\exp\left\{\sum_{j=1}^{N}\int d^3x\left[i\pi_j(\phi_{j+1}-\phi_j) - \Delta\tau\left(\frac{1}{2}\pi_j^2 + \frac{1}{2}(\nabla\phi_j)^2 + \frac{1}{2}m^2\phi_j^2 + U(\phi_j)\right)\right]\right\} \ . \end{aligned}$$

The momentum integrations can be performed since they are just products of Gaussian integrals.

Example: Neutral Scalar Field

Divide position space into M^3 little cubes with $V = L^3$, $L = aM$, $a \to 0$, $M \to \infty$, M an integer. For convenience, and to ensure that Z remains explicitely dimensionless at each step in the calculation, we write $\pi_j = A_j/(a^3\,\Delta\tau)^{1/2}$ and integrate A_j from $-\infty$ to $+\infty$. For each cube we obtain

$$\int_{-\infty}^{\infty} \frac{dA_j}{2\pi} \quad \exp\left[-\frac{1}{2}A_j^2 + i\left(\frac{a^3}{\Delta\tau}\right)^{1/2}(\phi_{j+1}-\phi_j)A_j\right] = (2\pi)^{-1/2}\exp\left[\frac{-a^3(\phi_{j+1}-\phi_j)^2}{2\Delta\tau}\right] .$$

Thus far we have

$$Z \;=\; \lim_{M,N\to\infty}\int\left[\Pi_{i=1}^{N}\frac{d\phi_i}{\sqrt{2\pi M^3}}\right]\exp\left\{\Delta\tau\sum_{j=1}^{N}\int d^3x\left[-\frac{1}{2}\left(\frac{(\phi_{j+1}-\phi_j)}{\Delta\tau}\right)^2 - \frac{1}{2}(\nabla\phi_j)^2 - \frac{1}{2}m^2\phi_j^2 - U(\phi_j)\right]\right\}$$

Returning to the continuum limit, we obtain

$$Z = N'\int_{\text{periodic}} \mathcal{D}\phi \exp\left(\int_0^\beta d\tau \int d^3x\mathcal{L}\right) .$$

The Lagrangian is expressed as a functional of ϕ and its first derivatives.
Z is expressed as a functional integral over ϕ of the exponential of the action in imaginary time.
The normalization constant is irrelevant, since multiplication of Z by any constant does not change the thermodynamics.

EXAMPLE: NEUTRAL SCALAR FIELD

Consider the case of noninteracting fields $U(\phi) = 0$. Interactions we discuss later. Define

$$S = \int_0^\beta d\tau \int d^3x \mathcal{L} = -\frac{1}{2}\int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2\right] .$$

Integrating by parts and taking note of the periodicity of ϕ, we obtain

$$S = -\frac{1}{2}\int_0^\beta d\tau \int d^3x \phi \left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right)\phi .$$

The field can be decomposed into a Fourier series according to

$$\phi(\vec{x},\tau) = \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty}\sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\phi_n(\vec{p}) ,$$

where $\omega_n = 2\pi n T$, due to the constraint of periodicity that $\phi(\vec{x},\beta) = \phi(\vec{x},0)$ for all $\vec{x}$.

$$S = -\frac{1}{2}\beta^2 \sum_n \sum_{\vec{p}} (\omega_n^2 + \omega^2)\phi_n(\vec{p})\phi_n^*(\vec{p}) \ , \ \ \omega = \sqrt{\vec{p}^2 + m^2}.$$

The phases can be integrated out to get

$$Z \ = \ N'\Pi_n\Pi_{\vec{p}}\left[\int_{-\infty}^{\infty} dA_n(\vec{p}) \exp\left[-\frac{1}{2}\beta^2(\omega_n^2+\omega^2)A_n^2(\vec{p})\right]\right] = N'\Pi_n\Pi_{\vec{p}}\left[2\pi/(\beta^2(\omega_n^2+\omega^2))\right]^{1/2} .$$

Example: Neutral Scalar Field

Ignoring an overall multiplicative factor independent of β and V,

$$Z = \Pi_n \Pi_{\vec{p}} \left[\beta^2(\omega_n^2 + \omega^2)\right]^{-1/2} .$$

More formally one can arrive at this result by using the general rules for Gaussian functional integrals over commuting (bosonic) variables,

$$Z = N' \int \mathcal{D}\phi \exp\left[-\frac{1}{2}(\phi, D\phi)\right] = N' \text{constant}(\det \ D)^{-1/2} ,$$

where $D = \beta^2(\omega_n^2 + \omega^2)$ in $(\vec{p}, \omega_n)$ space and $(\phi, D\phi)$ the inner product on function space.

$$\ln Z = -\frac{1}{2} \sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right] .$$

$$\text{Trick :} \quad \ln\left[(2\pi n)^2 + \beta^2\omega^2)\right] \ = \int_1^{\beta^2\omega^2} \frac{d\Theta^2}{\Theta^2 + (2\pi n)^2} + \ln\left[1 + (2\pi n)^2\right] ,$$

The $\beta-$ independent term can be ignored. Furthermore,

$$\sum_{-\infty}^{\infty} \frac{1}{n^2 + (\Theta/2\pi)^2} = \frac{2\pi^2}{\Theta}\left(1 + \frac{2}{e^{\Theta} - 1}\right) ,$$

Example: Neutral Scalar Field

Hence we arrive at

$$\ln Z = -\sum_{\vec{p}} \int_1^{\beta\omega} d\Theta \left(\frac{1}{2} + \frac{1}{e^{\Theta} - 1} \right) .$$

Carrying out the Θ integral, and throwing away a $\beta-$ independent piece, we finally arrive at

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[-\frac{1}{2}\beta\omega - \ln(1 - e^{-\beta\omega}) \right] ,$$

from which we obtain immediately the well-known expression for the ideal Bose gas ($\mu = 0$), once we subtract the divergent expressions for the zero-point energy

$$E_0 = -\frac{\partial}{\partial\beta} \ln Z_0 = V \int \frac{d^3p}{(2\pi)^3} \frac{\omega}{2} ,$$

and for the zero-point pressure

$$P_0 = T \frac{\partial}{\partial V} \ln Z_0 = -\frac{E_0}{V} ,$$

which are typical for the quantum field-theoretical treatment. With this subtraction the vacuum is defined as the state with zero energy and pressure.

Partition Function for Fermionic Fields

Dirac fermions are descibed by a four-spinor field ψ with a Lagrangian density

$$\mathcal{L} \;=\; \bar{\psi}(i\partial\!\!\!/ - m)\psi = \psi^\dagger\gamma^0\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi\;.$$

The momentum conjugate to this field is

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} = i\psi^\dagger\;,$$

because $\gamma^0\gamma^0 = 1$. Thus, ψ and $\psi^\dagger$ must be treated as independent entities in the Hamiltonian formulation.

$$\mathcal{H} = \Pi\frac{\partial\psi}{\partial t} - \mathcal{L} = \psi^\dagger\left(i\frac{\partial}{\partial t}\right)\psi - \mathcal{L} = \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi\;,$$

and the partition function is

$$Z = \mathrm{Tr}\;\mathrm{e}^{-\beta(\hat{H}-\mu\hat{Q})}\;,$$

with the conserved charge $Q = \int d^3x\psi^\dagger\psi$. The path integral representation is

$$Z = \int\mathcal{D}\psi^\dagger\mathcal{D}\psi\exp\left[\int_0^\beta d\tau\int d^3x\psi^\dagger\left(-\gamma^0\frac{\partial}{\partial\tau} + i\vec{\gamma}\cdot\vec{\nabla} - m + \mu\gamma^0\right)\psi\right]$$

Partition Function for Fermionic Fields

As with bosons, it is most convenient to work in $(\vec{p}, \omega_n)$ space instead of $(\vec{x}, \tau)$ space, i.e.,

$$\psi_\alpha(\vec{x}, \tau) = \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \mathrm{e}^{i(\vec{p}\vec{x}+\omega_n\tau)} \tilde{\psi}_{\alpha;n}(\vec{p}) ,$$

where now $\omega_n = (2n+1)\pi T$ due to the antiperiodicity of the (Grassmannian) Fermion field at the borders of the fundamental strip $0 \le \tau \le \beta$ in the imaginary time, $\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)$. Now we are ready to evaluate the fermionic partition function (2),

$$\begin{aligned} Z &= \left[\Pi_n \Pi_{\vec{p}} \Pi_\alpha \int i d\tilde{\psi}^\dagger_{\alpha;n}(\vec{p}) d\tilde{\psi}_{\alpha;n}(\vec{p})\right] \mathrm{e}^S , \quad S = \sum_n \sum_{\vec{p}} i\psi^\dagger_{\alpha;n}(\vec{p}) D_{\alpha\rho} \psi_{\rho;n}(\vec{p}) , \\ D &= -i\beta \left[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma} \cdot \vec{p} - m\gamma^0\right] , \end{aligned} \tag{2}$$

using Grassmannian integration of Gaussian functional integrals, we obtain

$$Z = \det\ D .$$

Employing the identity $\ln \det D = \mathrm{Tr} \ln D$, and evaluating the determinant in Dirac space explicitly (Exercise !), one finds

$$\ln Z = 2 \sum_n \sum_{\vec{p}} \ln \left\{\beta^2 \left[(\omega_n + i\mu)^2 + \omega^2\right]\right\} .$$

Exercise: Calculation of Dirac determinant $\det(\gamma_\mu p_\mu - m^*)$, p_0 = $i(\omega_n + i\mu)$

Exercise 2: Show that $2\sum_{n=-\infty}^{+\infty} \ln \beta^2[\omega^2 + (\omega_n + i\mu)^2] = \sum_{n=-\infty}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega - \mu)^2] + \ln \beta^2[\omega_n^2 + (\omega + \mu)^2]\right\}$

Partition Function for Fermionic Fields

Since both positive and negative frequencies have to be summed over, the latter expression can be put in a form analogous to the above expression in the bosonic case,

$$\ln Z = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2\left(\omega_n^2 + (\omega-\mu)^2\right)\right] + \ln\left[\beta^2\left(\omega_n^2 + (\omega+\mu)^2\right)\right]\right\} \ .$$

In the further evaluation we can go similar steps as in the bosonic case, with two exceptions: (1) the presence of a chemical potential, splitting the contributions of particles and antiparticles; (2) the Matsubara frequencies are now odd multiples of πT, so that the infinite sum to be exploited reads

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2\pi^2 + \Theta^2} = \frac{1}{\Theta}\left(\frac{1}{2} - \frac{1}{e^{\Theta}+1}\right) \ .$$

Integrating over the auxiliary variable Θ, and dropping terms independent of β and μ, we finally obtain

$$\ln Z = 2V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega + \ln(1 + e^{-\beta(\omega-\mu)}) + \ln(1 + e^{-\beta(\omega+\mu)})\right] \ .$$

Notice that the factor 2 corresponding to the spin-$\frac{1}{2}$ nature of the fermions comes out automatically. Separate contributions from particles (μ) and antiparticles (-μ) are evident. Finally, the zero-point energy of the vacuum also appears in this formula.

Interactions: Hubbard-Stratonovich Trick

A general class of interactions for which the Hubbard-Stratonovich (HS) transformation is immediately applicable, are four-fermion couplings of the current-current type

$$\mathcal{L}_{int} = G(\bar{\psi}\psi)^2 \; . \tag{3}$$

A Fermi gas with this typ of interaction serves as a model for electronic superconductivity (Bardeen-Cooper-Schrieffer (BCS) model, 1957) or for chiral symmetry breaking in quark matter (Nambu–Jona-Lasinio (NJL) model, 1961).

The HS-transformation for (3) reads

$$\exp\left[G(\bar{\psi}\psi)^2\right] = \mathcal{N}\int \mathcal{D}\sigma \exp\left[\frac{\sigma^2}{4G} + \bar{\psi}\psi\sigma\right]$$

and allows to bring the functional integral over fermionic fields into a quadratic (Gaussian) form so that fermions can be integrated out.

This is also called *Bosonization* procedure.

Walecka (σ - ω) model of asymmetric nuclear matter (Functional integral approach)

Walecka model for dense nuclear matter (I)

Meson exchange model

example: scalar (σ) meson

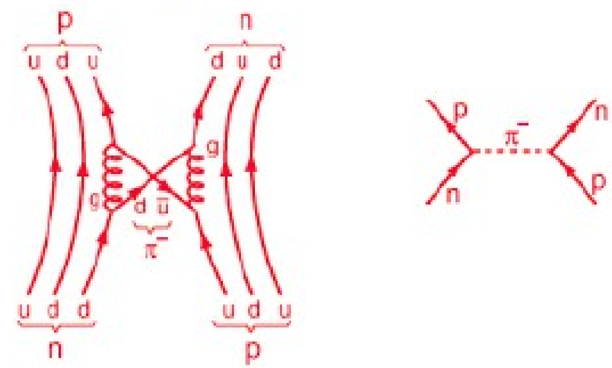
$$(-\triangle + m_\sigma^2)\sigma(\vec{r}) = -g_\sigma \delta(\vec{r})$$

$$\Rightarrow \quad \sigma(r) = -\frac{g_\sigma}{4\pi}\frac{e^{-m_\sigma r}}{r}$$

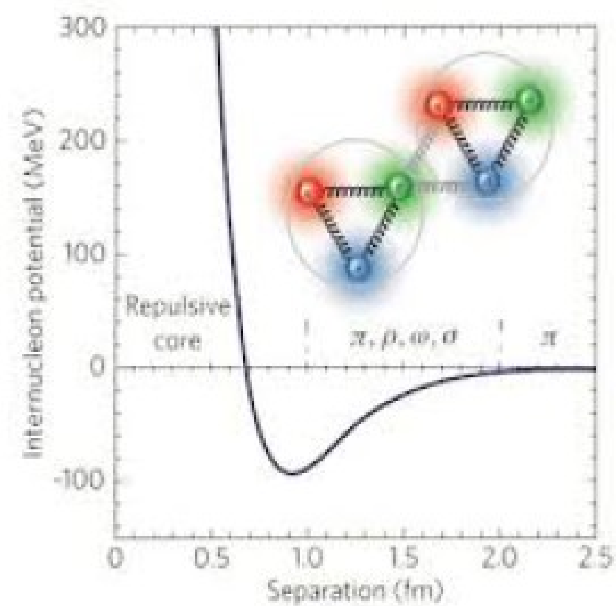
$$V_{NN}^{(\sigma)}(r) = g_\sigma \sigma(r) = -\frac{g_\sigma^2}{4\pi}\frac{e^{-m_\sigma r}}{r}$$

Meson	I^π	T	S	M[MeV]
$\pi^0, \pi^\pm$	0^-	1	0	140
σ	0^+	0	0	≈ 500
$K^0, K^\pm$	0^-	1/2	± 1	495
η	0^-	0	0	550
$\rho^0, \rho^\pm$	1^-	1	0	770
ω	1^-	0	0	780
δ	0^+	1	0	900

Feynman diagrams for π^- exchange



NN- Potential



Walecka model for dense nuclear matter (II)

Field theoretical formulation: Lagrangian and Path Integral for Partition Function

$$\mathcal{Z}_{gk}(T,V,\{\mu_i\}) = \int [d\overline{\Psi}][d\Psi] \exp\left\{ \int_0^{\beta=1/T} d\tau \int_V d^3\vec{x} \; \left(\mathcal{L}_0 + \mathcal{L}_I + \mu_p \Psi_p^+ \Psi_p + \mu_n \Psi_n^+ \Psi_n\right) \right\}$$

$$\mathcal{L}_0(\tau,\vec{x}) \quad = \quad \overline{\Psi}(\tau,\vec{x}) \left(i\gamma_\mu \partial_\mu - m_N\right) \Psi(\tau,\vec{x}) \; , \qquad \mathcal{L}_I(\tau,\vec{x}) = j_{\omega_\mu}(\tau,\vec{x}) \frac{G_\omega}{2} j_{\omega_\mu}(\tau,\vec{x}) - j_\sigma(\tau,\vec{x}) \frac{G_\sigma}{2} j_\sigma(\tau,\vec{x})$$

$$\begin{aligned} j_\sigma(\tau,\vec{x}) &= \overline{\Psi}(\tau,\vec{x})\Psi(\tau,\vec{x}) \\ j_{\omega_\mu}(\tau,\vec{x}) &= \overline{\Psi}(\tau,\vec{x})\gamma_\mu\Psi(\tau,\vec{x}) \end{aligned} \qquad \Psi = \begin{pmatrix} \psi_n \\ \psi_p \end{pmatrix}; \quad \psi_n = \begin{pmatrix} u_{n,\uparrow} \\ u_{n,\downarrow} \\ v_{n,\uparrow} \\ v_{n,\downarrow} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{Neutron} \\ \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{Antineutron} \end{matrix}$$

$$\begin{aligned} \mu_n = \mu_p \quad &\longrightarrow \quad \text{symmetric nuclear matter} \\ \mu_n \neq 0;\; \mu_p = 0 \quad &\longrightarrow \quad \text{pure neutron matter} \\ \mu_n = \mu_p + \mu_{e^-} \quad &\longrightarrow \quad \text{neutron star matter } (\beta\text{-equilibrium}) \end{aligned}$$

Walecka model for dense nuclear matter (III)

Evaluation of the Path Integral: Hubbard-Stratonovich trick

$$\exp\left(-\left(\overline{\Psi}\Psi\right)\frac{G_\sigma}{2}\left(\overline{\Psi}\Psi\right)\right) = \left(\det G_\sigma^{-1}\right)^{\frac{1}{2}}\int [d\sigma]\exp\left(\frac{\sigma^2}{2G_\sigma}+\sigma\overline{\Psi}\Psi\right)$$

Effective action quadratic $\Longrightarrow$ Gaussian Path Integral

$$\mathcal{S}\equiv\int_0^\beta d\tau\int d^3\vec{x}\left\{\overline{\Psi}(\vec{x},\tau)\left(-\gamma_0\frac{\partial}{\partial\tau}+i\vec{\gamma}\vec{\nabla}-m_N+\gamma_0\mu+\sigma-\gamma_\mu\omega_\mu\right)\Psi(\vec{x},\tau)+\frac{\sigma^2}{2G_\sigma}-\frac{\omega_\mu^2}{2G_{\omega_\mu}}\right\}$$

Fourier representation: $\Psi(\vec{x},\tau)=\sqrt{\frac{T}{V}}\sum_n\sum_{\vec{p}}e^{i(\vec{p}\vec{x}+\omega_n\tau)}\Psi_n(\vec{p})$, with $\omega_n\equiv\pi T(2n+1)$

$$\int_0^\beta d\tau\int d^3\vec{x}\,\overline{\Psi}(\vec{x},\tau)\left(-\gamma_0\frac{\partial}{\partial\tau}+i\vec{\gamma}\vec{\nabla}-m_N+\gamma_0\mu+\sigma-\gamma_0\omega_0\right)\Psi(\vec{x},\tau)$$

$$=\quad\frac{1}{\beta V}\int_0^\beta d\tau\int d^3\vec{x}\sum_{n,n'}\sum_{\vec{p},\vec{p}'}\overline{\Psi}_{n'}(\vec{p}')\left(-i\gamma_0\omega_n-\vec{\gamma}\vec{p}-m_N^*+\gamma_0\mu^*\right)\Psi_n(\vec{p})\,e^{i\{(\vec{p}-\vec{p}')\vec{x}+(\omega_n-\omega_{n'})\tau\}}$$

$$=\quad\beta\sum_n\sum_{\vec{p}}\overline{\Psi}_n(\vec{p})(-\gamma_\mu p_\mu-m_N^*)\Psi_n(\vec{p})=\sum_n\sum_{\vec{p}}\overline{\Psi}_n(\vec{p})G^{-1}[\sigma,\omega_0]\Psi_n(\vec{p})$$

Effective mass $m_N^*=m_N-\sigma$, chemical potential $\mu^*=\mu-\omega_0$ and quasiparticle propagator

$$G^{-1}[\sigma,\omega]\quad=\quad-\beta(\gamma_\mu p_\mu+m_N^*)\quad,\quad p_0=i\omega_n-\mu^*$$

Walecka model for dense nuclear matter (IV)

Evaluate fermionic Path Integral and mean field approximation:

$$
\begin{aligned}
\mathcal{Z}_{gk}(T,V,\{\mu_i\}) &= \mathcal{N}\prod_{n,\vec{p}}\int [d\overline{\Psi}_n(\vec{p})][d\Psi_n(\vec{p})][d\sigma][d\omega_0] e^{\left\{\frac{\sigma^2-\omega_0^2}{2G_{\omega_0}}+\sum_{n,\vec{p}}\overline{\Psi}_n(\vec{p})G^{-1}[\sigma,\omega_0]\Psi_n(\vec{p})\right\}} \\
&= \int [d\sigma][d\omega_0]\exp\left\{Tr\ \ln G^{-1}[\sigma,\omega_0]+\frac{\sigma^2}{2G_\sigma}-\frac{\omega_0^2}{2G_{\omega_0}}\right\} \\
&= \exp\left\{Tr\ \ln G^{-1}[\overline{\sigma},\overline{\omega}_0]+\frac{\overline{\sigma}^2}{2G_\sigma}-\frac{\overline{\omega}_0^2}{2G_{\omega_0}}\right\}
\end{aligned}
$$

Stationarity condition: $\partial \ln Z_{gk}/\partial\overline{\sigma} = \partial \ln Z_{gk}/\partial\overline{\omega}_0 = 0$ corresponds to "gap equations":

$$
\overline{\sigma} = -G_\sigma Tr\, G[\overline{\sigma},\overline{\omega}_0] = G_\sigma n_s\ , \quad \overline{\omega}_0 = -G_\omega Tr\,\gamma_0 G[\overline{\sigma},\overline{\omega}_0] = G_\omega n\ .
$$

Thermodynamics: $\Omega(T,V,\mu) = -T\ln\mathcal{Z}_{gk} = -pV$

$$
p(\mu,T) = \frac{1}{2}G_\omega n^2 - \frac{1}{2}G_\sigma n_s^2 + 4T\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\ln\left(1+e^{-\beta(E^*-\mu^*)}\right)+\ln\left(1+e^{-\beta(E^*+\mu^*)}\right)\right]
$$

$$
n = 4\int\frac{d^3\vec{p}}{(2\pi)^3}\left[f_-(E^*)-f_+(E^*)\right]\ ,\ n_s = 4\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{m_N^*}{E^*}\left[f_-(E^*)-f_+(E^*)\right]\ ,\ f_\pm(E^*) = \frac{1}{e^{\beta(E^*\mp\mu^*)}+1}
$$

Quasiparticle properties $E^* = \sqrt{\vec{p}^2+{m_N^*}^2}$, $m_N^* = m_n - G_\sigma n_s$, $\mu^* = \mu - G_\omega n$.

Walecka model for dense nuclear matter (V)

Evaluate Traces: $Tr \ln G^{-1} = tr_p tr_D \ln G^{-1} = tr_p \ln \det_D G^{-1} = \sum_n \sum_{\vec{p}} \ln \det_D G^{-1}$

Scalar mean field

$$\begin{aligned}
\overline{\sigma} &= -G_{\overline{\sigma}}\, Tr\, G[\overline{\sigma}, \overline{\omega}_0] \\
&= -2G_\sigma T \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} tr_D \left[\gamma_\mu p_\mu - (m - \overline{\sigma}) + i\gamma_0(\mu - \overline{\omega})\right]^{-1} \\
&= 2G_\sigma T \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{m^*}{\vec{p}^2 + m^{*2} + (\omega_n + i\mu^*)^2} \right) \\
&= G_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m^*}{E^*} \left(\frac{1}{e^{\beta(E^* - \mu^*)} + 1} + \frac{1}{e^{\beta(E^* + \mu^*)} + 1} \right) \\
&\equiv G_\sigma n_s
\end{aligned}$$

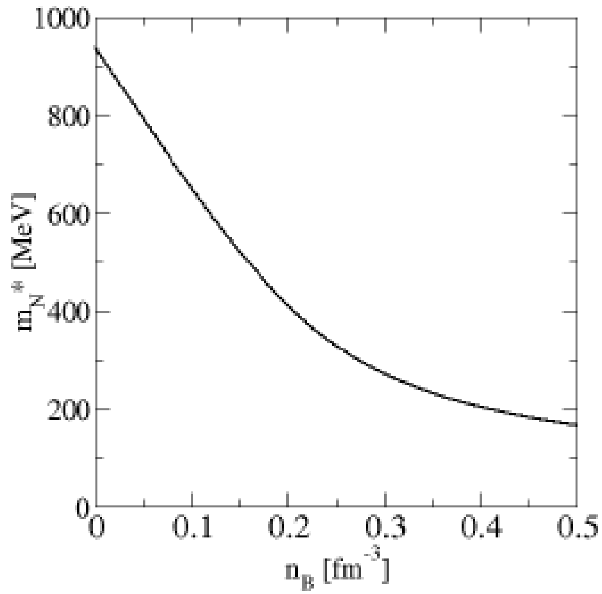
Vector mean field

$$\begin{aligned}
\overline{\omega}_0 &= -G_{\overline{\omega}_0}\, Tr\, \gamma_0 G[\overline{\sigma}, \overline{\omega}_0] \\
&= G_\omega \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{1}{e^{\beta(E^* - \mu^*)} + 1} - \frac{1}{e^{\beta(E^* + \mu^*)} + 1} \right) \\
&\equiv G_\omega n
\end{aligned}$$

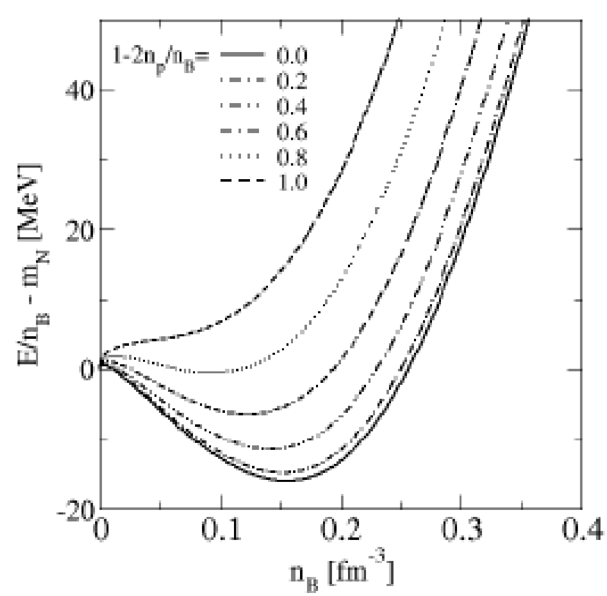
Matsubara sums $\longrightarrow$ Exercise!!

Walecka model for dense nuclear matter - results

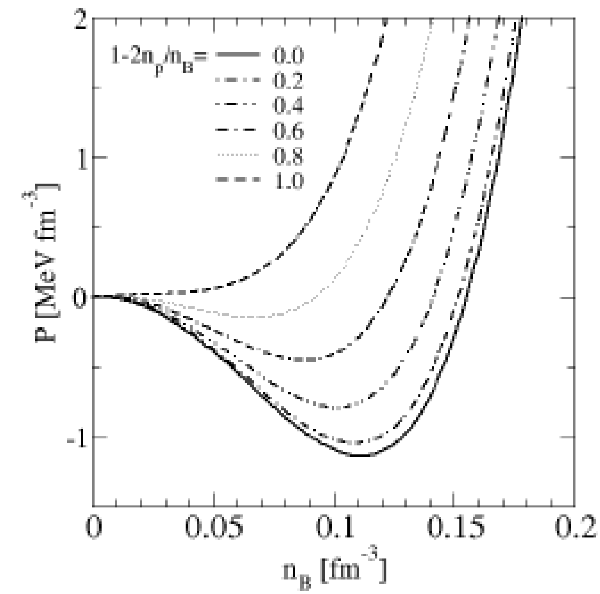
Effective mass



Energy per nucleon



Pressure



Symmetric nuclear matter ($n_p/n_B = 0.5$) saturates with a binding energy per nucleon of 16 MeV at $n_B = n_p + n_n = 0.16$ fm^{-3}. Increasing the asymmetry towards pure neutron matter ($n_p = 0$) makes the system unbound.

See, e.g., Kapusta's book "Finite temperature field theory" for the nuclear liquid-gas phase transition.

Relativistic density functional approach to quark matter - string-flip model (SFM)

M.A.R. Kaltenborn, N.-U.F. Bastian, D.B. Blaschke, arxiv:1701.04400 v3

PHYSICAL REVIEW D VOLUME 34, NUMBER 11 1 DECEMBER 1986

Pauli quenching effects in a simple string model of quark/nuclear matter

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(Received 16 December 1985)

Relativistic density functional approach* (I)

$$\mathcal{Z} = \int \mathcal{D}\bar{q}\mathcal{D}q \exp\left\{\int_0^\beta d\tau \int_V d^3x \left[\mathcal{L}_{\rm eff} + \bar{q}\gamma_0\hat{\mu}q\right]\right\}, \quad q = \begin{pmatrix} q_u \\ q_d \end{pmatrix}, \quad \hat{\mu} = \mathrm{diag}(\mu_u, \mu_d)$$

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm free} - \boxed{U(\bar{q}q, \bar{q}\gamma_0 q)}, \quad \mathcal{L}_{\rm free} = \bar{q}\left(-\gamma_0\frac{\partial}{\partial\tau} + i\vec{\gamma}\cdot\vec{\nabla} - \hat{m}\right)q, \ \hat{m} = \mathrm{diag}(m_u, m_d)$$

General nonlinear functional of quark density bilinears: scalar, vector, isovector, diquark ...
Expansion around the expectation values:

$$U(\bar{q}q,\, \bar{q}\gamma_0 q) = U(n_{\rm s}, n_{\rm v}) + (\bar{q}q - n_{\rm s})\Sigma_{\rm s} + (\bar{q}\gamma_0 q - n_{\rm v})\Sigma_{\rm v} + \ldots ,$$

$$\langle\bar{q}q\rangle = n_{\rm s} = \sum_{f=u,d} n_{{\rm s},f} = -\sum_{f=u,d}\frac{T}{V}\frac{\partial}{\partial m_f}\ln\mathcal{Z}, \quad \Sigma_{\rm s} = \left.\frac{\partial U(\bar{q}q, \bar{q}\gamma_0 q)}{\partial(\bar{q}q)}\right|_{\bar{q}q = n_{\rm s}} = \frac{\partial U(n_{\rm s}, n_{\rm v})}{\partial n_{\rm s}},$$

$$\langle\bar{q}\gamma_0 q\rangle = n_{\rm v} = \sum_{f=u,d} n_{{\rm v},f} = \sum_{f=u,d}\frac{T}{V}\frac{\partial}{\partial \mu_f}\ln\mathcal{Z}, \quad \Sigma_{\rm v} = \left.\frac{\partial U(\bar{q}q, \bar{q}\gamma_0 q)}{\partial(\bar{q}\gamma_0 q)}\right|_{\bar{q}\gamma_0 q = n_{\rm v}} = \frac{\partial U(n_{\rm s}, n_{\rm v})}{\partial n_{\rm v}}$$

$$\boxed{\mathcal{Z} = \int \mathcal{D}\bar{q}\mathcal{D}q \exp\left\{\mathcal{S}_{\rm quasi}[\bar{q}, q] - \beta V\,\Theta[n_{\rm s}, n_{\rm v}]\right\}, \quad \Theta[n_{\rm s}, n_{\rm v}] = U(n_{\rm s}, n_{\rm v}) - \Sigma_{\rm s} n_{\rm s} - \Sigma_{\rm v} n_{\rm v}}$$

$$\mathcal{S}_{\rm quasi}[\bar{q}, q] = \beta\sum_n\sum_{\vec{p}}\bar{q}\,G^{-1}(\omega_n, \vec{p})\,q, \quad G^{-1}(\omega_n, \vec{p}) = \gamma_0(-i\omega_n + \hat{\mu}^*) - \vec{\gamma}\cdot\vec{p} - \hat{m}^*$$

*This work was inspired by the textbook on “Thermodynamics and statistical mechanics” of the “red” series on Theoretical Physics by Walter Greiner and Coworkers.

Relativistic density functional approach (II)

$$\mathcal{Z} = \int \mathcal{D}\bar{q}\mathcal{D}q \exp\{\mathcal{S}_{\rm quasi}[\bar{q},q] - \beta V \Theta[n_{\rm s},n_{\rm v}]\}\ , \quad \Theta[n_{\rm s},n_{\rm v}] = U(n_{\rm s},n_{\rm v}) - \Sigma_{\rm s} n_{\rm s} - \Sigma_{\rm v} n_{\rm v}$$

$$\mathcal{Z}_{\rm quasi} = \int \mathcal{D}\bar{q}\mathcal{D}q \exp\{\mathcal{S}_{\rm quasi}[\bar{q},q]\} = \det[\beta G^{-1}]\,, \qquad \ln\det A = {\rm Tr}\ln A$$

$$P_{\rm quasi} = \frac{T}{V}\ln \mathcal{Z}_{\rm quasi} = \frac{T}{V}{\rm Tr}\ln[\beta G^{-1}]$$

"no sea" approximation ...

$$= 2N_c \sum_{f=u,d}\int\frac{d^3p}{(2\pi)^3}\left\{T\ln\left[1+{\rm e}^{-\beta(E_f^*-\mu_f^*)}\right]+T\ln\left[1+{\rm e}^{-\beta(E_f^*+\mu_f^*)}\right]\right\}$$

$$P_{\rm quasi} = \sum_{f=u,d}\int\frac{dp}{\pi^2}\frac{p^4}{E_f^*}\left[f(E_f^*-\mu_f^*)+f(E_f^*+\mu_f^*)\right]$$

$$E_f^* = \sqrt{p^2+{m_f^*}^2}$$

$$f(E) = 1/[1+\exp(\beta E)]$$

$$P = \sum_{f=u,d}\int_0^{p_{{\rm F},f}}\frac{dp}{\pi^2}\frac{p^4}{E_f^*} - \Theta[n_{\rm s},n_{\rm v}]\,, \quad p_{{\rm F},f} = \sqrt{{\mu_f^*}^2 - {m_f^*}^2}$$

$$\hat{m}^* = \hat{m} + \Sigma_{\rm s}$$
$$\hat{\mu}^* = \hat{\mu} - \Sigma_{\rm v}$$

Selfconsistent densities

$$n_{\rm s} = -\sum_{f=u,d}\frac{\partial P}{\partial m_f} = \frac{3}{\pi^2}\sum_{f=u,d}\int_0^{p_{{\rm F},f}} dp\, p^2\frac{m_f^*}{E_f^*}\,, \quad n_{\rm v} = \sum_{f=u,d}\frac{\partial P}{\partial \mu_f} = \frac{3}{\pi^2}\sum_{f=u,d}\int_0^{p_{{\rm F},f}} dp\, p^2 = \frac{p_{{\rm F},u}^3 + p_{{\rm F},d}^3}{\pi^2}\,.$$

Relativistic density functional approach (III)

Density functional for the SFM

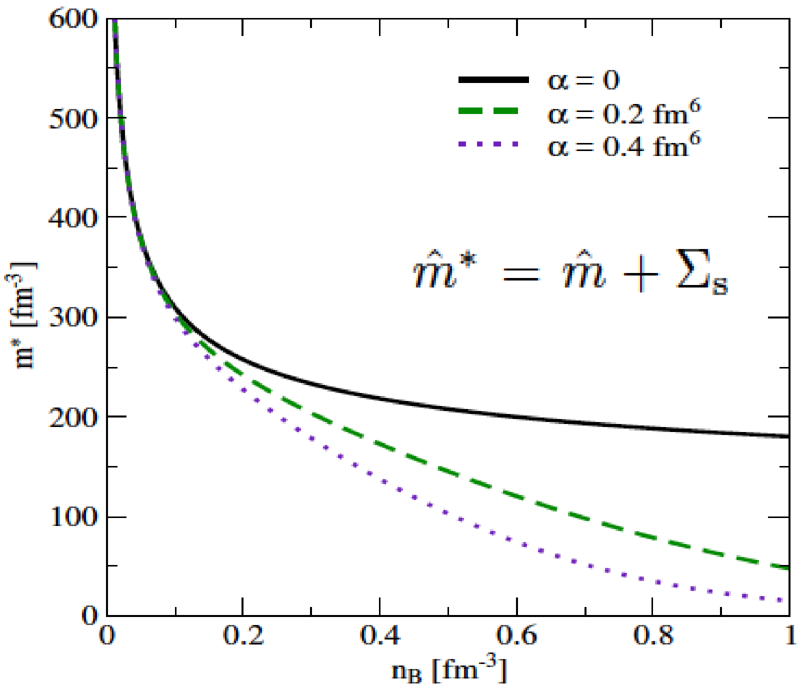
$$U(n_s, n_v) = D(n_v) n_s^{2/3} + a n_v^2 + \frac{b n_v^4}{1 + c n_v^2} ,$$

Quark selfenergies

$$\Sigma_s = \frac{2}{3} D(n_v) n_s^{-1/3} ,$$

Quark "confinement"

$$\Sigma_v = 2 a n_v + \frac{4 b n_v^3}{1 + c n_v^2} - \frac{2 b c n_v^5}{(1 + c n_v^2)^2} + \frac{\partial D(n_v)}{\partial n_v} n_s^{2/3}$$

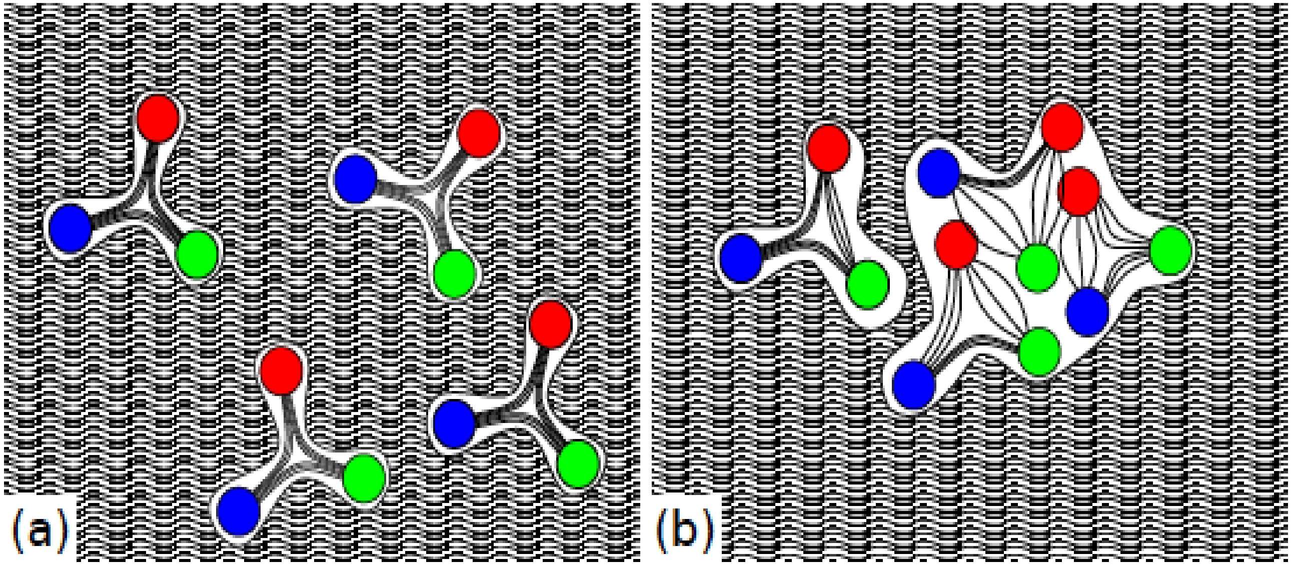


String tension & confinement due to dual Meissner effect (dual superconductor model)

$$D(n_v) = D_0 \Phi(n_v)$$

Effective screening of the string tension in dense matter by a reduction of the available volume $\alpha = v|v|/2$

$$\Phi(n_B) = \begin{cases} 1, & \text{if } n_B < n_0 \\ e^{-\alpha(n_B - n_0)^2}, & \text{if } n_B > n_0 \end{cases}$$

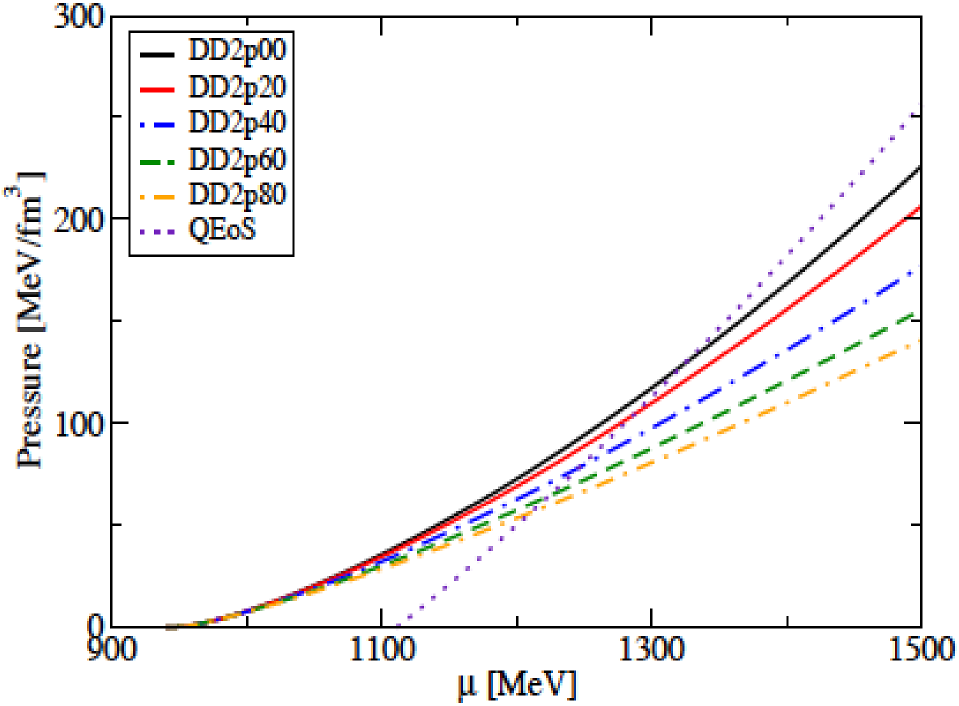


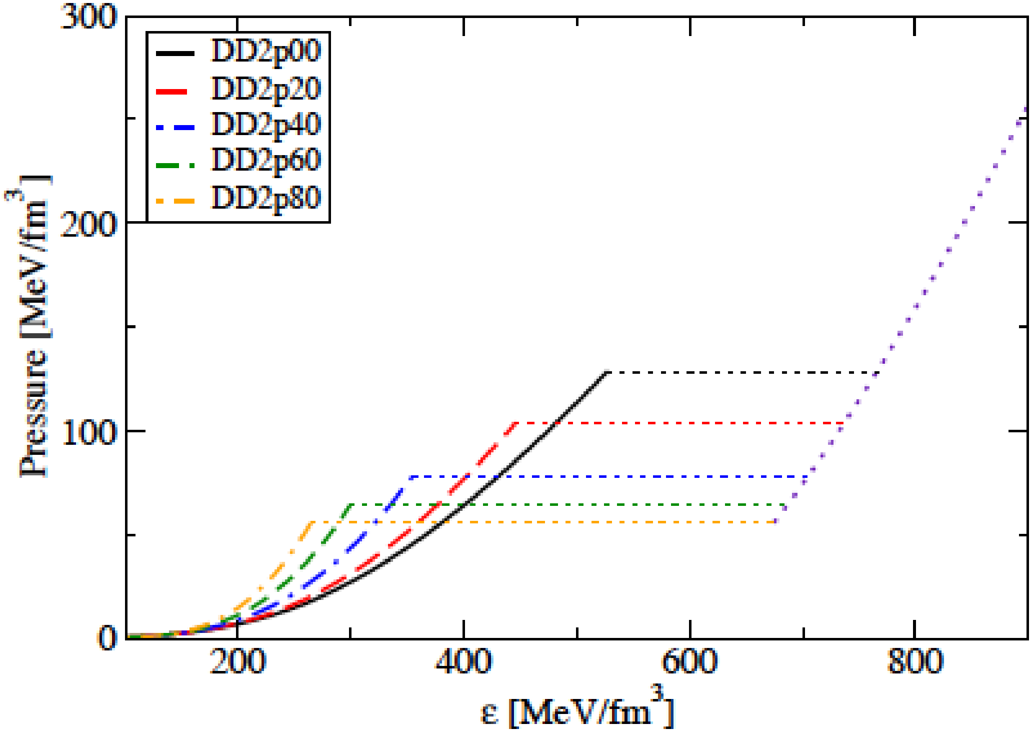
3. Phase transition to SFM quark matter

Hadronic matter: DD2 with excluded volume **[S. Typel, EPJA 52 (3) (2016)]**

$$\Phi_n = \Phi_p = \begin{cases} 1, & \text{if } n_B < n_0 \\ e^{-\frac{v|v|}{2}(n_B - n_0)^2}, & \text{if } n_B > n_0 \end{cases}$$

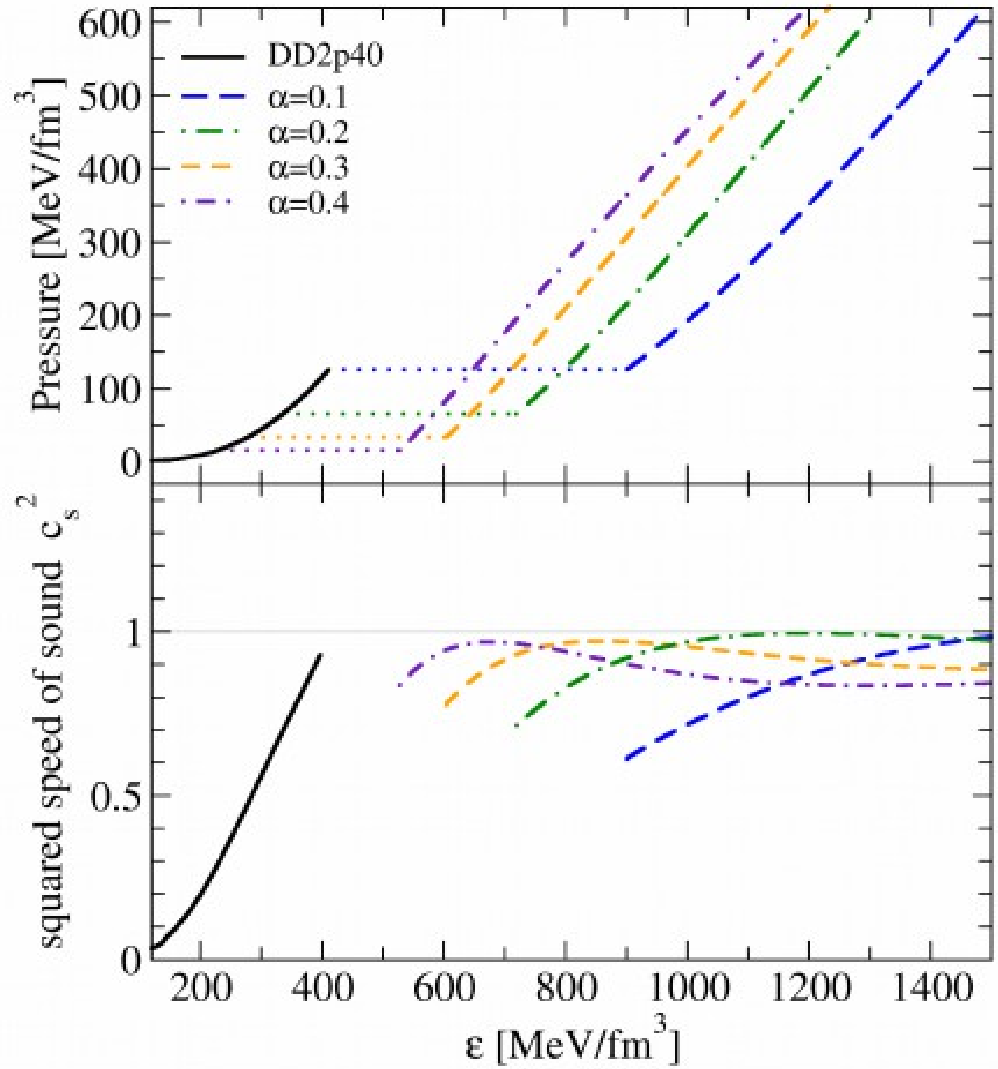
Varying the hadronic excluded volume parameter, p00 → v=0, … , p80 → v=8 fm^3

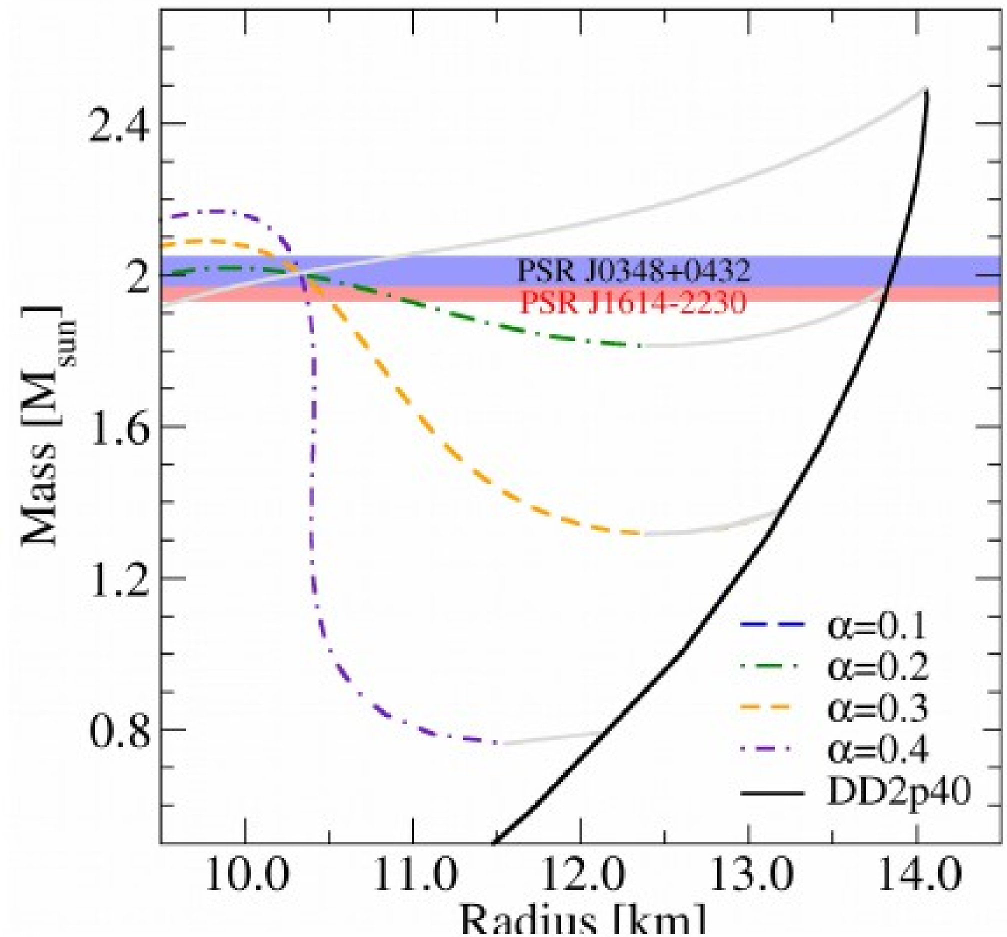




Hybrid EOS - parameters

$\underline{\alpha}, a, b$

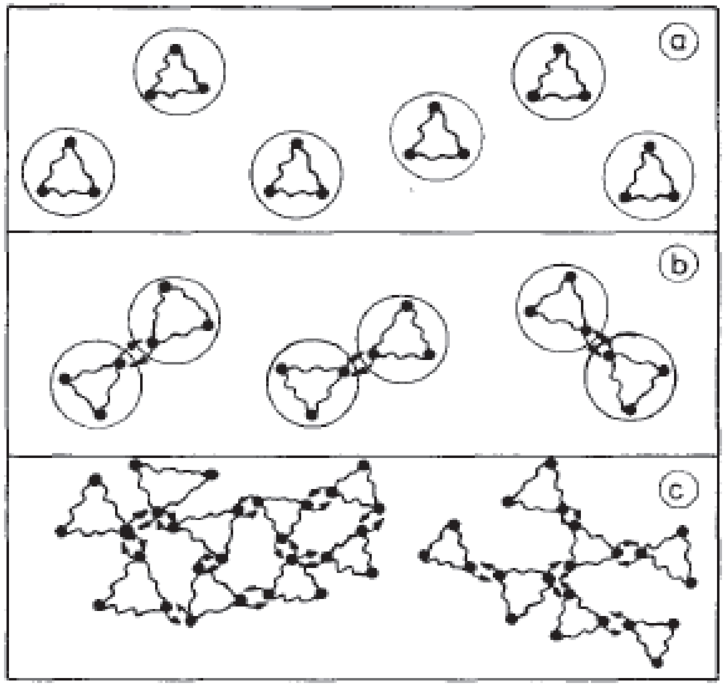




Kaltenborn, Bastian, Blaschke, arXiv:1701.04400

Quark Pauli Blocking in Nuclear Matter

2. Pauli blocking among baryons

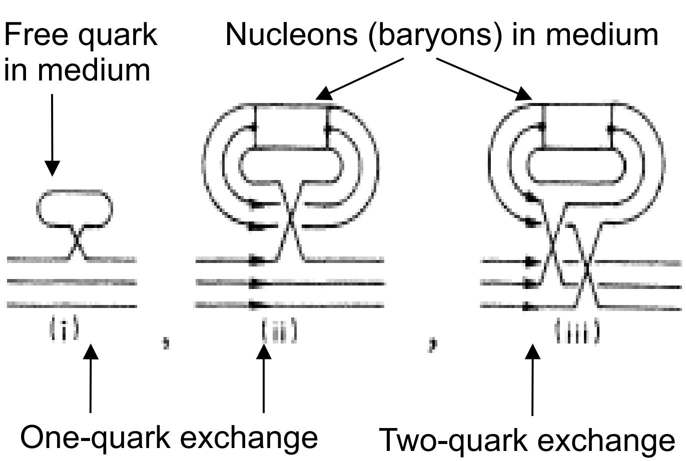


a) Low density: Fermi gas of nucleons (baryons)

b) ~ saturation: Quark exchange interaction and Pauli blocking among nucleons (baryons)

c) high density: Quark cluster matter (string-flip model ...)

Roepke & Schulz, Z. Phys. C 35, 379 (1987); Roepke, DB, Schulz, PRD 34, 3499 (1986)



Nucleon (baryon) self-energy --> Energy shift

$$\Delta E^{\rm Pauli}_{\nu P} = \sum_{123} |\psi_{\nu P}(123)|^2 [E(1)+E(2)+E(3)-E^0_{\nu P}][f_{\alpha_1}(1)+f_{\alpha_2}(2)+f_{\alpha_3}(3)]$$
$$+ \sum_{123}\sum_{456}\sum_{\nu' P'} \psi^*_{\nu P}(123)\psi_{\nu' P'}(456) f_3(E^0_{\nu' P'})\{\delta_{36}\psi_{\nu P}(123)\psi^*_{\nu' P'}(456) - \psi_{\nu P}(453)\psi^*_{\nu' P'}(126)\}$$
$$\times [E(1)+E(2)+E(3)+E(4)+E(5)+E(6)-E^0_{\nu P}-E^0_{\nu' P'}]$$
$$= \Delta E^{\rm Pauli,free}_{\nu P} + \Delta E^{\rm Pauli,bound}_{\nu P} .$$

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Pauli quenching effects in a simple string model of quark/nuclear matter

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(Received 16 December 1985)

2. Pauli blocking among baryons - details

$$\Sigma_\nu(p,p_{Fn},p_{Fp}) = \sum_{\nu'=\{n,p\}}\sum_{\alpha=1,2} C^{(\alpha)}_{\nu\nu'} W_\alpha(p_{F\nu'},p)$$

$$W_\alpha(p_{F\nu'},p) = \frac{\Omega}{2\pi^2}\int_0^{p_{F\nu'}} p'^2 \bar{V}^{(\alpha)}(p,p')dp';$$

$$\bar{V}^{(\alpha)}(p,p') = \frac{1}{2}\int_{-1}^{1} V^{(\alpha)}(\vec{p},\vec{p}')dz;$$

$$V^{(\alpha)}(\vec{p},\vec{p}') = \frac{V_0 b}{\Omega m}\left(\frac{15}{2} - \lambda_\alpha^2(\vec{p}-\vec{p}')^2\right)\exp(-\lambda_\alpha^2(\vec{p}-\vec{p}')^2).$$

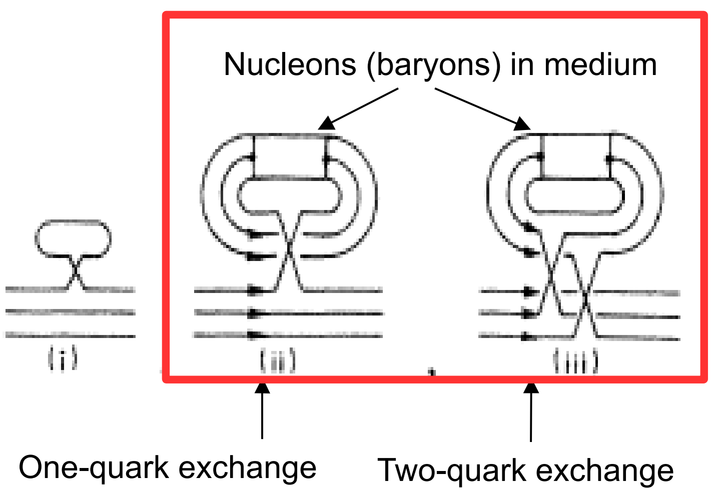
	$C^{(1)}_{n\ \nu}$	$C^{(2)}_{n\ \nu}$
neutron	$-\frac{96}{243}$	$\frac{10}{27}$
proton	$-\frac{28}{81}$	$\frac{8}{27}$

$$b^{-2} = \sqrt{3}m\omega$$

$$\omega = 178.425 \text{ MeV}$$

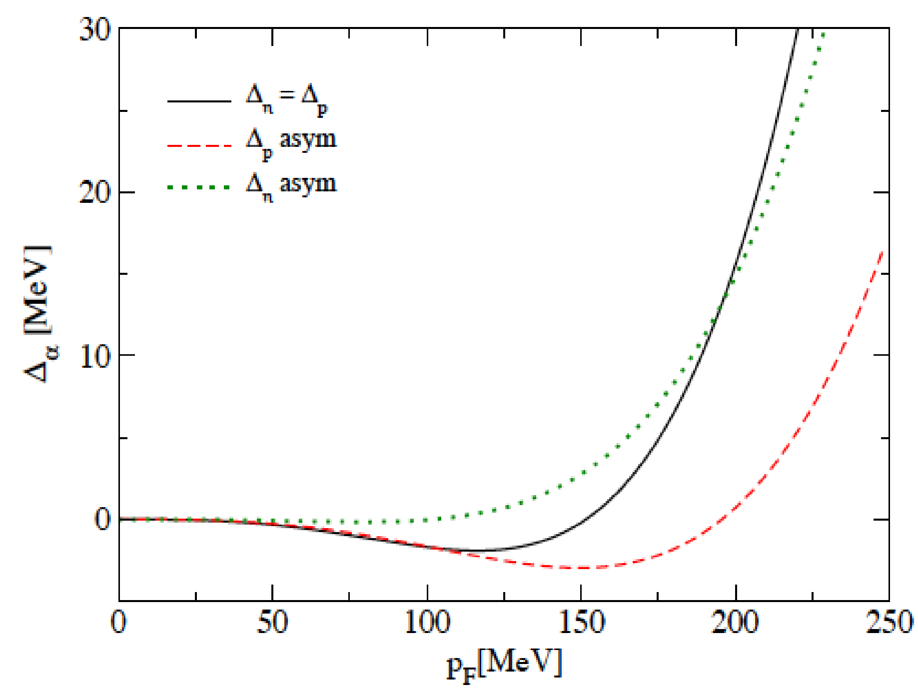
$$m = 350 \text{ MeV} \qquad b = 0.6 \text{ fm}$$

$$V_0 = \frac{9\sqrt{3}\pi^{3/2}}{2} \text{ and } \lambda_\alpha = \frac{b}{\sqrt{3}\alpha}.$$



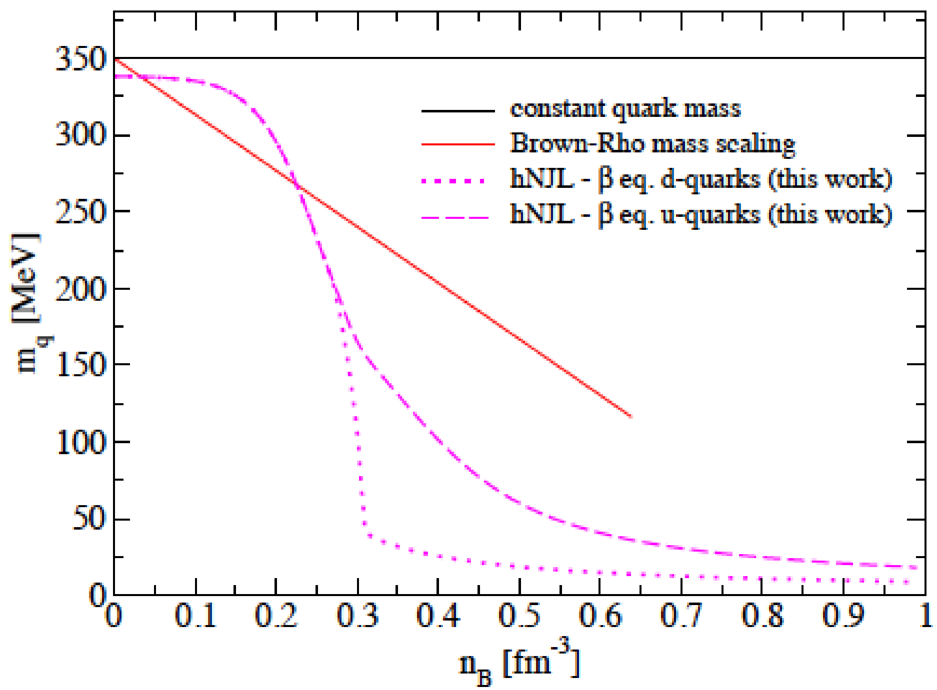
$$W_\alpha(p_{F\nu'},p) = \frac{V_0 b}{32\pi^2\lambda_\alpha^4 m}\{12\lambda_\alpha\sqrt{\pi}\left[\text{erf}\left(\lambda_\alpha(p_{F\nu'}-p)\right) + \text{erf}\left(\lambda_\alpha(p_{F\nu'}+p)\right)\right] + \frac{1}{p}\left[\left(11 - 2\lambda_\alpha^2\, p_{F\nu'}(p_{F\nu'}+p)\right) e^{-\lambda_\alpha^2(p_{F\nu'}+p)^2} + \left(11 - 2\lambda_\alpha^2\, p_{F\nu'}(p_{F\nu'}-p)\right) e^{-\lambda_\alpha^2(p_{F\nu'}-p)^2}\right]\}$$

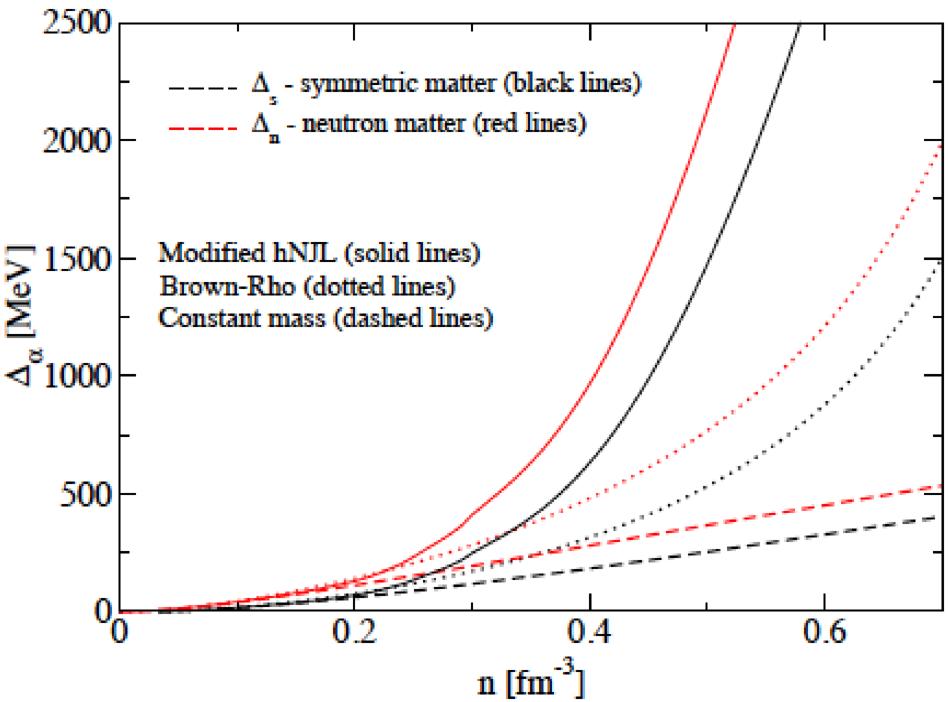
$$\Delta^{Pauli}_{\nu_A,P} = \frac{1}{24\sqrt{3\pi}}\frac{b}{m}\sum_{\nu'}[15a_{\nu,\nu'}P_F(\nu')^3 + \frac{17}{12}b_{\nu,\nu'}b^2(P^2+P_F(\nu')^2)P_F(\nu')^3]$$



2. Pauli blocking in NM – details

New aspect: chiral restoration --> dropping quark mass





Increased baryon swelling at supersaturation densities:
--> dramatic enhancement of the Pauli repulsion !!

D.B., H. Grigorian, G. Roepke: "Quark exchange effects in dense nuclear matter", in prep. (2017)

2. Pauli blocking among baryons – results

New EoS: Joining RMF (Linear Walecka) for pointlike baryons with chiral Pauli blocking

$$p = \frac{1}{4\pi^2}\sum_\nu \left[-E_\nu^* m_\nu^{*2} p_{F\nu} + \frac{2}{3}E_\nu^* p_{F\nu}^3 + m_\nu^{*4}\log\left(\frac{E_\nu^* + p_{F\nu}}{m_\nu^*}\right)\right] + \frac{1}{2}\left(\frac{g_\omega}{m_\omega}\right)^2 n^2 - \frac{1}{2}\left(\frac{g_\sigma}{m_\sigma}\right)^2 n_s^2 + p_{ex};$$

$$\epsilon = \frac{1}{4\pi^2}\sum_\nu \left[2\,E_\nu^{*3} p_{F\nu} - E_\nu^* m_\nu^{*2} p_{F\nu} - m_\nu^{*4}\log\left(\frac{E_I^* + p_{F\nu}}{m_\nu^*}\right)\right] + \frac{1}{2}\left(\frac{g_\omega}{m_\omega}\right)^2 n^2 + \frac{1}{2}\left(\frac{g_\sigma}{m_\sigma}\right)^2 n_s^2 + \epsilon_{ex},$$

$$\mu_{ex,\nu} = \Delta_\nu(n,x) = \Sigma_\nu(p_{F,\nu}; p_{Fn,}, p_{Fp}),$$

$$\epsilon_{ex} = \sum_\nu \int_0^n dn' \{x\Delta_p(n',x) + (1-x)\Delta_n(n',x)\},$$

$$p_{ex} = \sum_\nu \mu_{ex,\nu}\, n_\nu - \epsilon_{ex},$$

$$n_{s,\nu} = \frac{m_\nu^*}{\pi^2}\left[E_\nu^* p_{F\nu} - m_\nu^{*2}\log\left(\frac{E_\nu^* + p_{F\nu}}{m_\nu^*}\right)\right],$$

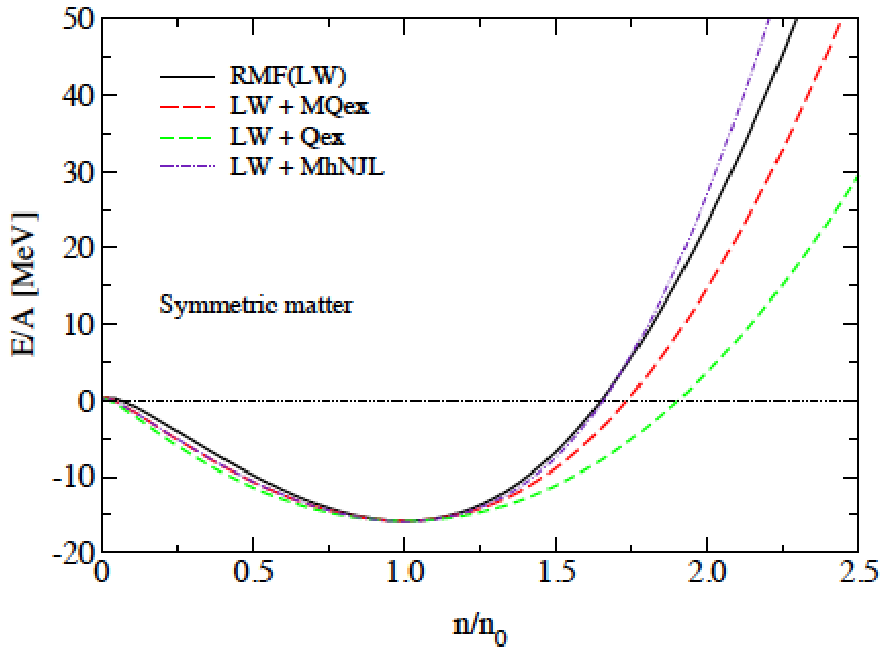
$$E_\nu^* = \sqrt{m_\nu^{*2} + p_{F\nu}^2}$$

$$n_\nu = \frac{p_{F\nu}^3}{3\pi^2},$$

$$m_\nu^* = m_\nu - \left(\frac{g_\sigma}{m_\sigma}\right)^2 n_{s,\nu},$$

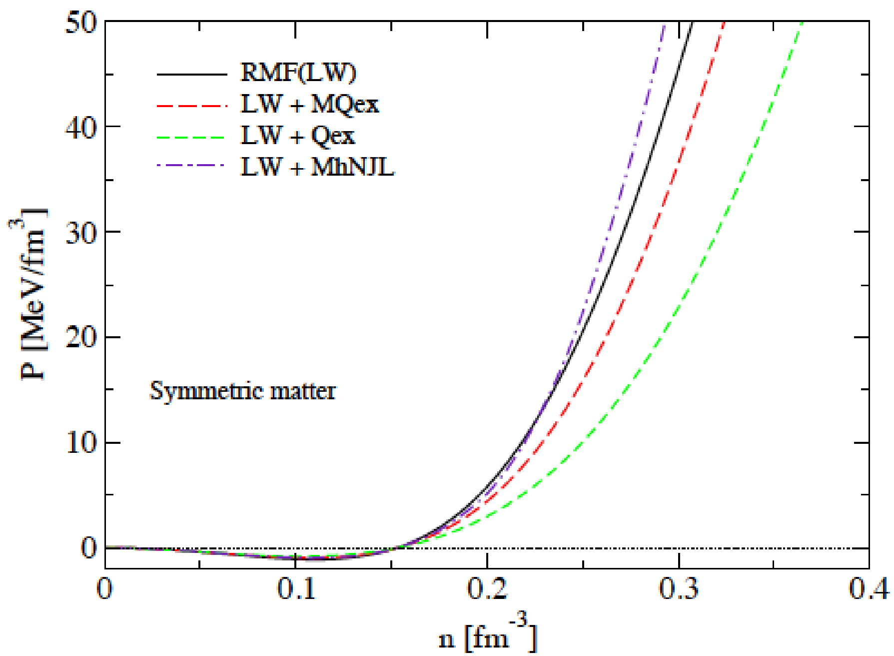
$$\mu_\nu = E_\nu^* + \left(\frac{g_\omega}{m_\omega}\right)^2 n_\nu + \mu_{ex,\nu}.$$

2. Pauli blocking among baryons – results

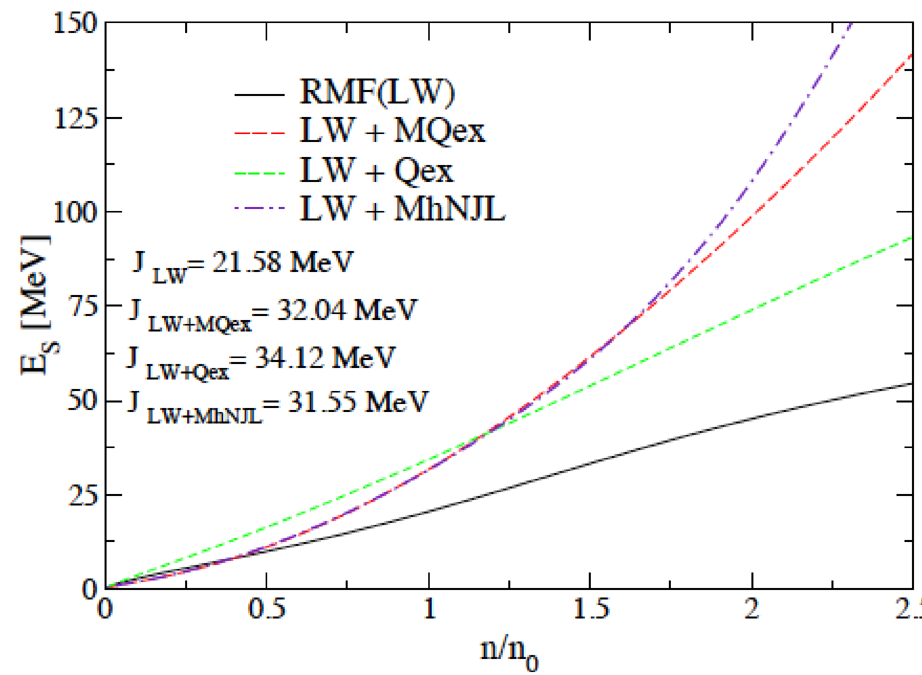


Parametrization: from saturation properties

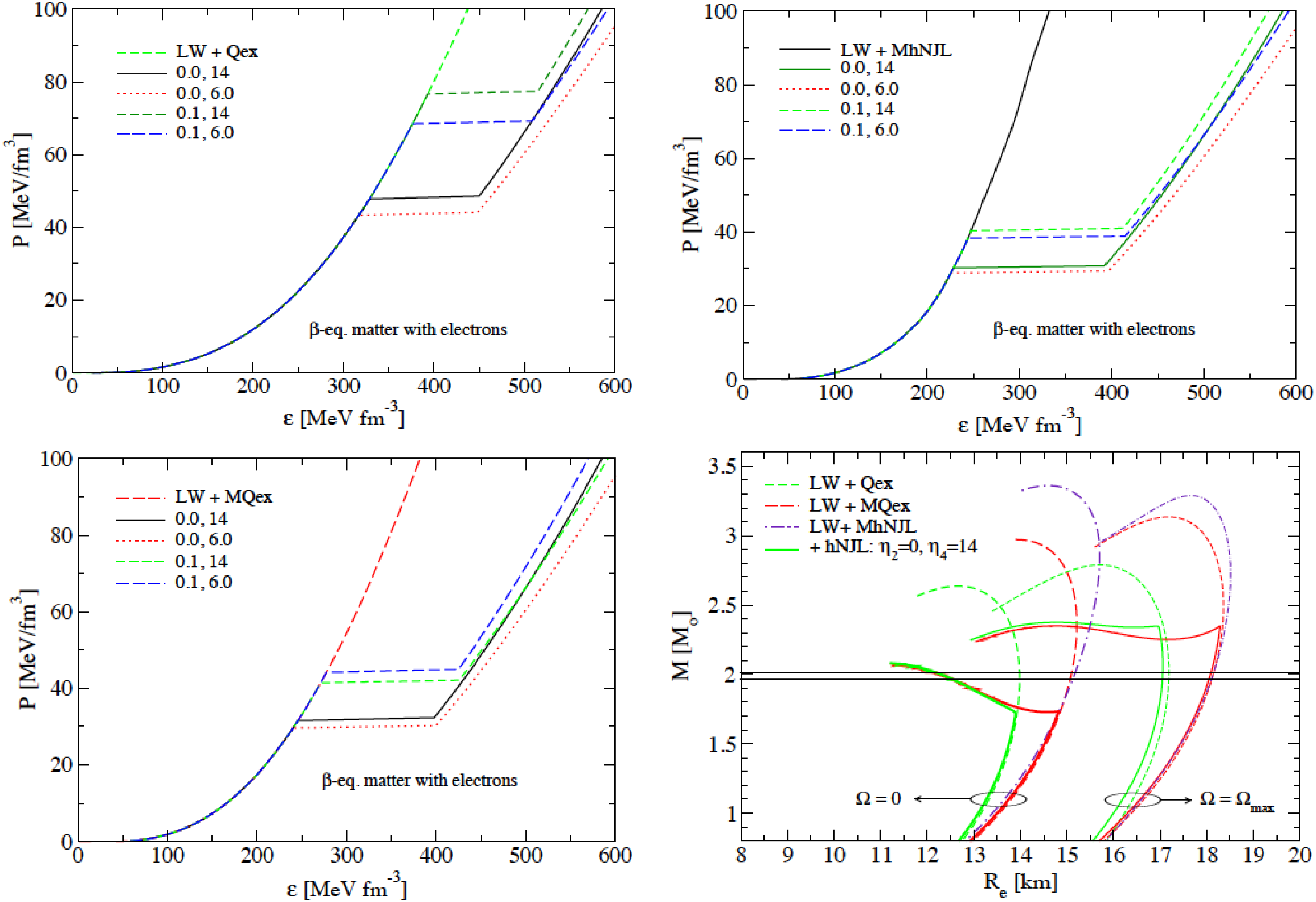
	$(g_\omega/m_\omega)^2$[fm^2]	$(g_\sigma/m_\sigma)^2$ [fm^2]
RMF (LW)	11.6582	15.2883
LW+Qex	6.11035	9.91197
LW+MQex	6.59170	13.29118
LW+MhNJL	9.25683	13.9474



Prediction: symmetry energy

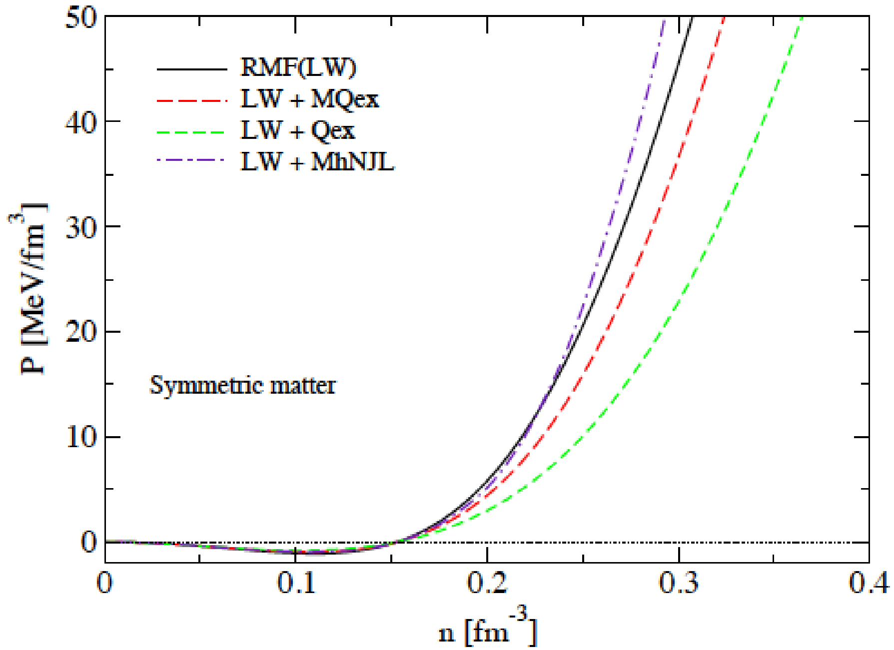


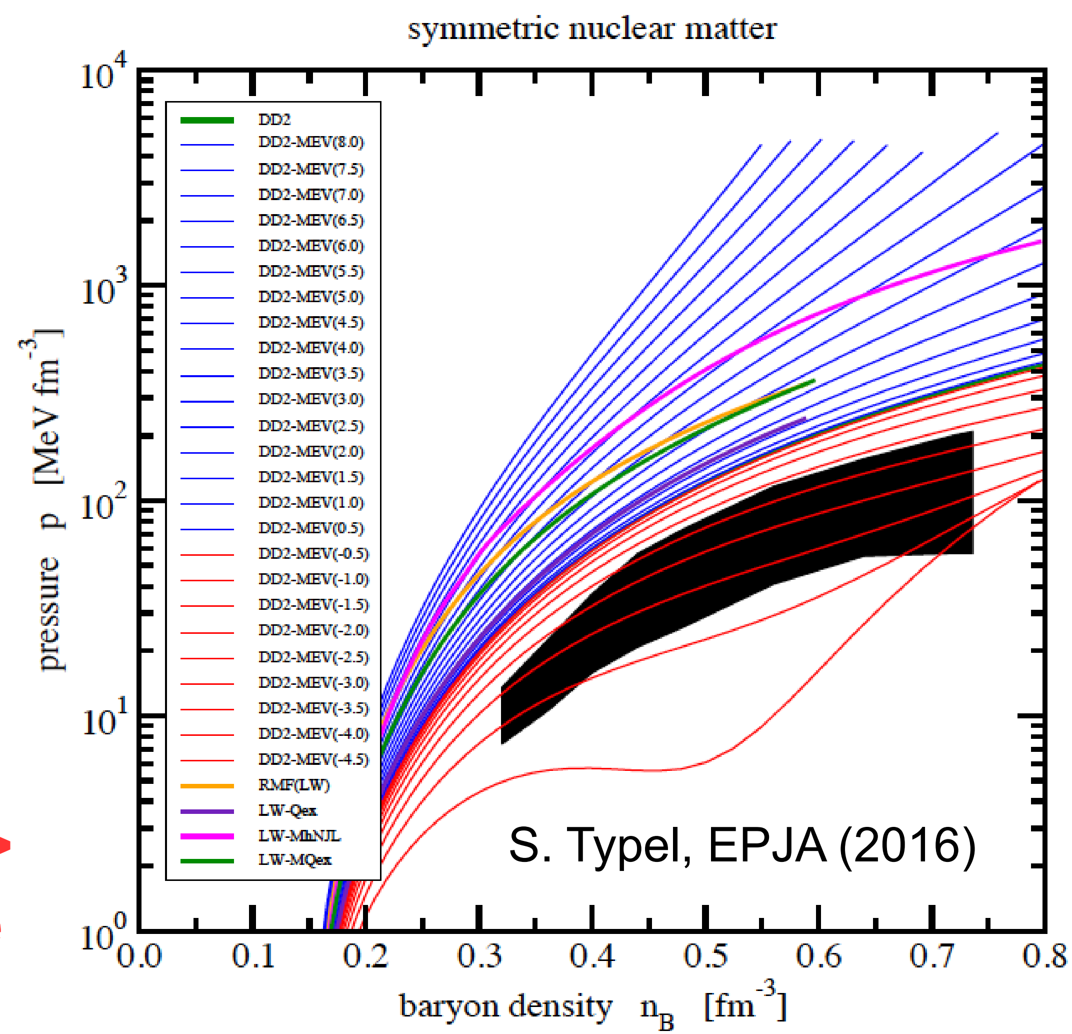
2. Pauli blocking in NM – results neutron stars



2. Pauli blocking in NM – nucleon excluded V.

New aspect: chiral restoration --> dropping quark mass





Increased baryon swelling at supersaturation densities: --> dramatic enhancement of the Pauli repulsion !!

D.B., H. Grigorian, G. Roepke: "Quark exchange effects in dense nuclear matter", in prep. (2017)

2. Pauli blocking in NM – Summary

Pauli blocking selfenergy (cluster meanfield) calculable in potential models for baryon structure

Partial replacement of other short-range repulsion mechanisms (vector meson exchange)

Modern aspects:
- onset of chiral symmetry restoration enhances nucleon swelling and Pauli blocking at high n
- quark exchange among baryons -> six-quark wavefunction -> “bag melting” -> deconfinement

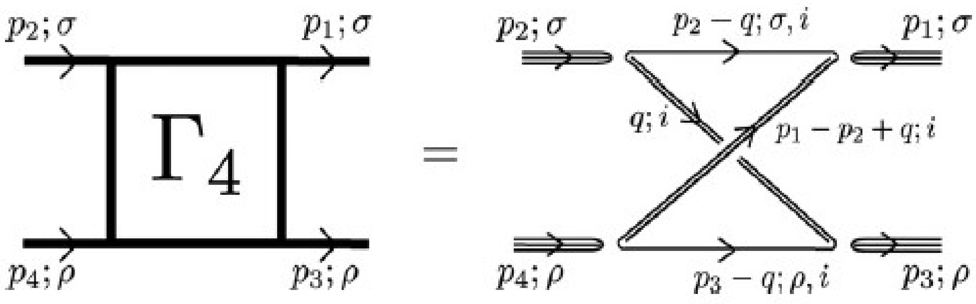
Chiral stiffening of nuclear matter --> reduces onset density for deconfinement

Hybrid EoS:
Convenient generalization of RMF models,
Take care: eventually aspects of quark exchange already in density dependent vertices!

Other baryons:
- hyperons
- deltas
Again calculable, partially done in nonrelativistic quark exchange models, chiral effects not yet!

Relativistic generalization:
Box diagrams of quark-diquark model ...

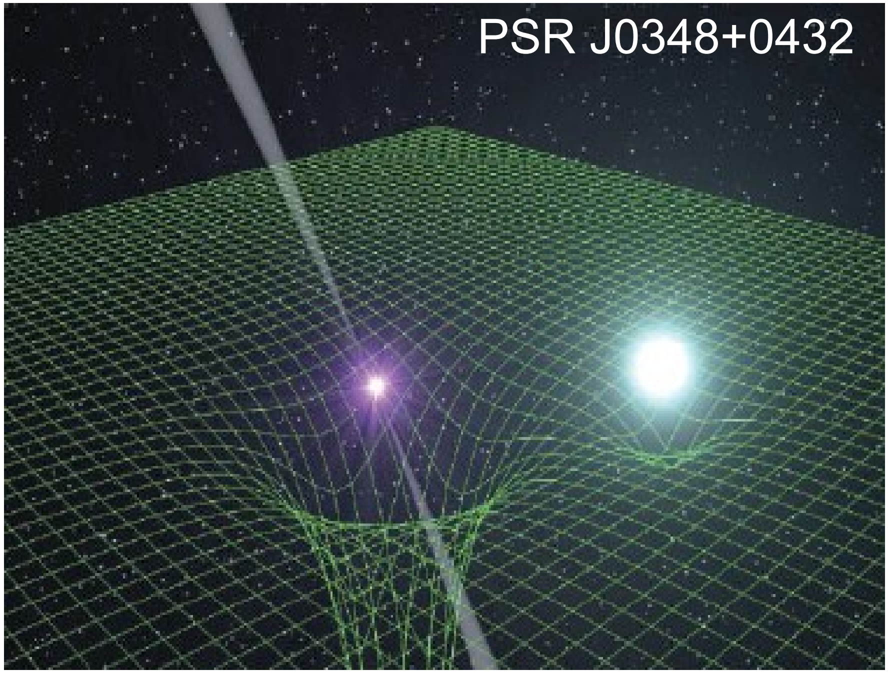
K. Maeda, Ann. Phys. 326 (2011) 1032



M-R relationships for hybrid stars – High-mass “twin” stars

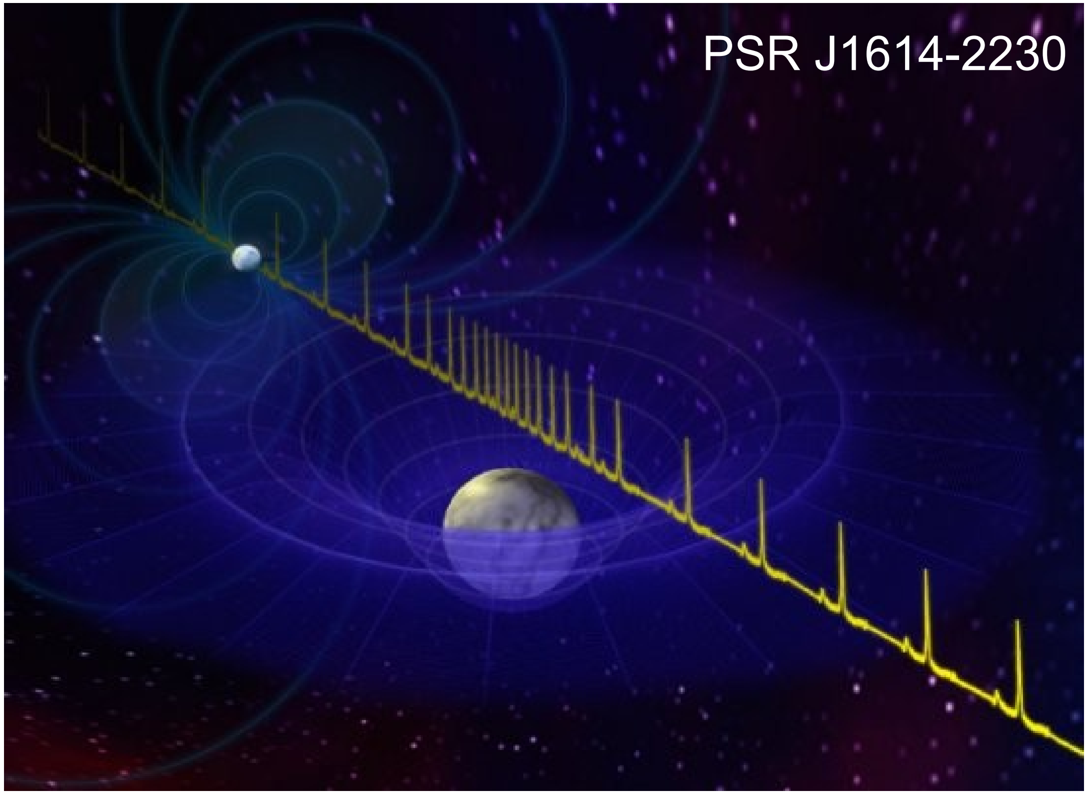
Two high-mass pulsars with $M \sim 2M_{sun}$

M=2.01 +/- 0.04 Msun



Antoniadis et al., Science 340 (2013) 448
Demorest et al., Nature 467 (2010) 1081
Fonseca et al., arxiv:1603.00545

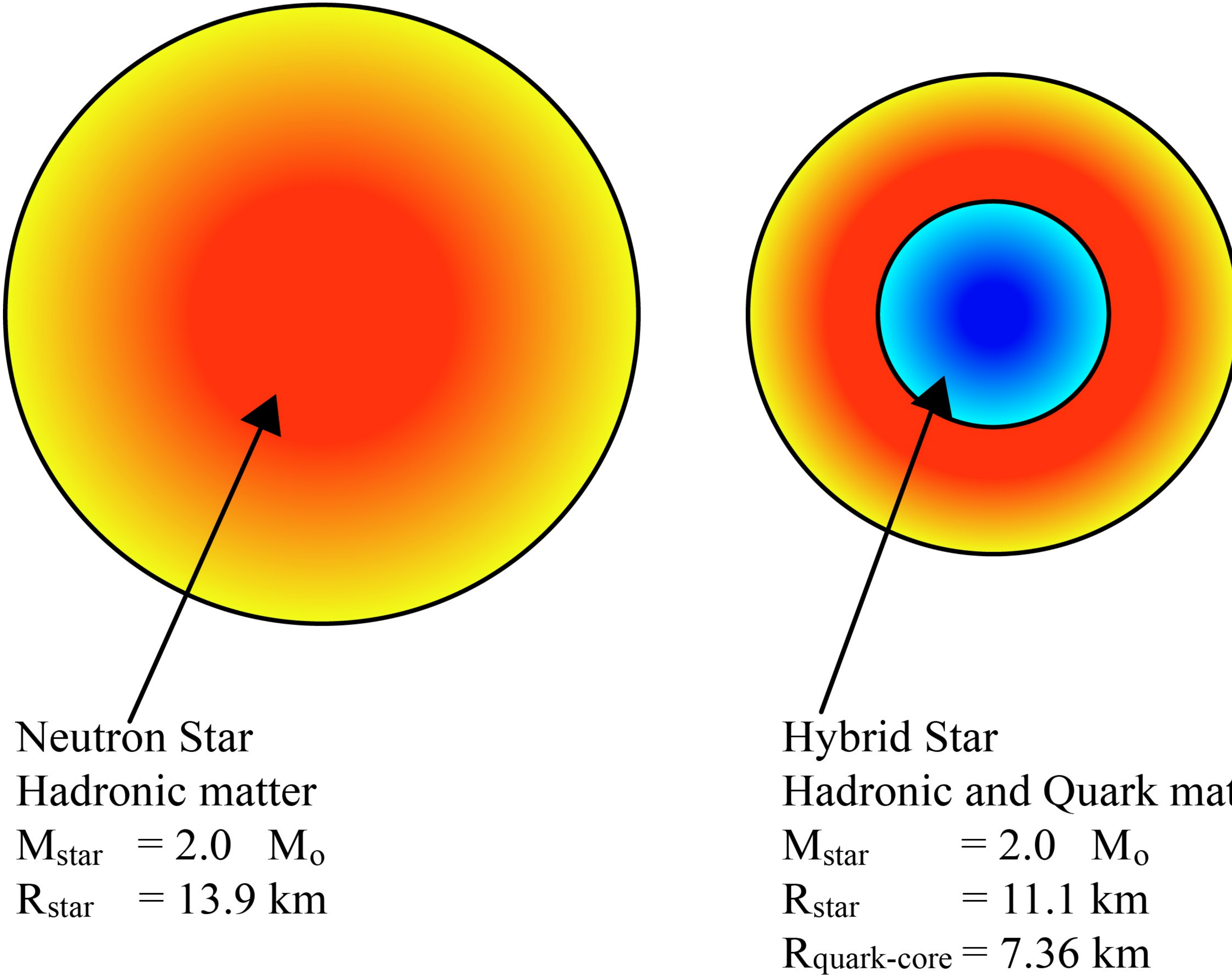
M=1.928 +/- 0.017 Msun



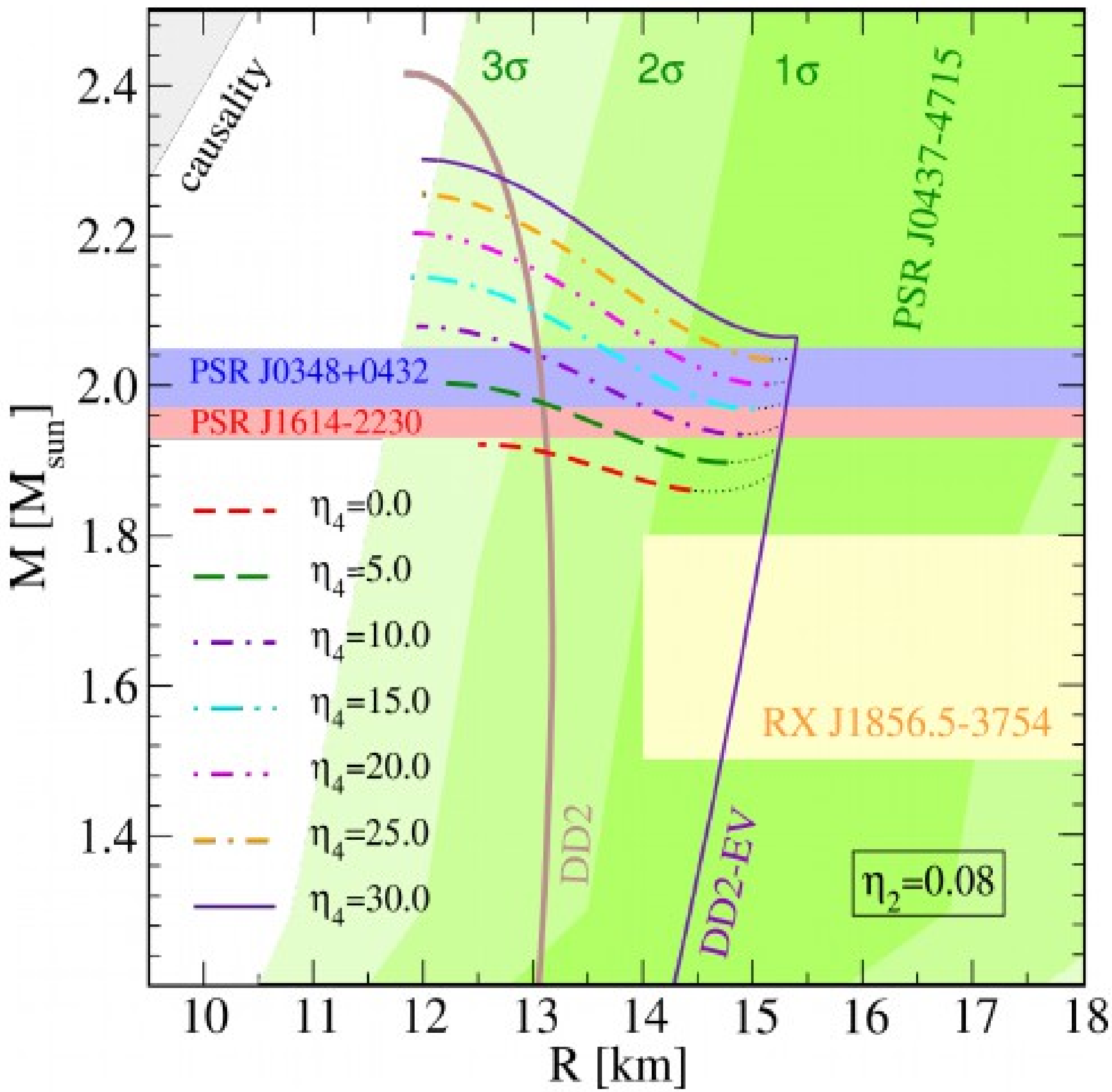
What if they were high-mass twin stars?

→ radius measurement required ! → NICER (2017)

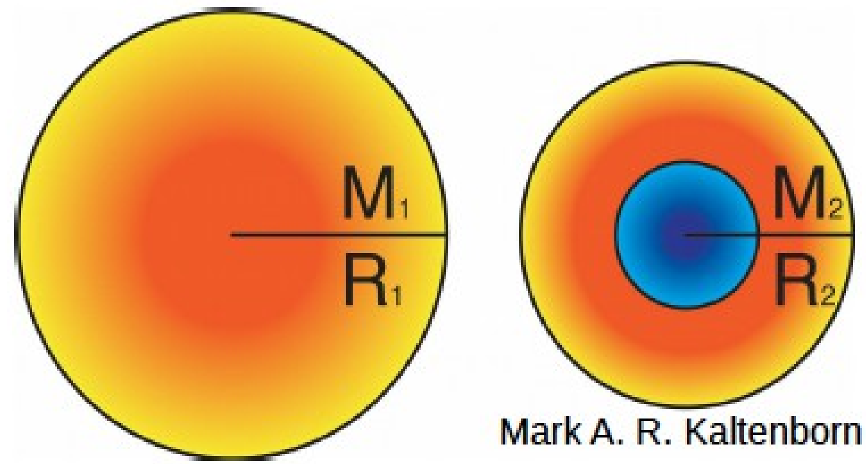
Two high-mass pulsars with $M \sim 2M_{sun}$



Motivation – Neutron stars (Twins?)



- Star configurations with same masses, but different radii

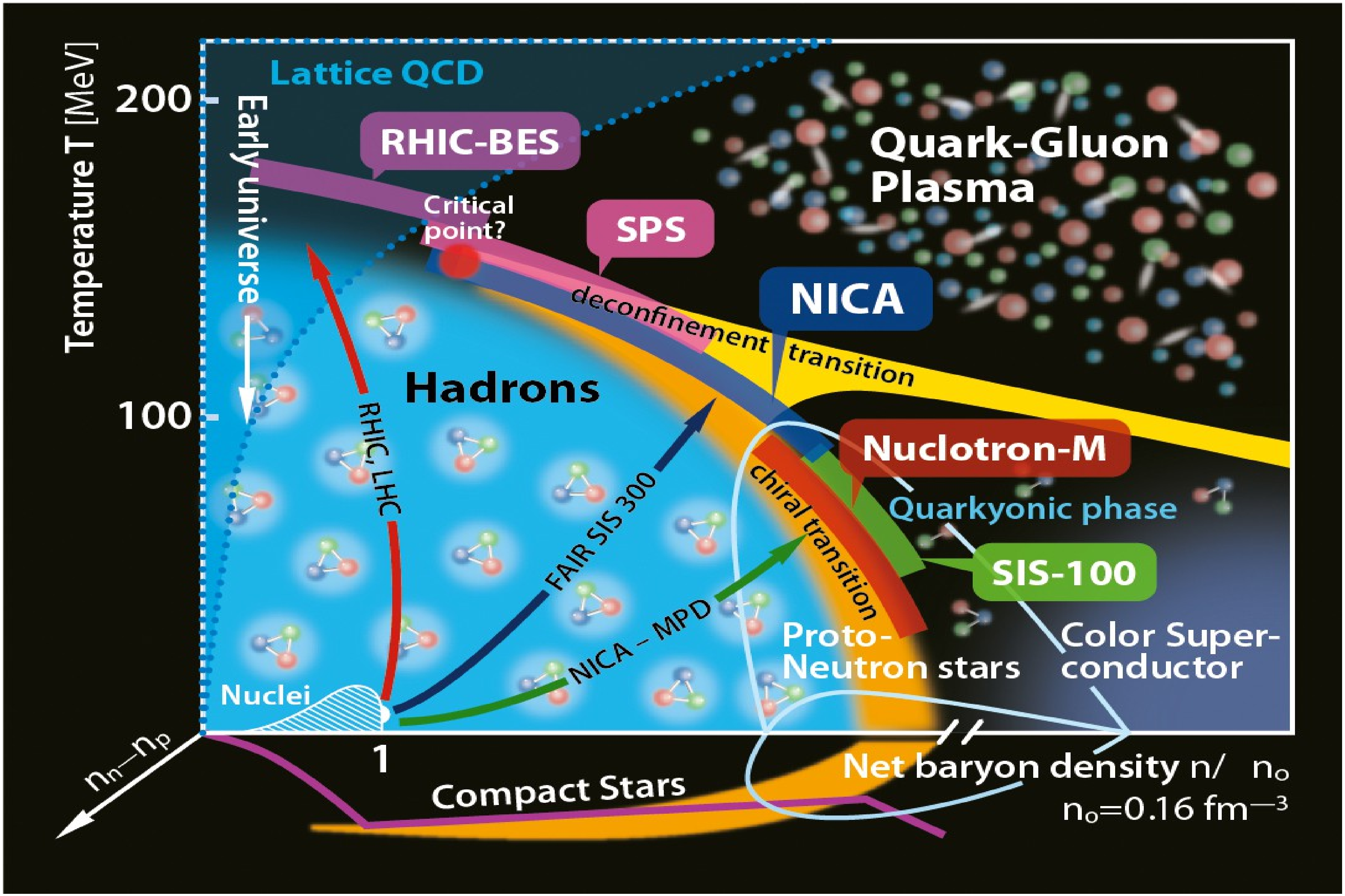


Mark A. R. Kaltenborn

- **New class of EOS, that features high mass twins**
- NASA NICER mission: radii measurements ~ 0.5 km
- Existence of twins implies 1st order phase-transition and hence a critical point

Benic, Blaschke, Alvarez-Castillo, Fischer, Typel, A&A 577, A40 (2015)

Support a CEP in QCD phase diagram with Astrophysics?



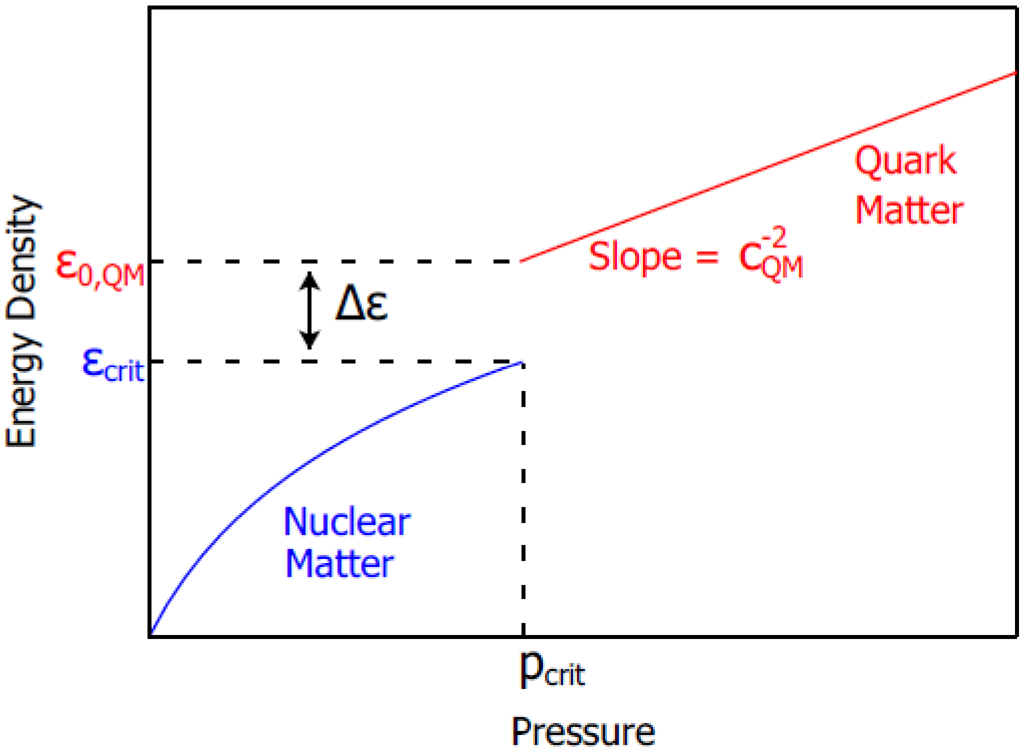
S. Benic et al., A&A 577, A40 (2015)

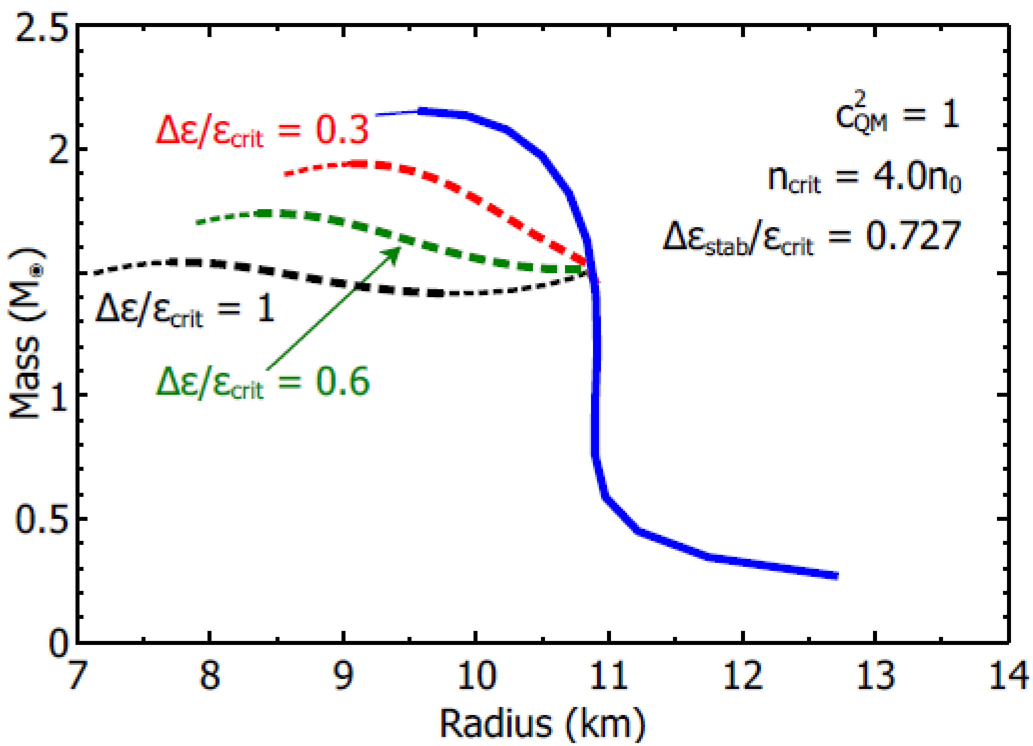
Crossover at finite T (Lattice QCD) + First order at zero T (Astrophysics) = Critical endpoint exists!

1. Constant speed of sound (css) model

Alford, Han, Prakash, arxiv:1302.4732

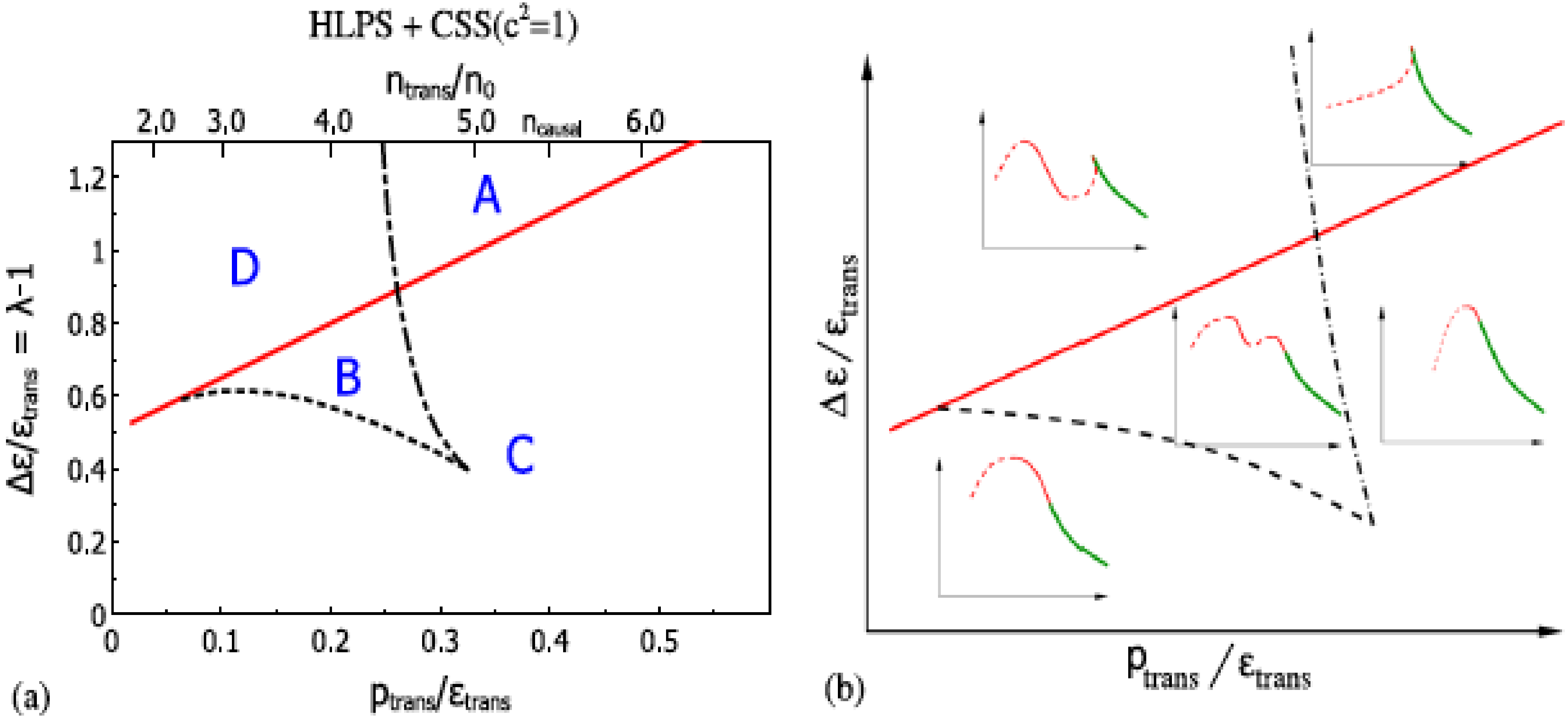
First order PT can lead to a stable branch of hybrid stars with quark matter cores which, depending on the size of the "latent heat" (jump in energy density), can even be disconnected from the hadronic one by an unstable branch → **"third family of CS"**.





Measuring two **disconnected populations** of compact stars in the M-R diagram would be the **detection of a first order phase transition** in compact star matter and thus the indirect proof for the existence of a **critical endpoint (CEP) in the QCD phase diagram**!

Key fact: Mass "twins" ↔ 1st order PT



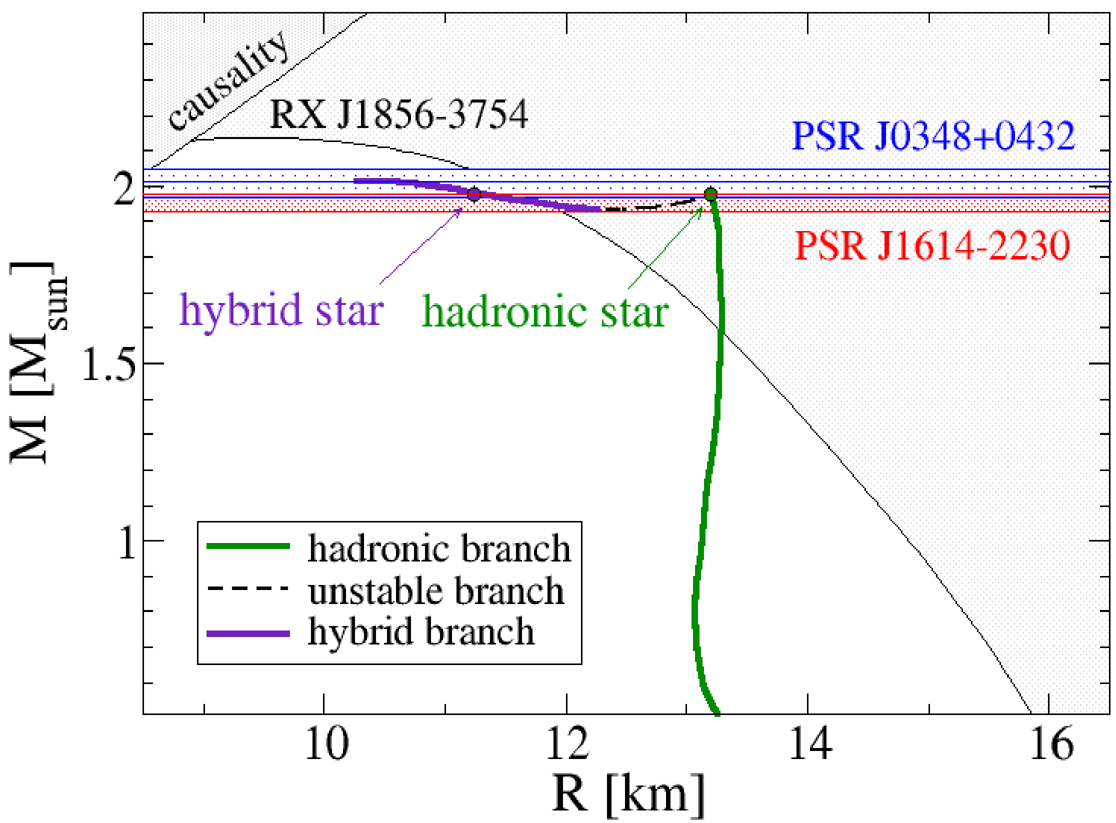
Systematic Classification [Alford, Han, Prakash: PRD88, 083013 (2013)]

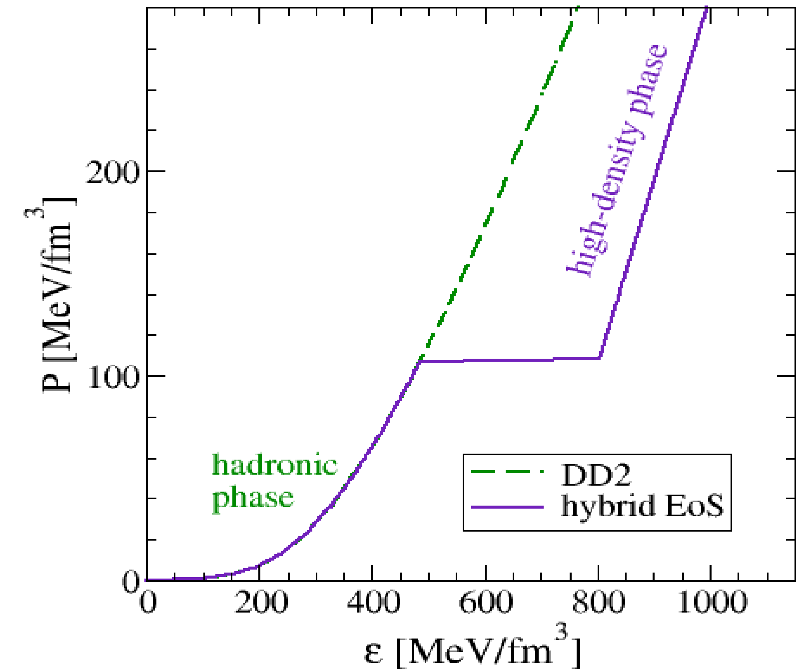
EoS P(ε) <--> Compact star phenomenology M(R)

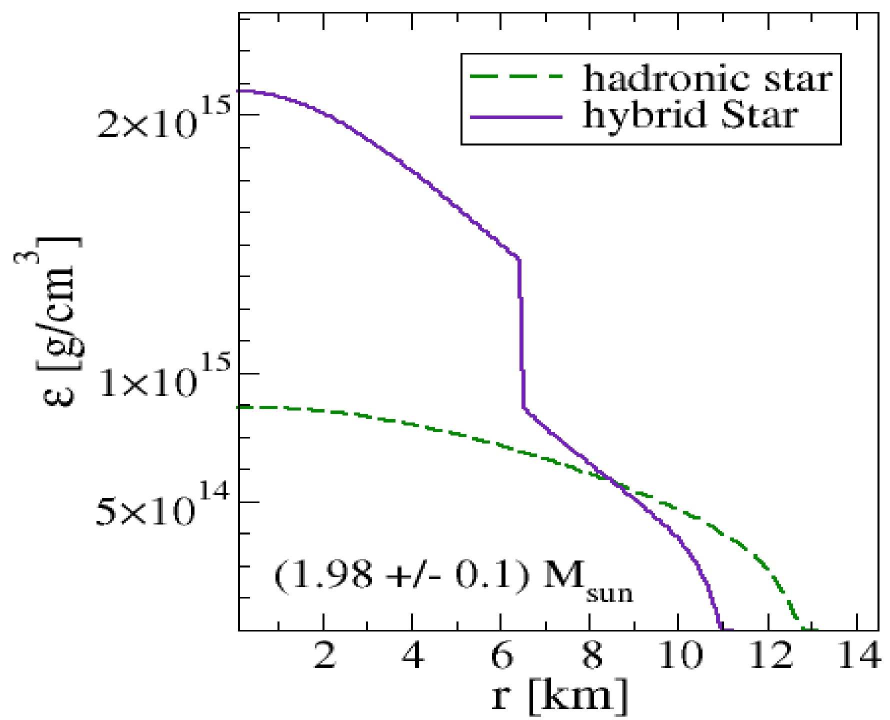
Most interesting and clear-cut cases: (D)isconnected and (B)oth – high-mass twins!

"Holy Grail" - High-Mass Twin Stars

Twins prove exitence of **disconnected populations** (third family) in the M-R diagram
Consequence of a **first order phase transition**
Question: Do twins prove the 1st order phase trans.?

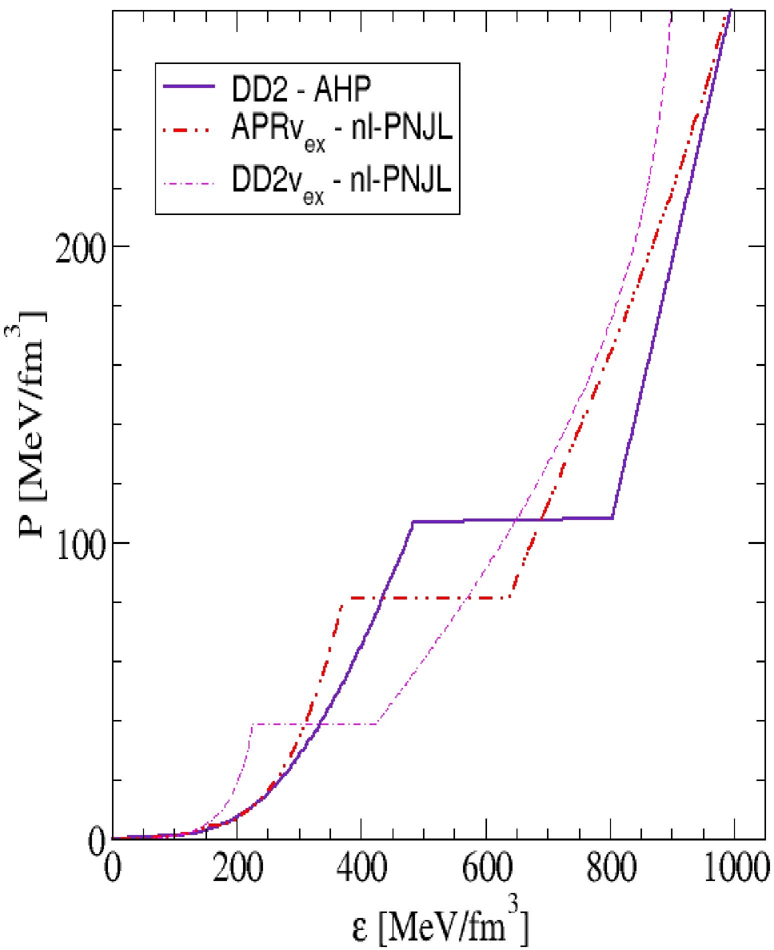


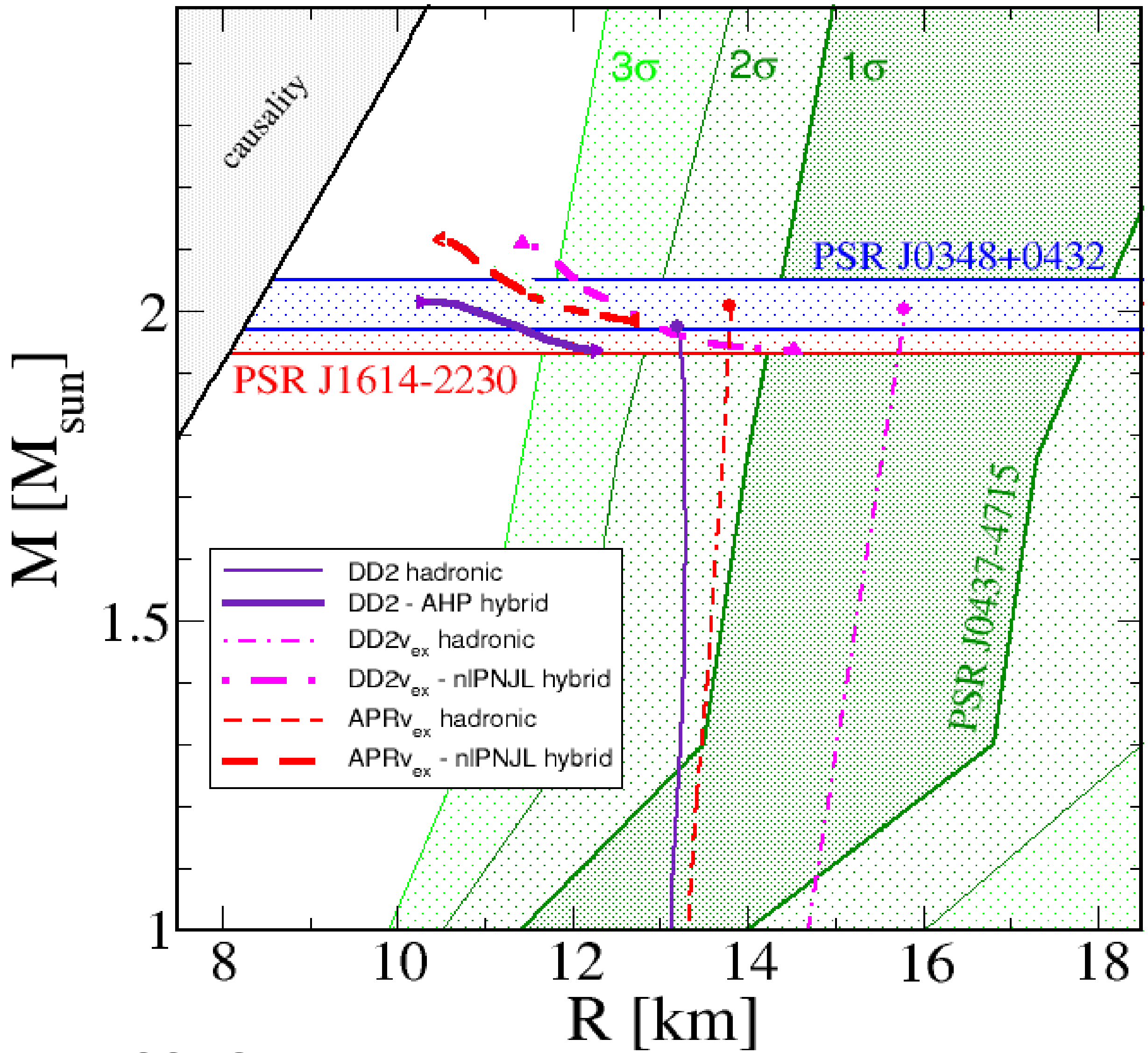




Alvarez & Blaschke, arxiv:1304.7758

High mass twins: more examples !





DB, Alvarez, Benic,
arxiv:1310.3803
Proceedings of CPOD 2013

MESSAGE:

- excluded volume (quark Pauli blocking) important
- high-density quark matter slightly stiffer eta_v=0.25
- the scaled energy density jump (0.65) fulfills the twin condition of the schematic model by Alford et al. (2013)

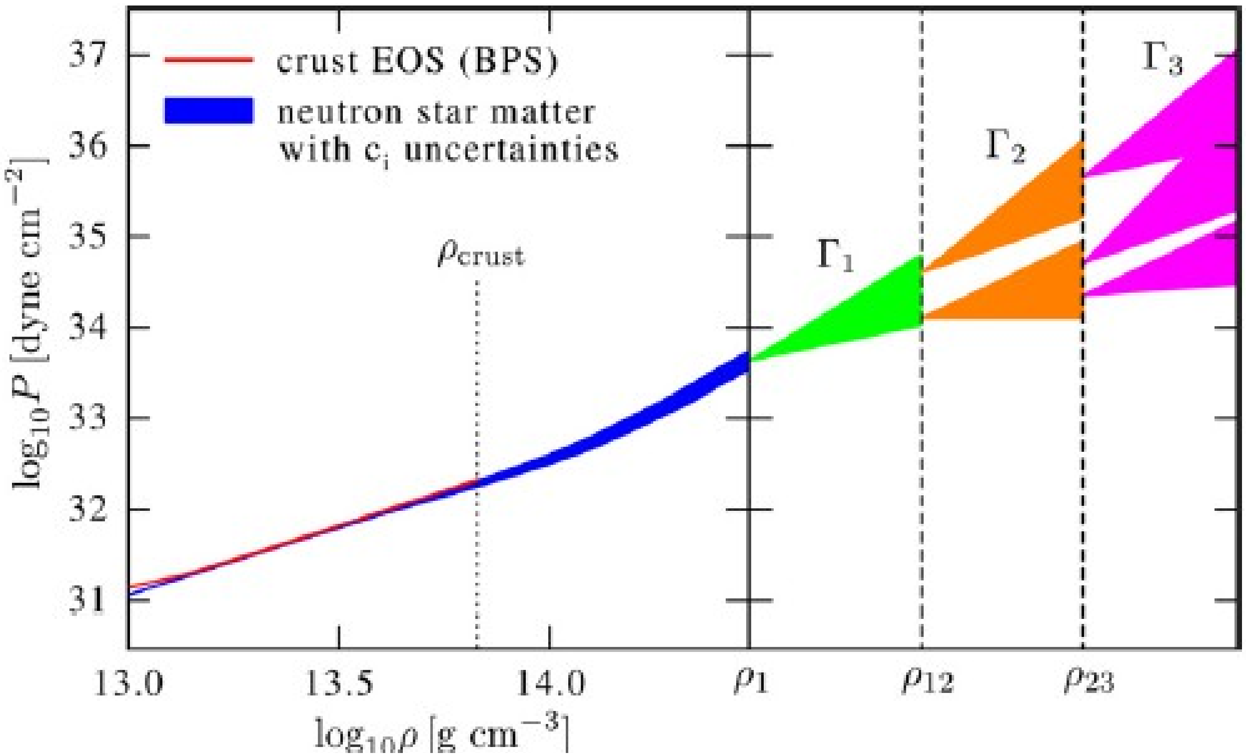
→ **Astronomers: Find disconnected star branches !!**

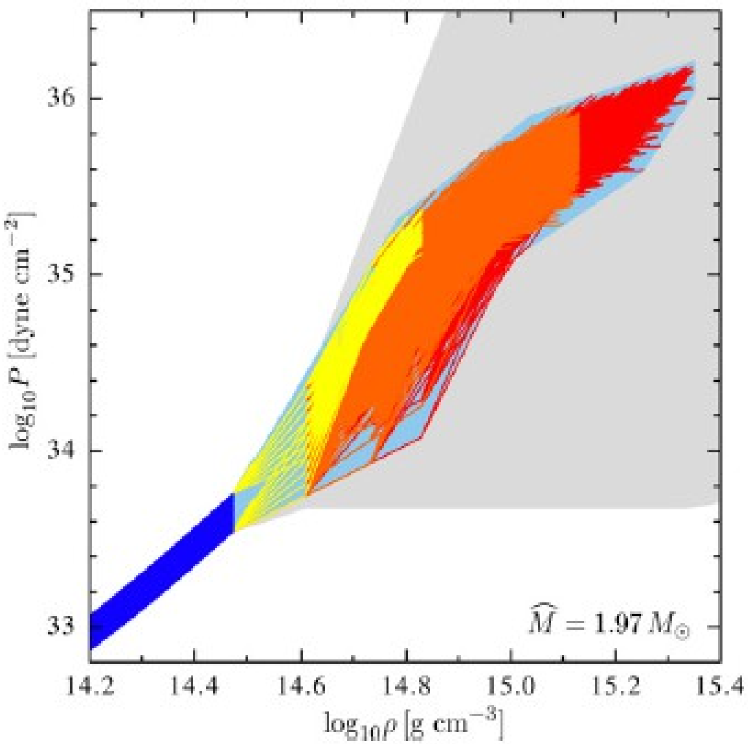
2. Piecewise polytrope EoS – high mass twins?

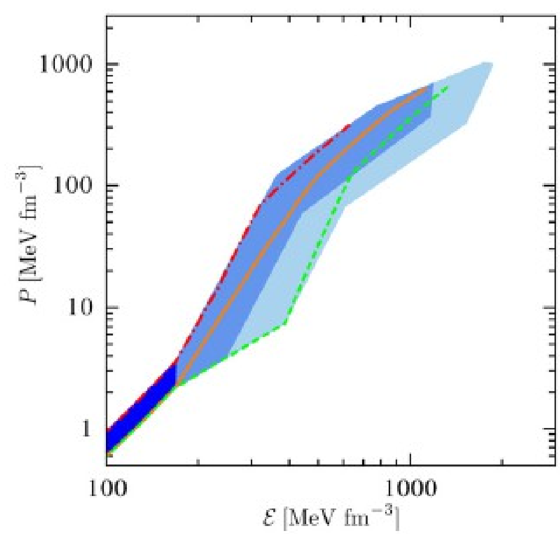
Hebeler et al., ApJ 773, 11 (2013)

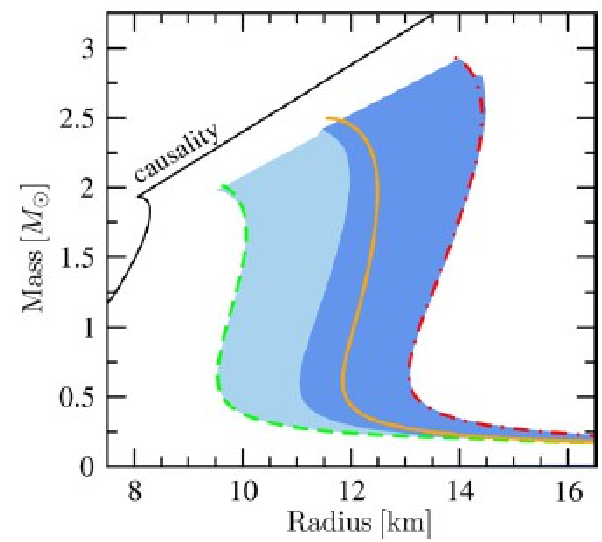
$$P_i(n) = \kappa_i n^{\Gamma_i}$$

$$i = 1 \; : \; n_1 \leq n \leq n_{12}$$
$$i = 2 \; : \; n_{12} \leq n \leq n_{23}$$
$$i = 3 \; : \; n \geq n_{23} \; ,$$









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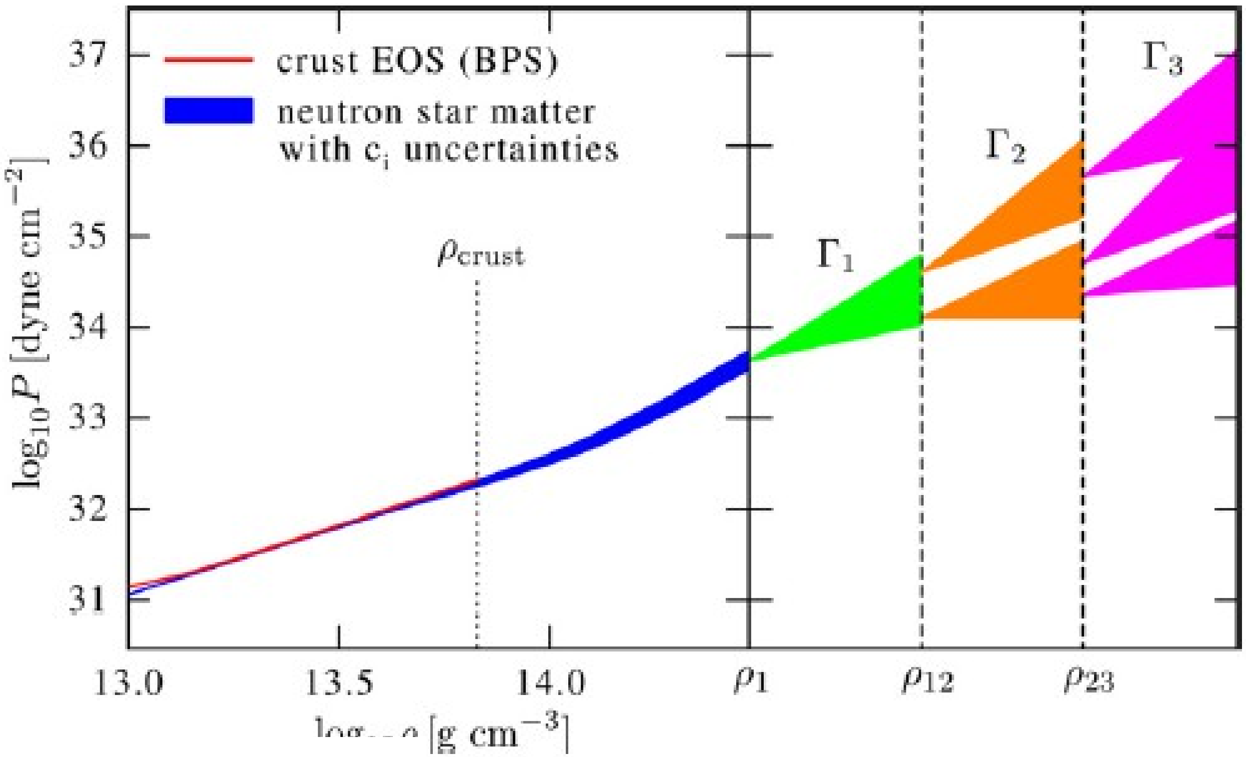


Here, 1[st] order PT in region 2:

$$\Gamma_2 = 0 \text{ and } P_2 = \kappa_2 = P_{crit}$$

$$P(n) = n^2 \frac{d(\varepsilon(n)/n)}{dn},$$

$$\varepsilon(n)/n = \int dn \frac{P(n)}{n^2} = \int dn\, \kappa n^{\Gamma-2} = \frac{\kappa\, n^{\Gamma-1}}{\Gamma-1} + C,$$

$$\mu(n) = \frac{P(n)+\varepsilon(n)}{n} = \frac{\kappa\Gamma}{\Gamma-1} n^{\Gamma-1} + m_0,$$

$$n(\mu) = \left[(\mu - m_0)\frac{\Gamma-1}{\kappa\Gamma}\right]^{1/(\Gamma-1)}$$

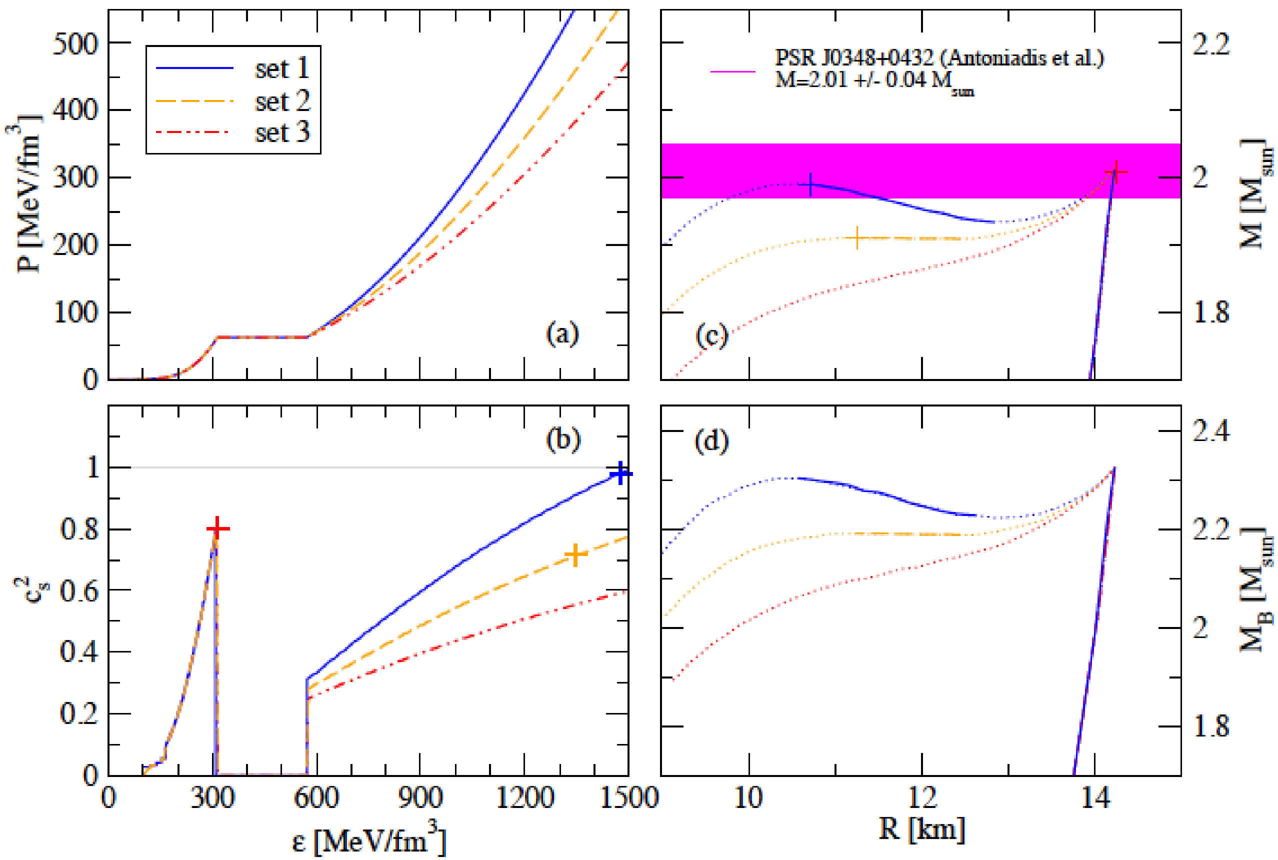
$$P(\mu) = \kappa\left[(\mu - m_0)\frac{\Gamma-1}{\kappa\Gamma}\right]^{\Gamma/(\Gamma-1)}$$

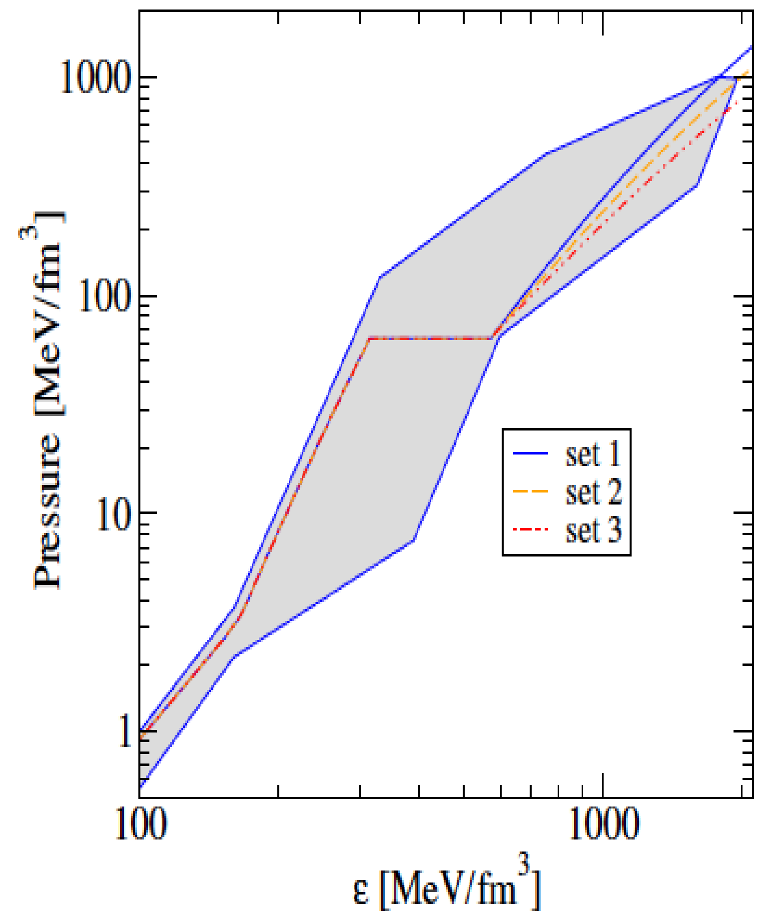
Maxwell construction:

$$P_1(\mu_{crit}) = P_3(\mu_{crit}) = P_{crit}$$
$$\mu_{crit} = \mu_1(n_{12}) = \mu_3(n_{23})$$

Seidov criterion for instability: $$\frac{\Delta\varepsilon}{\varepsilon_{crit}} \geq \frac{1}{2} + \frac{3}{3}\frac{P_{crit}}{\varepsilon_{crit}}$$

2. Piecewise polytrope EoS – high mass twins?





	Γ_3	κ_3 [MeV fm$^{3(\Gamma_3-1)}$]	$m_{0,3}$ [MeV]	M^{NS}_{max} [$M_\odot$]	M^{HS}_{max} [$M_\odot$]	M^{HS}_{min} [$M_\odot$]
set 1	3.12	447.16	1014.87	2.01	1.991	1.934
set 2	2.80	365.12	1004.88	2.01	1.910	1.909
set 3	2.50	302.56	991.75	2.01	-	-

Set with same onset of Phase transition:
P_crit = 68.18 MeV/fm^3
eps_crit = 318.26 MeV/fm^3
Δeps = 253.89 MeV/fm^3
n_12 = 0.32 fm^-3 ; n_23 = 0.53 fm^-3

Third family solutions in the 2M_sun mass range (HMT) exist !!

[D. Alvarez-Castillo & D.B. arxiv:1703.02681]

3. Density functional approach to quark matter

[M.A. Kaltenborn, N.-U. Bastian & D.B., arxiv:1701.04400]

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int} = \bar{q}\left(i\partial\!\!\!/ - m_0\right) q + \mu_0\, \bar{q}\gamma^0 q + U\left(\bar{q}q, \bar{q}\gamma_0 q\right)$$

$$U\left(\bar{q}q, \bar{q}\gamma_0 q\right) = U\left(\langle\bar{q}q\rangle, \langle\bar{q}\gamma_0 q\rangle\right) + \left.\frac{\partial U\left(\bar{q}q, \bar{q}\gamma_0 q\right)}{\partial \bar{q}q}\right|_{\substack{\bar{q}q=\langle\bar{q}q\rangle \\ \bar{q}\gamma_0 q=\langle\bar{q}\gamma_0 q\rangle}} \left(\bar{q}q - \langle\bar{q}q\rangle\right) + \left.\frac{\partial U\left(\bar{q}q, \bar{q}\gamma_0 q\right)}{\partial \bar{q}\gamma_0 q}\right|_{\substack{\bar{q}q=\langle\bar{q}q\rangle \\ \bar{q}\gamma_0 q=\langle\bar{q}\gamma_0 q\rangle}} \left(\bar{q}\gamma_0 q - \langle\bar{q}\gamma_0 q\rangle\right) + \dots$$

$$\mathcal{L}_{\rm eff} \approx \mathcal{L}_{\rm free} + U + U_s \bar{q}q - U_s\langle\bar{q}q\rangle + U_v \bar{q}\gamma_0 q - U_v\langle\bar{q}\gamma_0 q\rangle$$

$$\bar{n}_s = \langle\bar{q}q\rangle \quad , \quad \bar{n}_v = \langle\bar{q}\gamma^0 q\rangle \; , \quad \left.\frac{\partial U\left(\bar{q}q, \bar{q}\gamma_0 q\right)}{\partial \bar{q}q, \bar{q}\gamma_0 q}\right|_{\substack{\bar{q}q=\langle\bar{q}q\rangle \\ \bar{q}\gamma_0 q=\langle\bar{q}\gamma_0 q\rangle}} = U_{s,v}$$

$$P = -\left.\frac{\partial\Omega}{\partial V}\right|_{\mu,T} = g\int\frac{d^3p}{(2\pi)^3}\left[T\ln(1+e^{-\beta(E-\mu^*)}) + T\ln(1+e^{-\beta(E+\mu^*)})\right] + \Theta \qquad E = \sqrt{\vec{p}^2 + m^{*2}}.$$

$$m_i^* = m_{0,i} - \Sigma_{s,i}(\bar{n}_s) = m_{0,i} + \underbrace{D(\bar{n}_s)\bar{n}_s^{-\frac{1}{3}}}_{\text{confinement}} \qquad D(\bar{n}_s, \bar{n}_v) = D_0\Phi(\bar{n}_s, \bar{n}_v)$$

Available volume fraction:

$$\Phi(\bar{n}_s, \bar{n}_v) = e^{-\alpha(\bar{n}_v - n_0)^2}$$

$$\mu_i^*(\bar{n}_v) = \mu_{0,i} - \Sigma_{v,i}(\bar{n}_v) \qquad \Sigma_{v,i}(\bar{n}_v) = a\bar{n}_v + \frac{b\bar{n}_v^3}{1 + c\bar{n}_v^2}$$

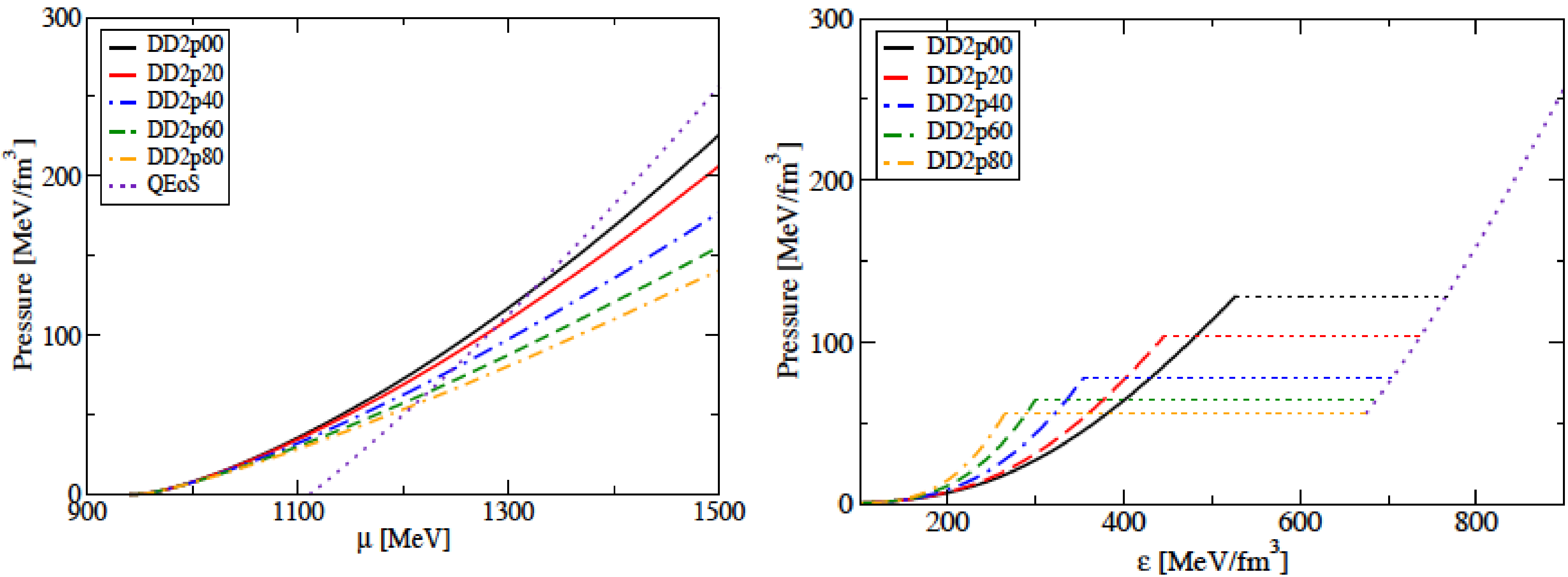
Thermodynamic consistency -->
Rearrangement selfenergies

3. Density functional approach to quark matter

Hadronic matter: DD2 with excluded volume **[S. Typel, EPJA 52 (3) (2016)]**

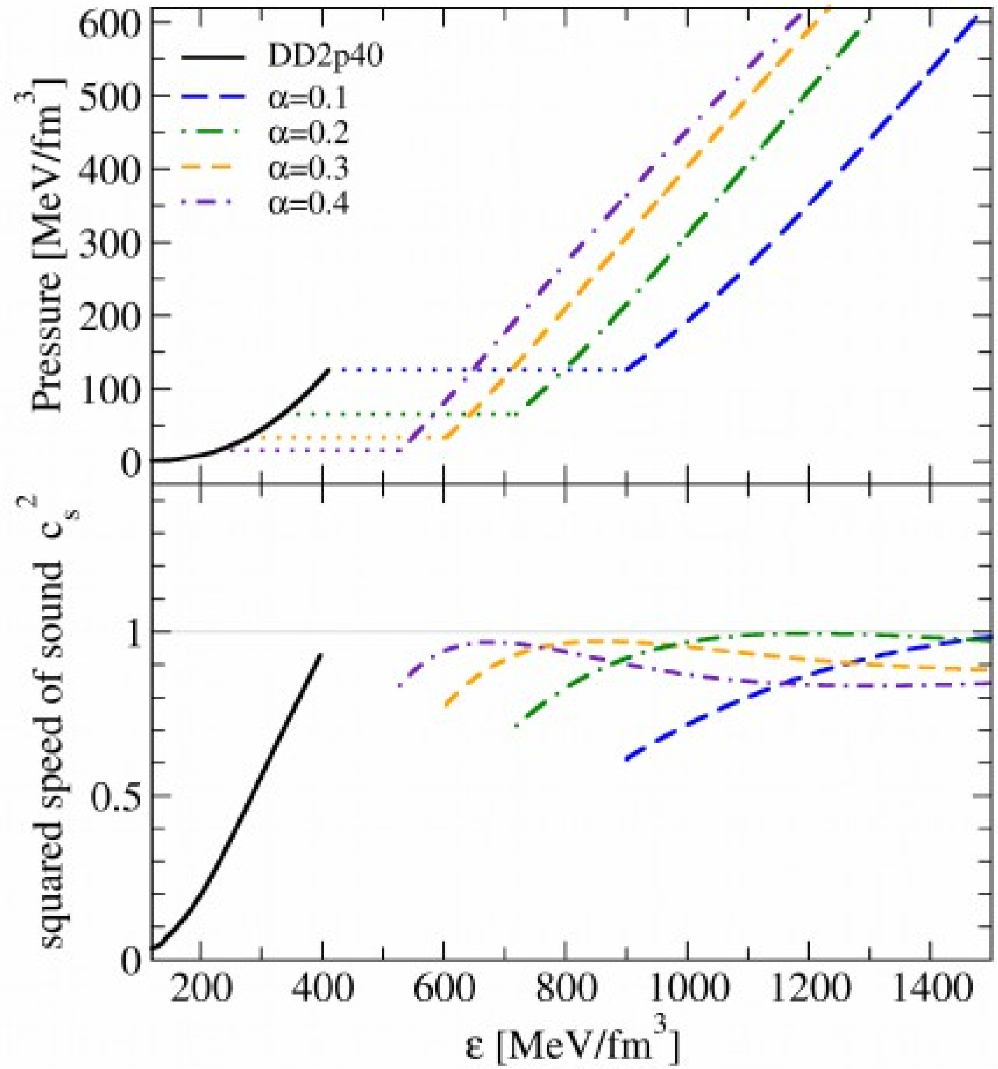
$$\Phi_n = \Phi_p = \begin{cases} 1, & \text{if } n_B < n_0 \\ e^{-\frac{v|v|}{2}(n_B - n_0)^2}, & \text{if } n_B > n_0 \end{cases}$$

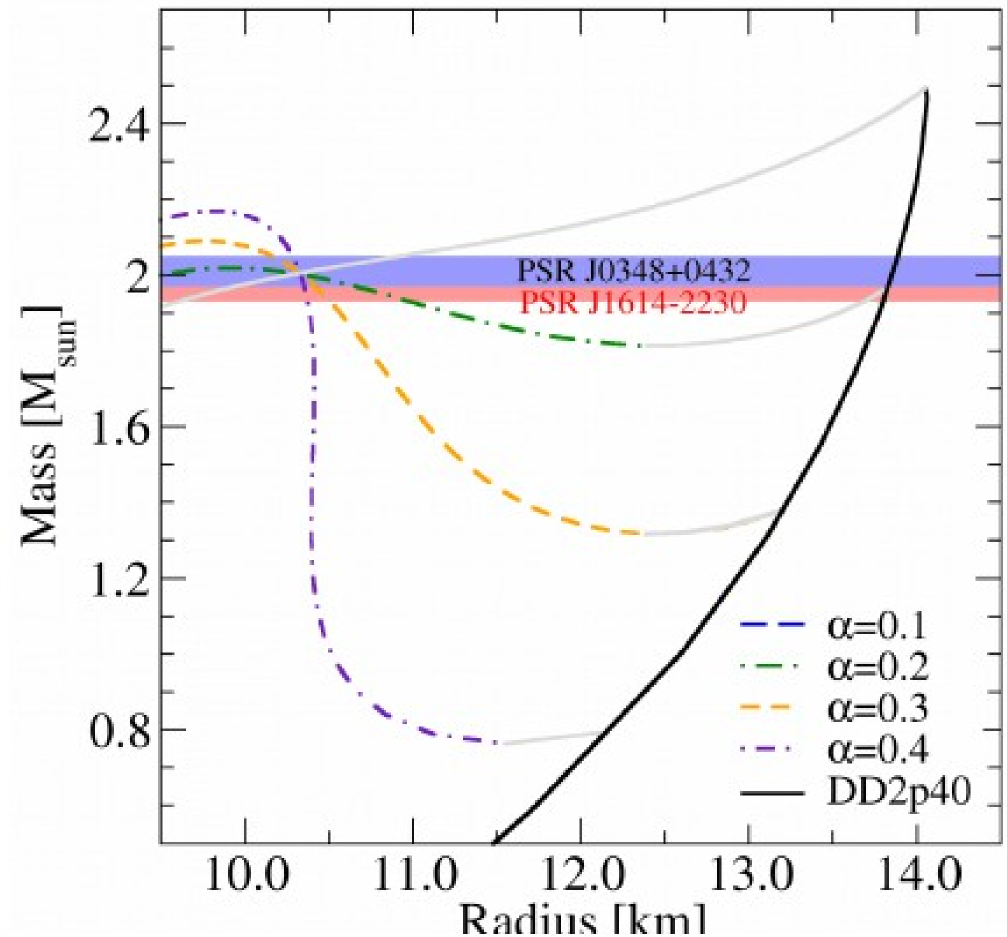
Varying the hadronic excluded volume parameter, p00 → v=0, … , p80 → v=8 fm^3



Hybrid EOS - parameters

$\underline{\alpha}, a, b$





Kaltenborn, Bastian, Blaschke, arXiv:1701.04400

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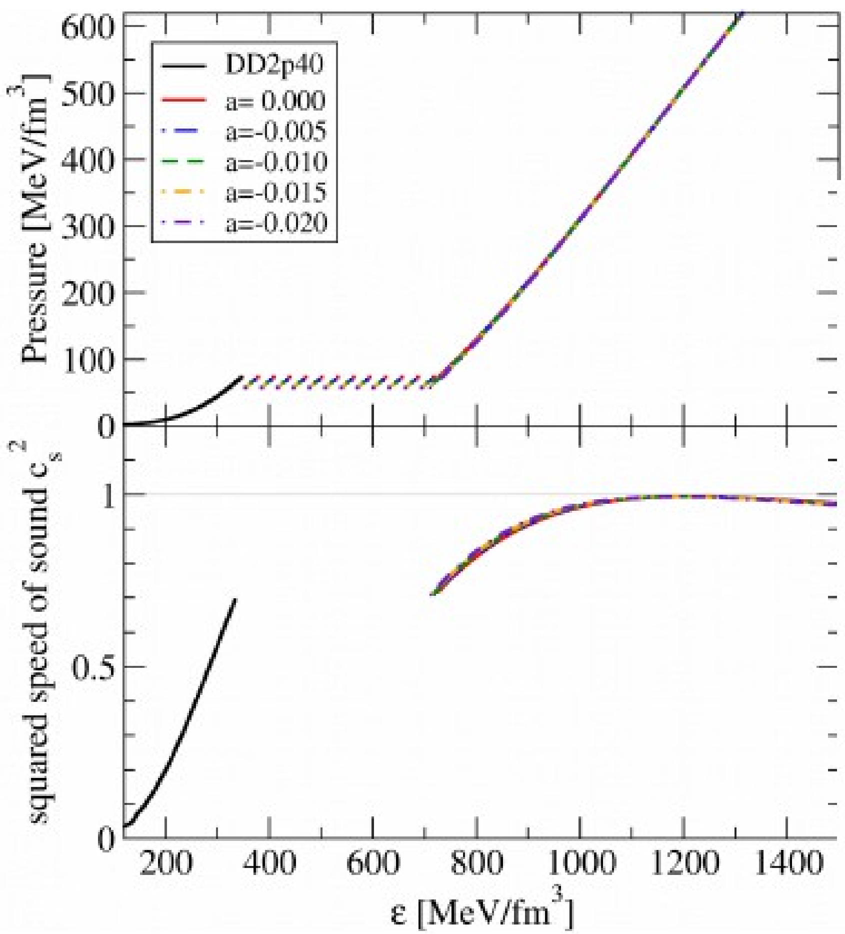

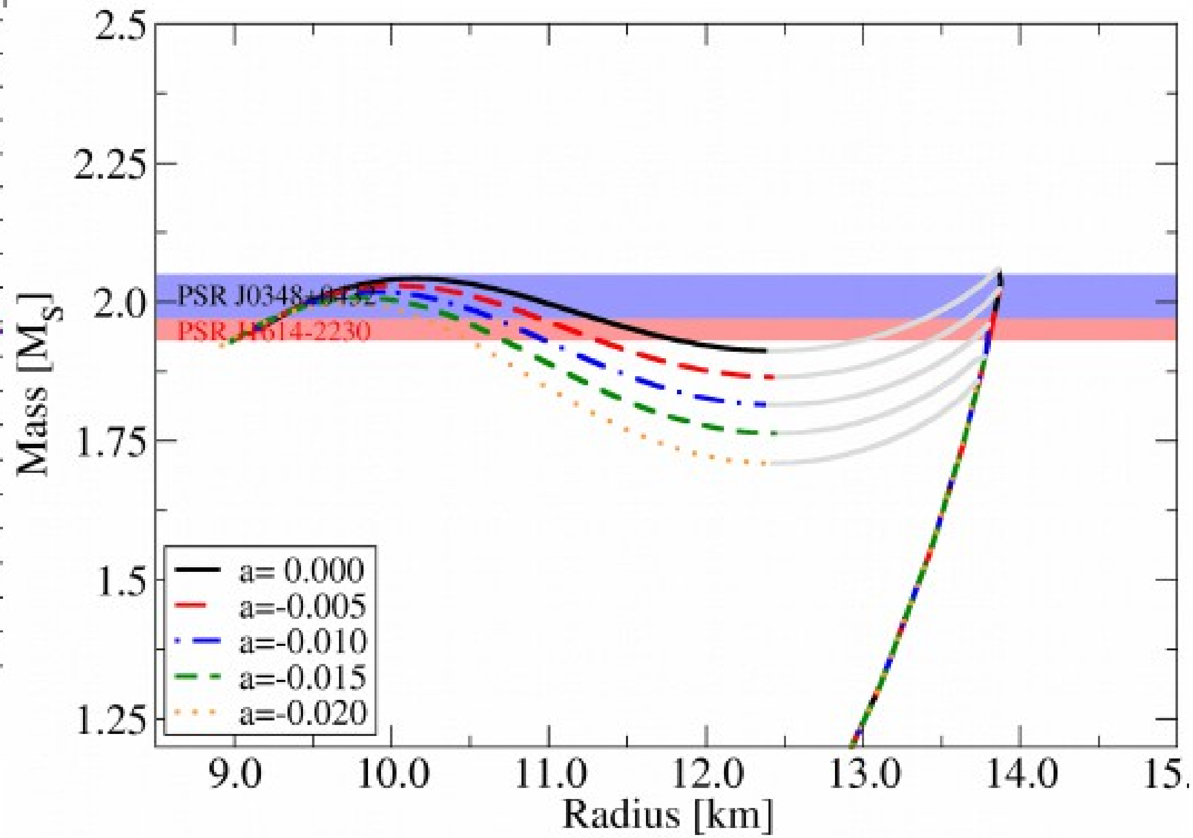

Kaltenborn, Bastian, Blaschke, arXiv:1701.04400

Hybrid EOS - parameters

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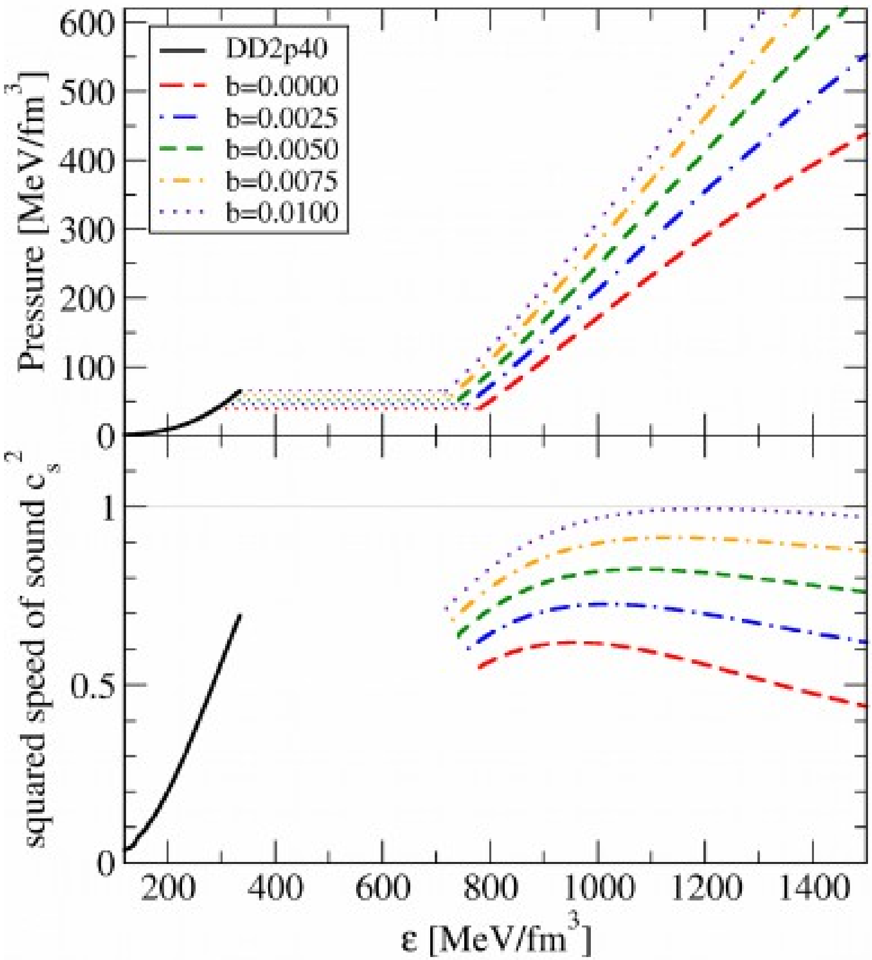
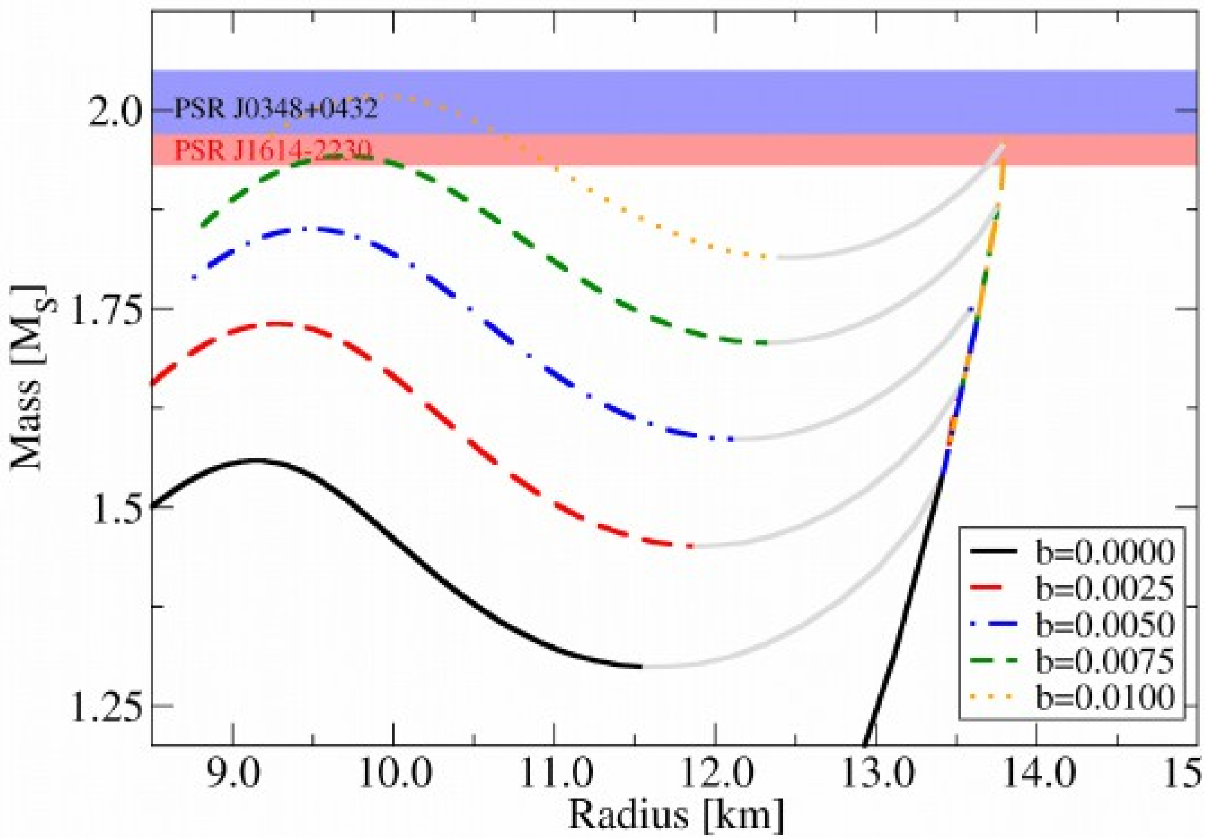
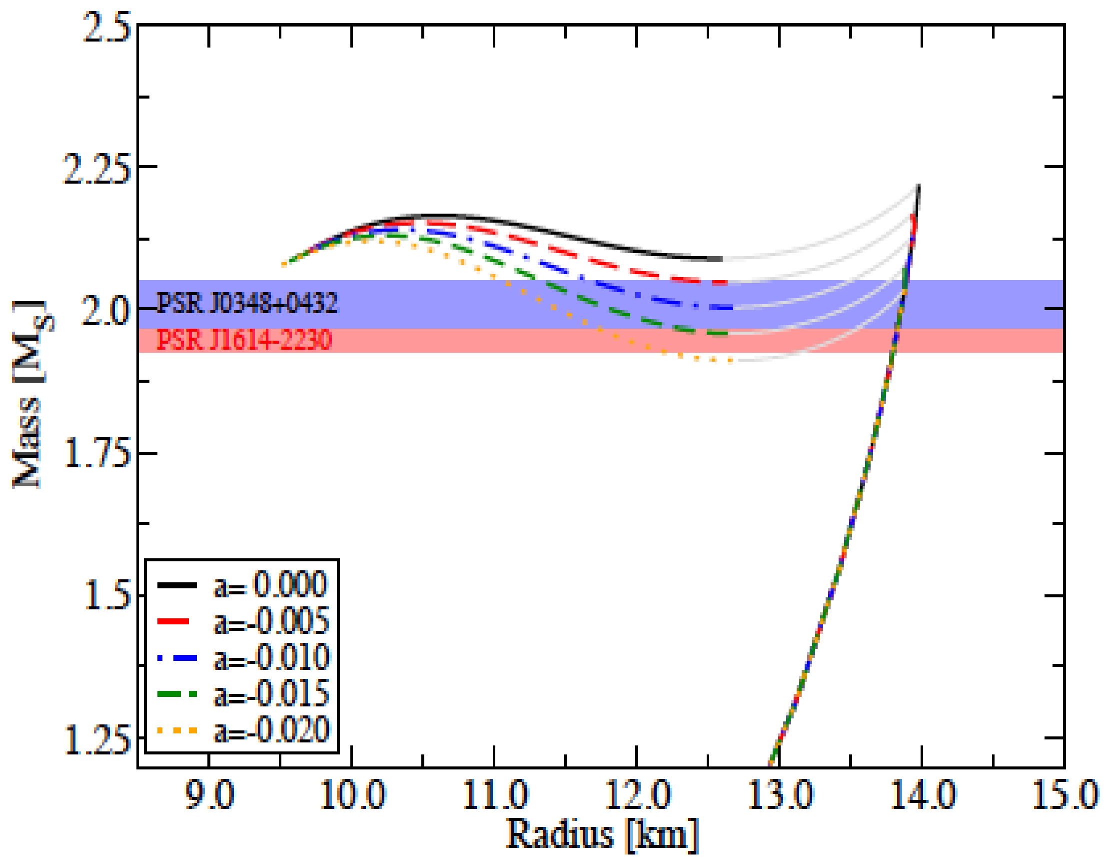


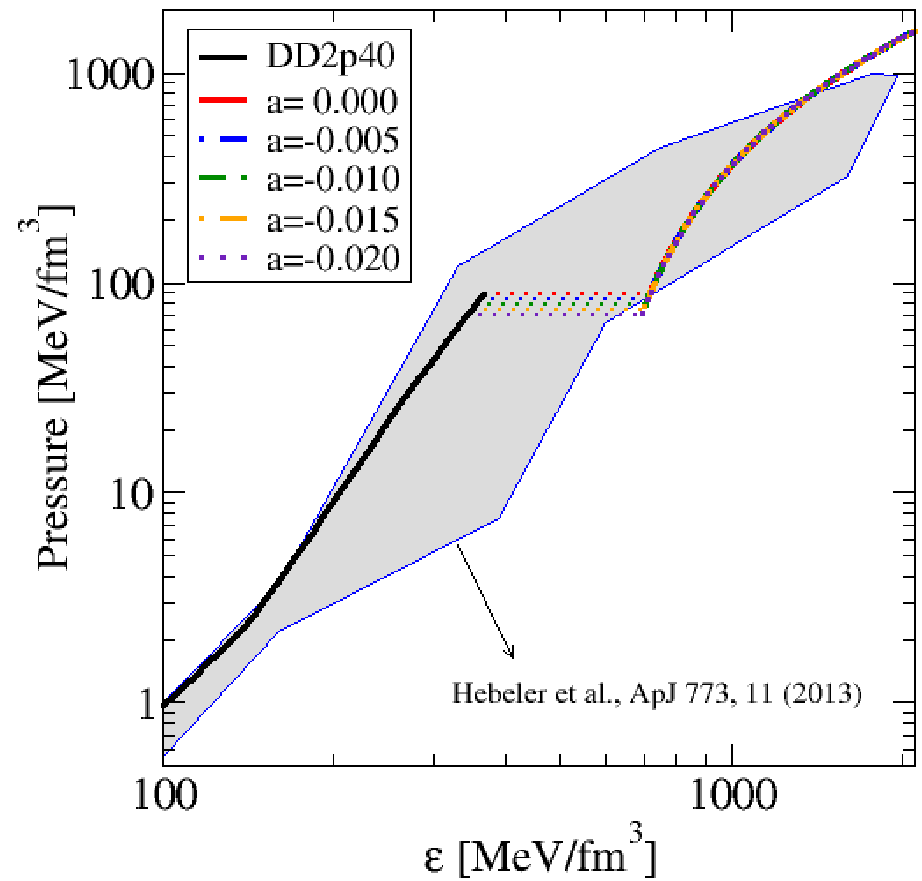


Kaltenborn, Bastian, Blaschke, arXiv:1701.04400

3. Density functional approach to quark matter

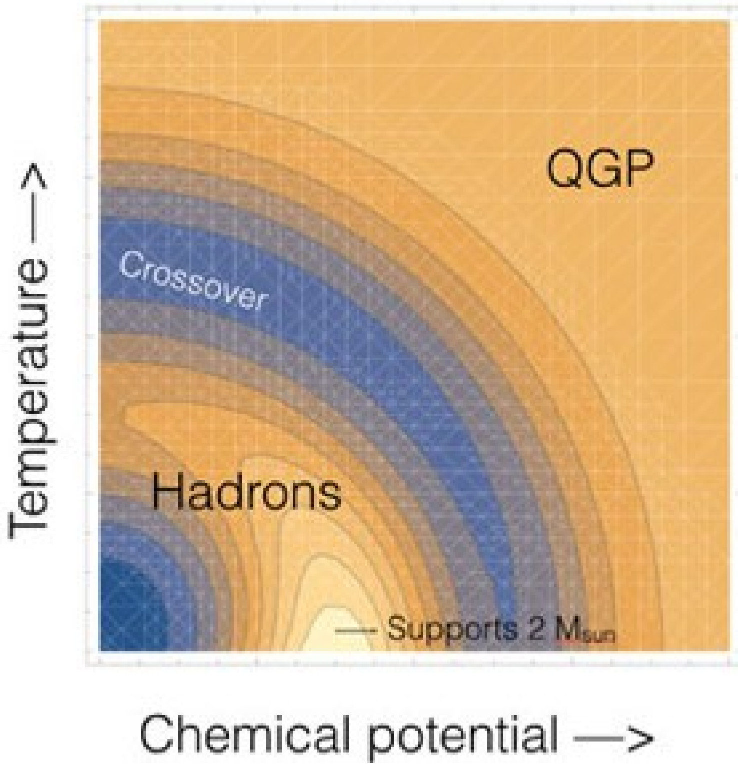
Varying the 4-quark coupling parameter a

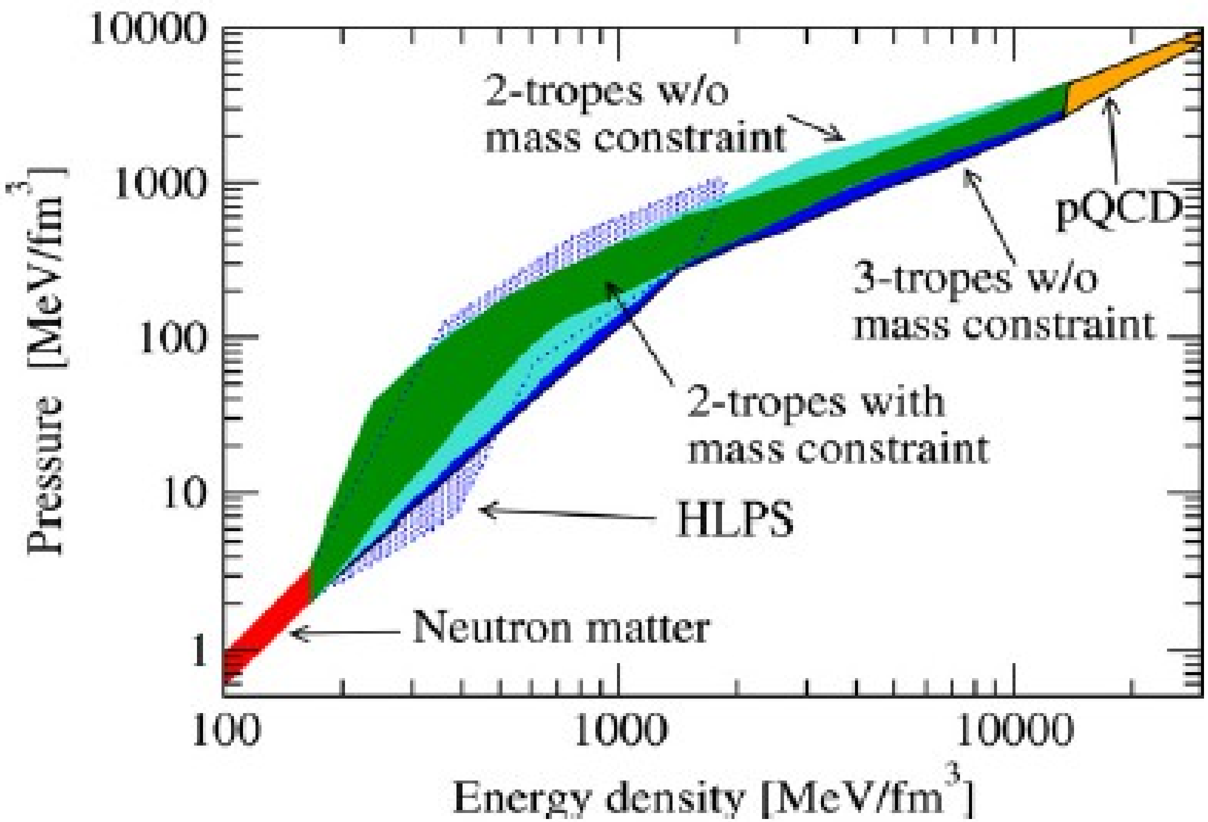




[M.A. Kaltenborn, N.-U. Bastian & D.B., arxiv:1701.04400]

Towards “measuring” the EoS in the T – mu plane (QCD phase diagram)



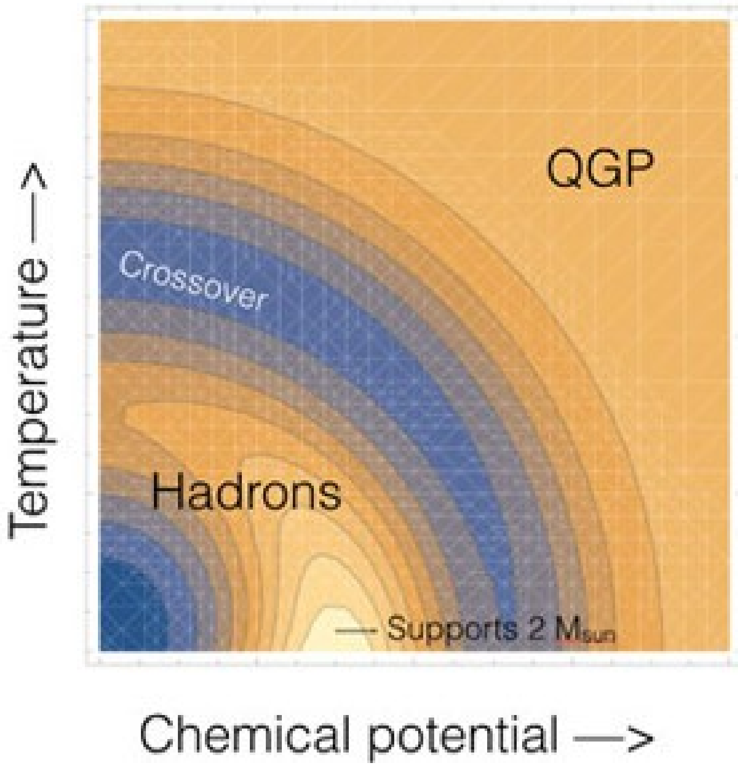


Speed-of-sound diagram from the INT program in Seattle, Summer 2016

Interpolation between lattice QCD and Compact star physics (2 M_sun)

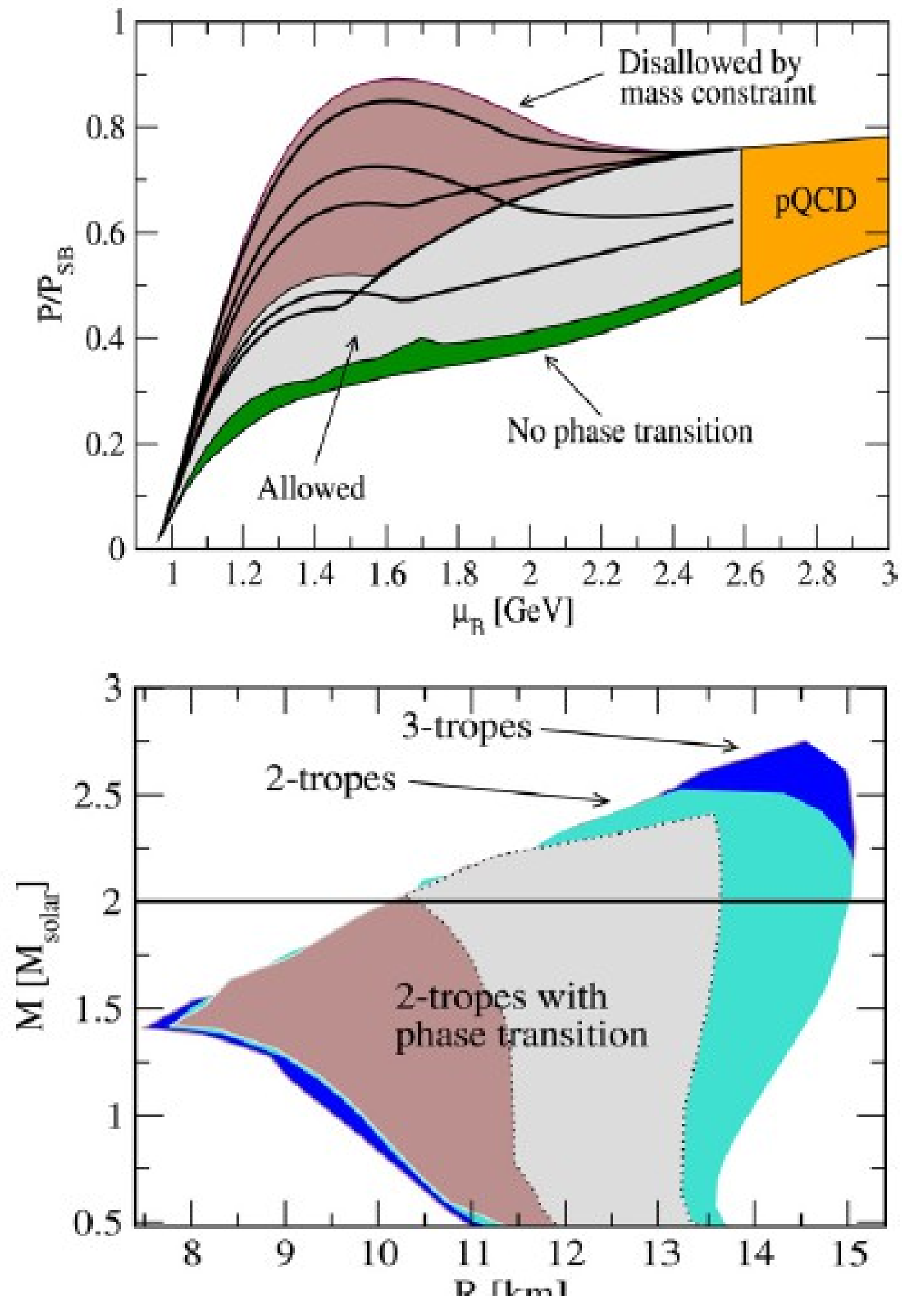
A. Kurkela, E. Fraga, J. Schaffner-Bielich, A. Vuorinen, Astrophys. J. 789 (2014) 127

Towards “measuring” the EoS in the T – mu plane (QCD phase diagram)



Speed-of-sound diagram from the INT program in Seattle, Summer 2016

Interpolation between lattice QCD and Compact star physics (2 M_sun)

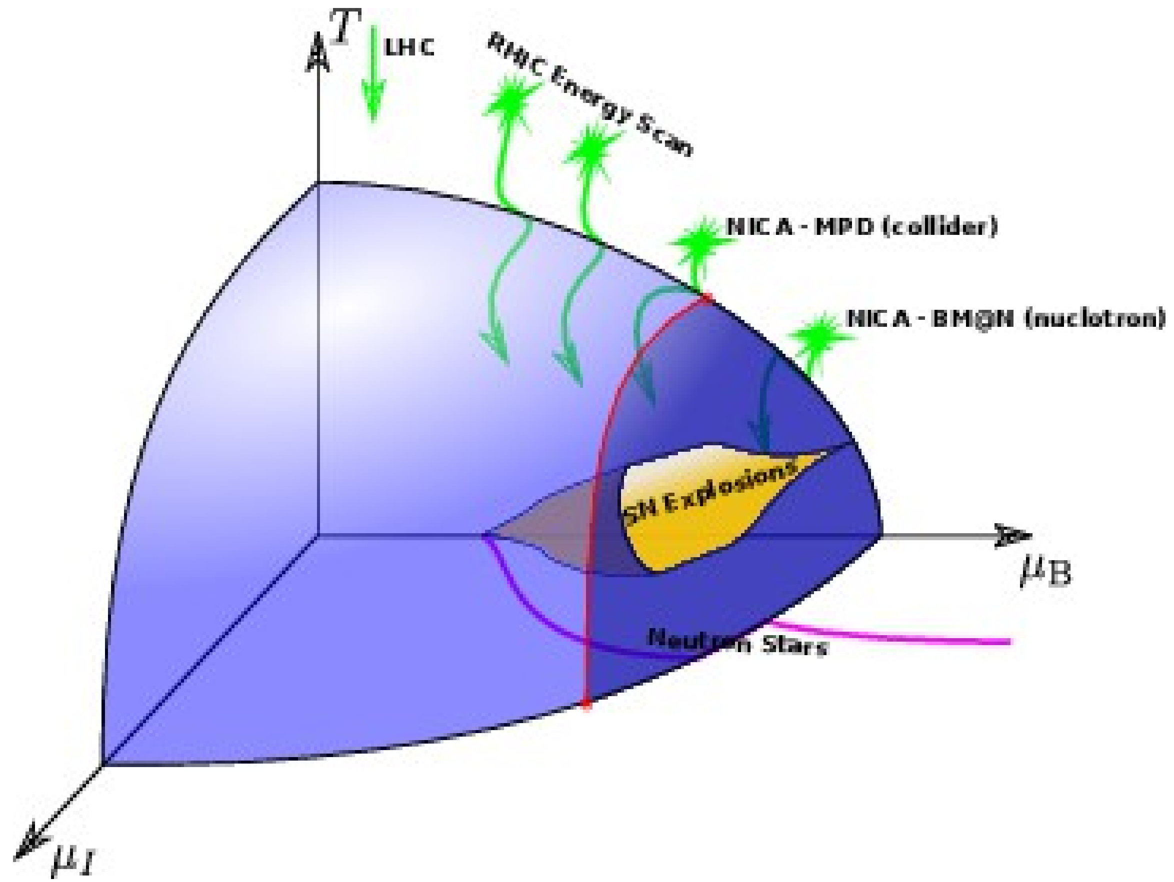


A. Kurkela, E. Fraga, J. Schaffner-Bielich, A. Vuorinen, Astrophys. J. 789 (2014) 127

Conclusion:

High-mass twins (HMTs) with quark matter cores can be obtained within different hybrid star EoS models, e.g.,
- constant speed of sound
- higher order NJL
- piecewise polytrope
- density functional

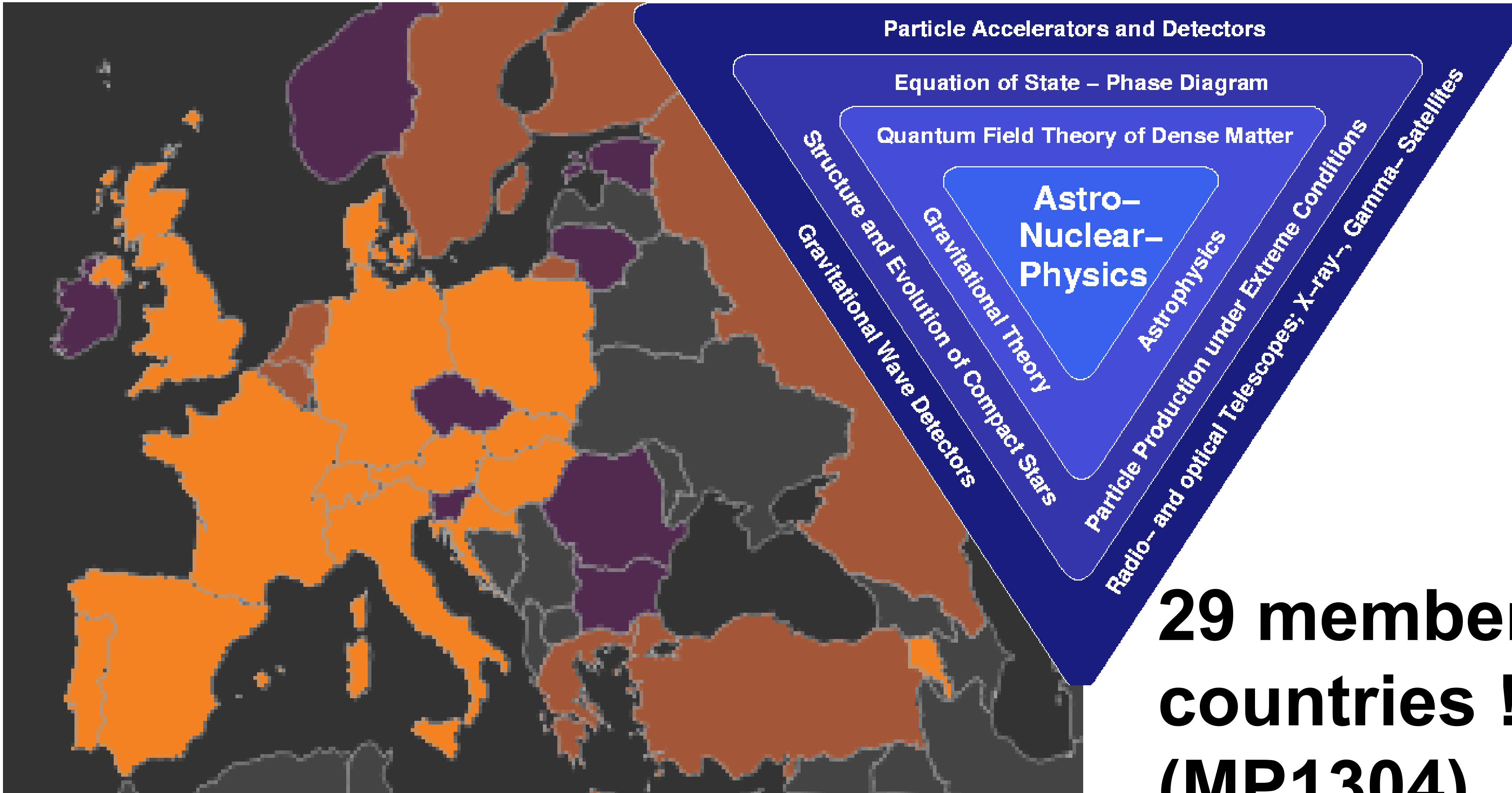
HMTs require stiff hadronic and quark matter EoS with a strong phase transition (PT)



Existence of HMTs can be verified, e.g., by precise compact star mass and radius observations (and a bit of good luck) → Indicator for strong PT !!

Extremely interesting scenarios possible for dynamical evolution of isolated (spin-down and accretion) and binary (NS-NS merger) compact stars

Critical endpoint search in the QCD phase diagram with Heavy-Ion Collisions goes well together with Compact Star Astrophysics

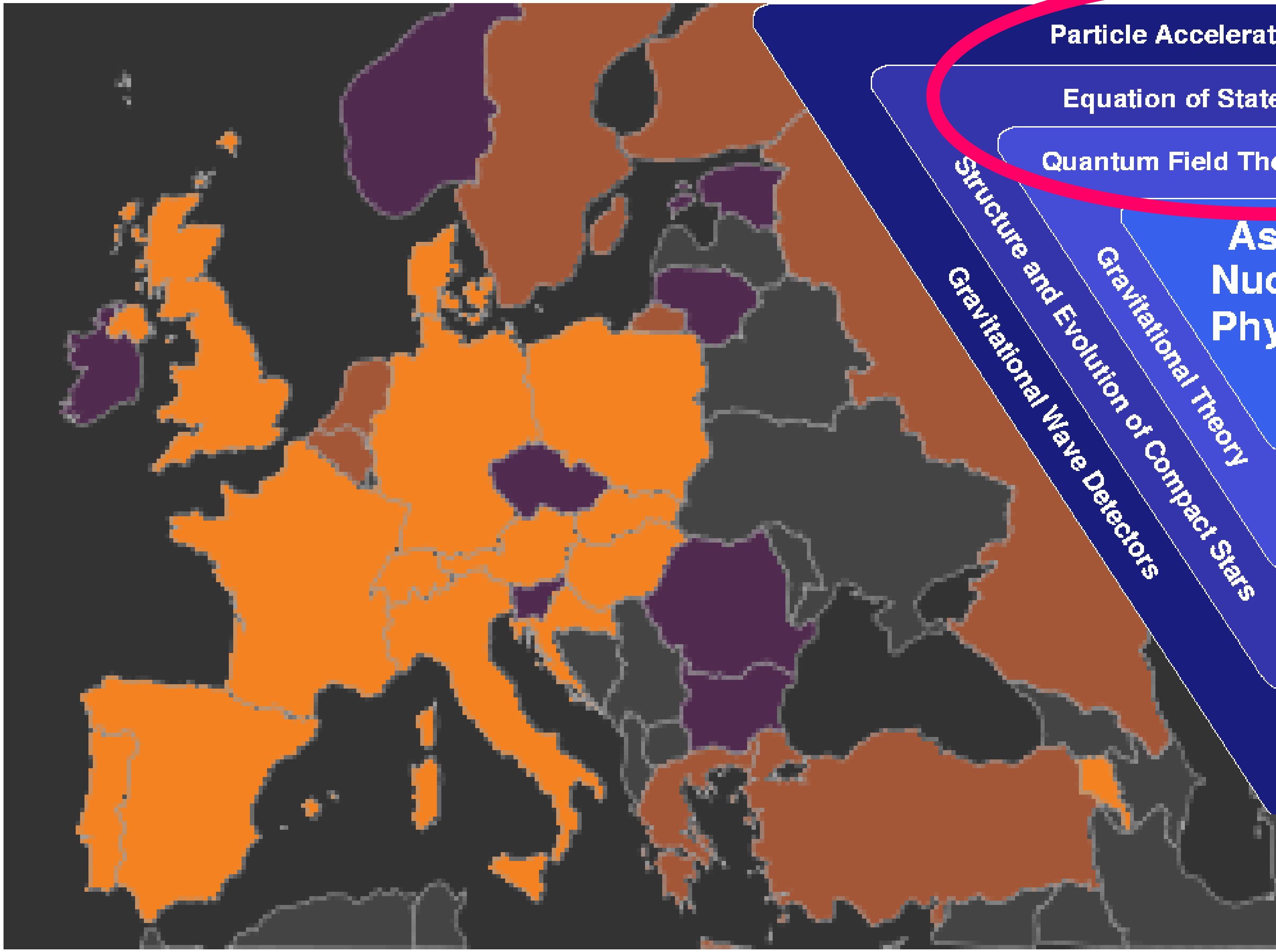


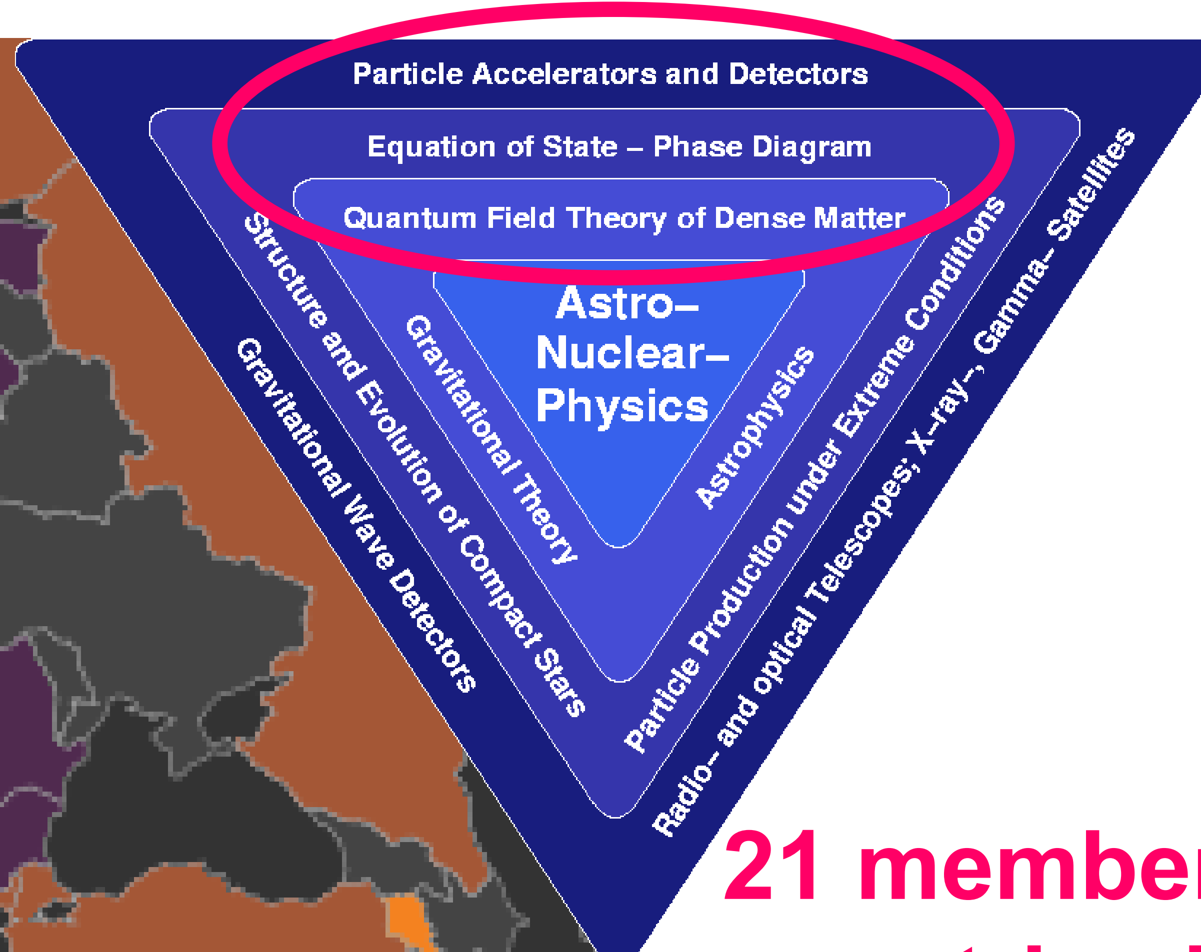
29 member countries !! (MP1304)

New compstar **!**

cost
EUROPEAN COOPERATION
IN SCIENCE AND TECHNOLOGY

Kick-off: Brussels, November 25, 2013





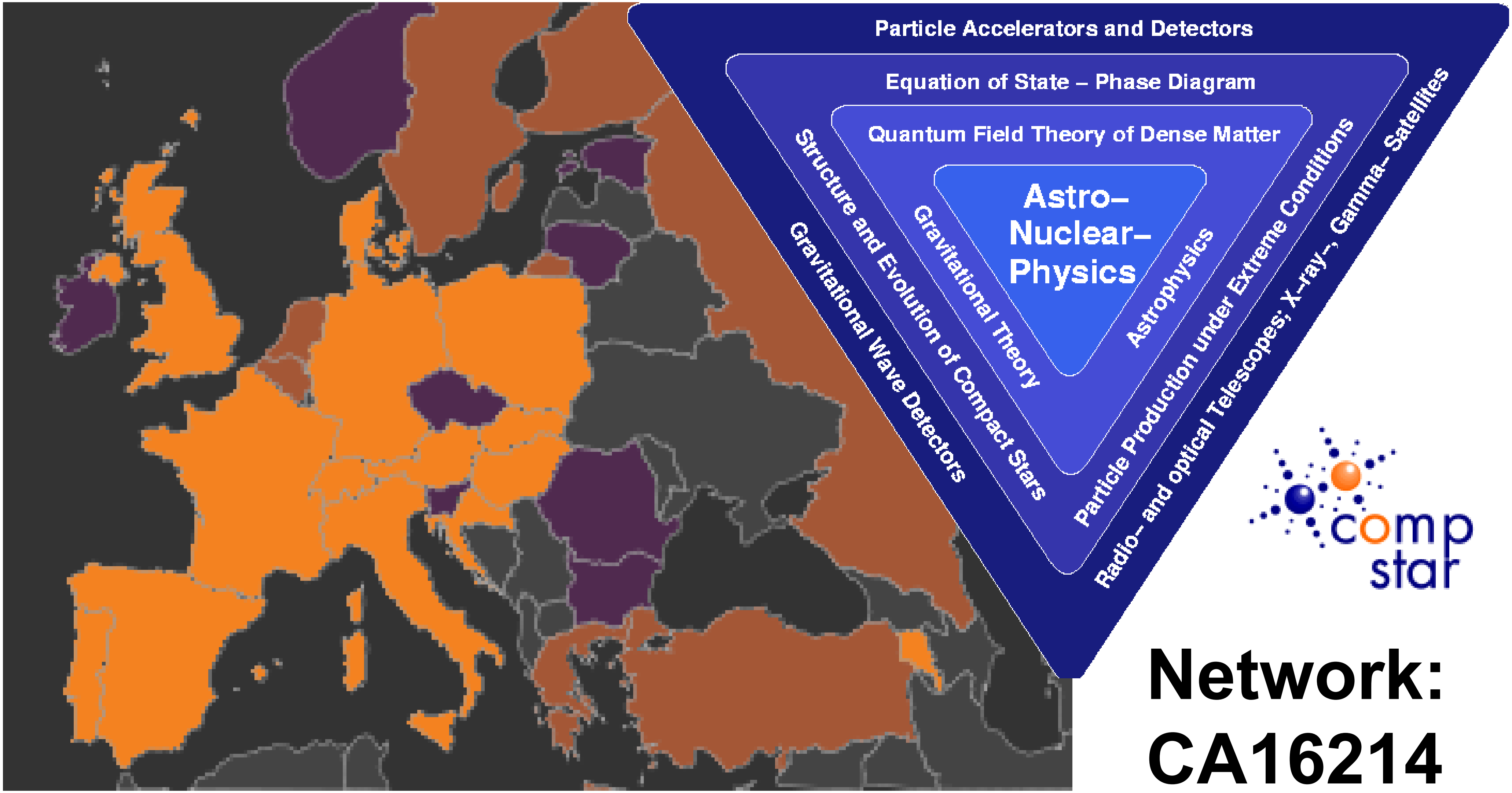
21 member countries ! (CA15213)

"Theory of HOt Matter in Relativistic Heavy-Ion Collisions"

New: THOR !

EUROPEAN COOPERATION
IN SCIENCE AND TECHNOLOGY

Kick-off: Brussels, October 17, 2016



comp star

Network: CA16214

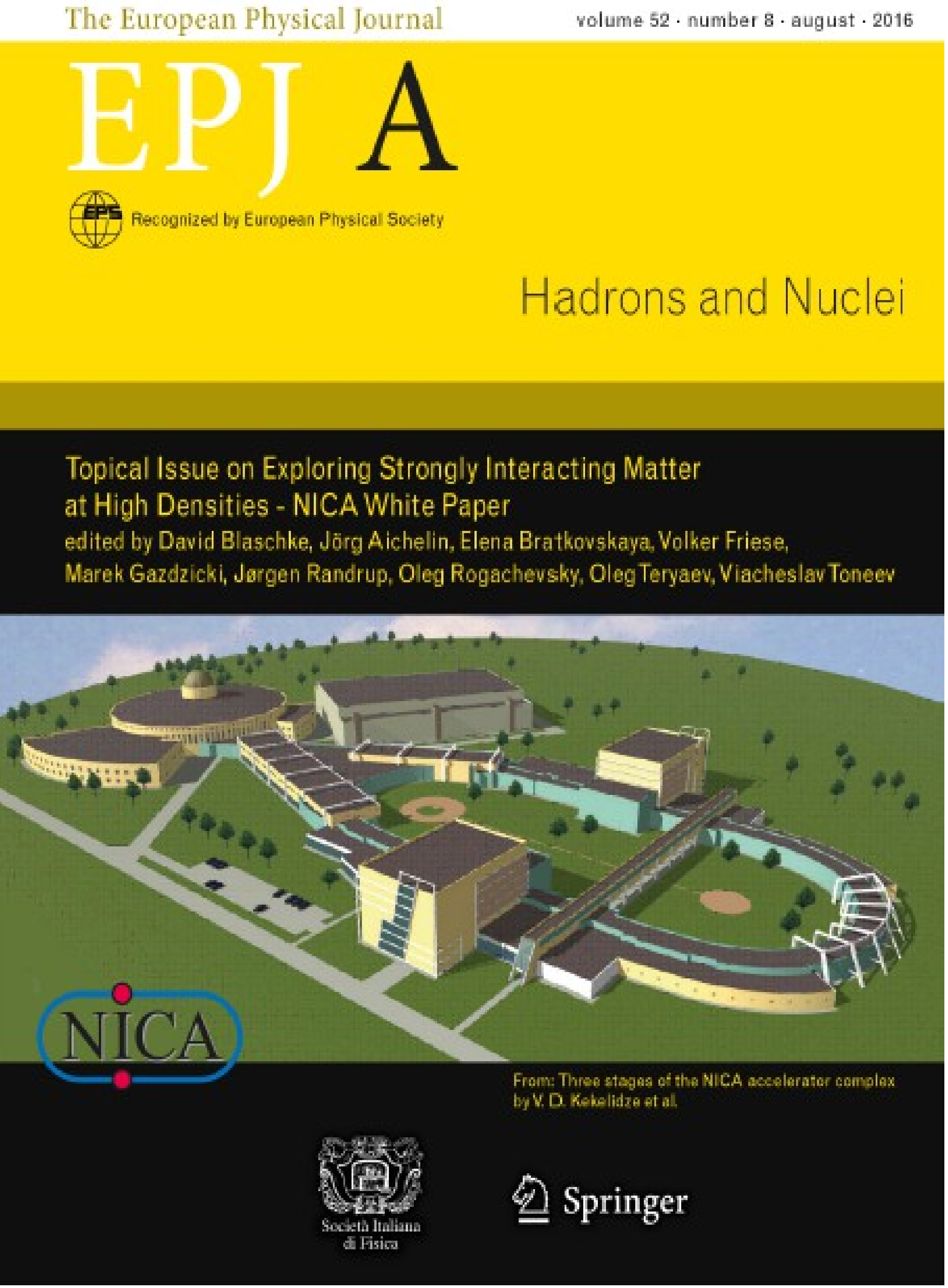
Newest:

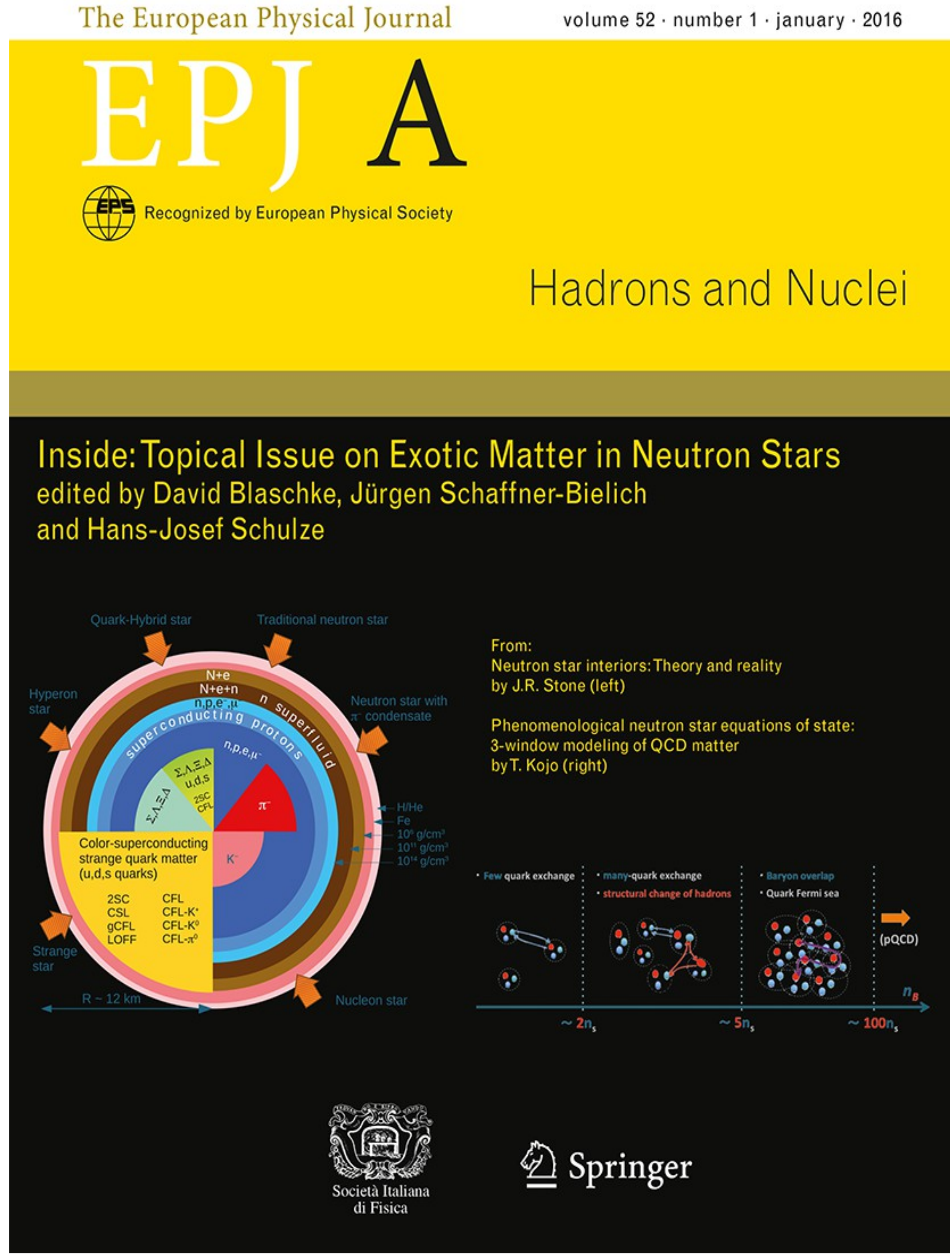
PHAROS

cost
EUROPEAN COOPERATION
IN SCIENCE AND TECHNOLOGY

Accepted; Kick-off: Brussels, late 2017

International Conference “Critical Point and Onset of Deconfinement”
University of Wroclaw, May 29 – June 4, 2016





EPJA Topical Issues can be found at → http://epja.epj.org/component/list/?task=topic

Solution of tasks

Exercise: Calculation of Dirac determinant $\det(\gamma_\mu p_\mu - m^*)$, p_0 = $i(\omega_n + i\mu)$

Solution: 1. Use explicit form of gamma matrices (and Pauli matrices)

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. Write down the determinant

$$\|\gamma_\mu p_\mu - m^*\| = \begin{vmatrix} (p_0 - m^*) & 0 & p_3 & (p_1 - ip_2) \\ 0 & (p_0 - m^*) & (p_1 + ip_2) & -p_3 \\ -p_3 & (-p_1 + ip_2) & (-p_0 - m^*) & 0 \\ (-p_1 - ip_2) & p_3 & 0 & (-p_0 - m^*) \end{vmatrix}$$

3. Determine the subdeterminants

$$\begin{aligned} D_{11} &= -(p_0 + m^*) \left(\vec{p}^2 + m^2 - p_0^2\right) \\ D_{13} &= p_3 \left(\vec{p}^2 + m^2 - p_0^2\right) \\ D_{14} &= -(p_1 + ip_2) \left(\vec{p}^2 + m^{*2} - p_0^2\right) \end{aligned}$$

4. Calculate the determinant according to standard rules

$$\begin{aligned} \|\gamma_\mu p_\mu - m^*\| &= (p_0 - m^*) D_{11} + p_3 D_{13} - (p_1 - ip_2) D_{14} \\ &= \left(-p_0^2 + p_1^2 + p_2^2 + p_3^2 + m^{*2}\right)\left(\vec{p}^2 + m^{*2} - p_0^2\right) \\ &= \left(\vec{p}^2 + m^{*2} - p_0^2\right)^2 \end{aligned}$$

5. Result:

$$= \underline{\underline{\left[\omega^2 + (\omega_n + i\mu)^2\right]^2}}, \qquad \omega^2 = \vec{p}^2 + m^{*2}$$

Exercise 2: Show that $2\sum_{n=-\infty}^{+\infty} \ln \beta^2[\omega^2 + (\omega_n + i\mu)^2] = \sum_{n=-\infty}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega - \mu)^2] + \ln \beta^2[\omega_n^2 + (\omega + \mu)^2]\right\}$

<u>Solution:</u> 1. Consider an analytic function $F(z_n)$ where $z_n = (i\omega_n - \mu)$, with $\omega_n = (2n+1)\pi T$

$$\sum_{n=-\infty}^{+\infty} F\left((\omega_n + i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n + i\mu)^2\right) + \sum_{n=-\infty}^{-1} F\left((\omega_n + i\mu)^2\right)$$

$$\sum_{n=-\infty}^{-1} F\left((\omega_n + i\mu)^2\right) = \sum_{n=1}^{\infty} F\left((\omega_{-n} + i\mu)^2\right) = \sum_{n=0}^{\infty} F\left((\omega_{-n-1} + i\mu)^2\right) = \sum_{n=0}^{\infty} F\left((-\omega_{-n-1} - i\mu)^2\right)$$

2. For the fermionic Matsubara frequencies holds $-\omega_{-n-1} = -\pi T(2(-n-1)+1) = \pi T(2n+1) = \omega_n$

$$\sum_{n=-\infty}^{+\infty} F\left((\omega_n + i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n + i\mu)^2\right) + \sum_{n=0}^{+\infty} F\left((\omega_n - i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n + i\mu)^2\right) + \sum_{n=0}^{+\infty} F^*\left((\omega_n + i\mu)^2\right)$$

$$= 2\sum_{n=0}^{+\infty} Re\ F\left((\omega_n + i\mu)^2\right)$$

3. Using this relationship based on the symmetry of the Matsubara frequencies, transform:

$$\begin{aligned}
2\sum_{n=-\infty}^{+\infty} \ln \beta^2[\omega^2 + (\omega_n + i\mu)^2] &= 4\sum_{n=0}^{+\infty} Re\ \ln \beta^2[(\omega^2 + \omega_n^2 - \mu^2) + i(2\omega_n\mu)] \\
&= 2\sum_{n=0}^{+\infty} \ln \beta^2[(\omega^2 + \omega_n^2 - \mu^2)^2 + (2\omega_n\mu)^2] \\
&= 2\sum_{n=0}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega - \mu)^2] + \ln \beta^2[\omega_n^2 + (\omega + \mu)^2]\right\} \\
&= \sum_{n=-\infty}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega - \mu)^2] + \ln \beta^2[\omega_n^2 + (\omega + \mu)^2]\right\}
\end{aligned}$$

Bose-Einstein Condensation: Charged Scalar Field

Consider a complex scalar field (two real components ϕ_1, ϕ_2):
$\Phi = (\phi_1 + i\phi_2)/\sqrt{2}\,, \quad \Phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 \,,$$

with U(1) symmetry: $\Phi \longrightarrow \Phi \mathrm{e}^{-i\alpha}$, where α is a real constant.
Noether theorem: continuous symmetry $\rightarrow$ conserved current

$$\begin{aligned} \mathcal{L} \rightarrow \mathcal{L}' &= \partial_\mu (\Phi^* \mathrm{e}^{i\alpha(x)})(\partial^\mu \Phi \mathrm{e}^{-i\alpha(x)}) - m^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 \,, \\ &= \mathcal{L} + \Phi^* \Phi \partial_\mu \alpha \partial^\mu \alpha + i \partial_\mu \alpha (\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*) \end{aligned}$$

Equation of motion for the "field" $\alpha(x)$

$$\partial^\mu \frac{\partial \mathcal{L}'}{\partial(\partial^\mu \alpha)} = \frac{\partial \mathcal{L}'}{\partial \alpha}$$

Since $\partial \mathcal{L}'/\partial \alpha = 0$ follows a conserved "current": $\partial \mathcal{L}'/\partial(\partial^\mu \alpha) = \Phi^* \Phi \partial_\mu \alpha - i \Phi \partial_\mu \Phi^* + i \Phi^* \partial_\mu \Phi$.
Recover original field theory by setting $\alpha =$constant. Then

$$j_\mu = i(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) \,, \quad \partial^\mu j_\mu = 0$$

Full current: $J_\mu = \int d^3x\, j_\mu(x)$; conserved charge: $Q = \int d^3x\, j_0(x)$

Bose-Einstein Condensation: Charged Scalar Field (2)

Decompose the complex $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ into real and imaginary parts: ϕ_1, ϕ_2.
Conjugate momenta: $\pi_1 = \partial\phi_1/\partial t$, $\pi_2 = \partial\phi_2/\partial t$
Hamiltonian density and charge:

$$\mathcal{H} = \frac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2\phi_1^2 + m^2\phi_2^2\right] + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \,, \quad Q = \int d^3x(\phi_2\pi_1 - \phi_1\pi_2)$$

The partition function is

$$Z = \int [d\pi_1][d\pi_2] \int_{\text{periodic}} [d\phi_1][d\phi_2] \exp\left[\int^\beta d^4x \left(i\pi_1\frac{\partial\phi_1}{\partial\tau} + i\pi_2\frac{\partial\phi_2}{\partial\tau} - \mathcal{H}(\pi_1,\pi_2,\phi_1,\phi_2) + \mu(\phi_2\pi_1 - \phi_1\pi_2)\right)\right]$$

Integration over conjugate field momenta can be done with the result:

$$\begin{aligned} Z \;=\; & (N')^2 \int_{\text{periodic}} [d\phi_1][d\phi_2] \exp\Big\{ \int^\beta d^4x\Big[-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2 - \frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1\right)^2 \\ & -\frac{1}{2}(\nabla\phi_1)^2 - \frac{1}{2}(\nabla\phi_2)^2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2\Big]\Big\} \,. \end{aligned}$$

Differs from naïve expectation $\mathcal{L}(\phi_1, \phi_2, \partial_\mu\phi_1, \partial_\mu\phi_2; \mu = 0) + \mu j_0(\phi_1, \phi_2, \partial\phi_1/\partial\tau, \partial\phi_2/\partial\tau)$ by $\mu^2\Phi^*\Phi$

Bose-Einstein Condensation: Charged Scalar Field (3)

In the following: ideal gas ($\lambda = 0$). For $\lambda \neq 0$, perform HS-transformation! (Exercise)
Expand components of $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ in Fourier series:

$$\begin{aligned}
\phi_1(\vec{x},\tau) &= \sqrt{2}\zeta\cos\theta + \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \mathrm{e}^{i(\vec{p}\vec{x}+\omega_n\tau)} \phi_{1;n}(\vec{p}) \,, \\
\phi_2(\vec{x},\tau) &= \sqrt{2}\zeta\sin\theta + \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \mathrm{e}^{i(\vec{p}\vec{x}+\omega_n\tau)} \phi_{2;n}(\vec{p}) \,.
\end{aligned}$$

Infrared character of Φ carried by ζ and θ, independent of $(\vec{x},\tau)$, so $\phi_{1;0}(\vec{0}) = \phi_{2;0}(\vec{0}) = 0$. Possibility of condensation of bosons into the zero-momentum state: finite fraction of particles in $n=0, \vec{p}=\vec{0}$ state.

$$\begin{aligned}
Z &= (N')^2 \left[\Pi_n \Pi_{\vec{p}} \int i d\phi_{1;n}(\vec{p}) d\phi_{2;n}(\vec{p})\right] \mathrm{e}^{S} \,, \\
S &= \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2}\sum_n \sum_{\vec{p}} (\phi_{1;-n}(-\vec{p}), \phi_{2;-n}(-\vec{p})) D \begin{pmatrix} \phi_{1;n}(\vec{p}) \\ \phi_{2;n}(\vec{p}) \end{pmatrix} \,, \\
D &= \beta^2 \begin{pmatrix} \omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2 \end{pmatrix} .
\end{aligned}$$

Bose-Einstein Condensation: Charged Scalar Field (4)

Carrying out integrations yields:

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 + \ln(\det D)^{-1}$$

Second term can be handled as:

$$\begin{aligned}\ln\det D &= \ln\left\{\Pi_n\Pi_{\vec{p}}\beta^4\left[(\omega_n^2+\omega^2-\mu^2)^2+4\mu^2\omega_n^2\right]\right\}\\ &= \ln\left\{\Pi_n\Pi_{\vec{p}}\beta^2\left[\omega_n^2+(\omega-\mu)^2\right]\right\}+\ln\left\{\Pi_n\Pi_{\vec{p}}\beta^2\left[\omega_n^2+(\omega-\mu)^2\right]\right\}\end{aligned}$$

Putting all together and evaluating the Matsubara sums, we obtain

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - V\int\frac{d^3p}{(2\pi)^3}[\beta\omega + \ln(1-\mathrm{e}^{-\beta(\omega-\mu)}) + \ln(1-\mathrm{e}^{-\beta(\omega+\mu)})]$$

Result independent of θ because of $U(1)$ symmetry. Parameter ζ from variation of $\ln Z$.

$$\frac{\partial\ln Z}{\partial\zeta} = 2\beta V(\mu^2 - m^2)\zeta = 0$$

implies that $\zeta = 0$ unless $|\mu| = m$, when ζ is determined from $\rho = Q/V$

$$\rho = \frac{T}{V}\left(\frac{\partial\ln Z}{\partial\mu}\right)_{\mu=m} = 2m\zeta^2 + \rho^*(\beta,\mu=m)\ ;\ \ \rho^* = \int\frac{d^3p}{(2\pi)^3}\left(\frac{1}{\mathrm{e}^{\beta(\omega-\mu)}-1} - \frac{1}{\mathrm{e}^{\beta(\omega+\mu)}-1}\right)$$

Critical temperature T_c for condensation from $\rho = \rho^*(\beta_c, \mu = m)$.

BOSE-EINSTEIN CONDENSATION: EXERCISES

1. Find the critical temperature T_c for condensation from $\rho = \rho^*(\beta_c, \mu = m)$.
   Show that the nonrelativistic (NR, $\rho \ll m^3$) and ultrarelativistic (UR, $\rho \gg m^3$) limits are given by

$$T_{c,\mathrm{NR}} = \frac{2\pi}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3} \;\; ; \;\; T_{c,\mathrm{UR}} = \left(\frac{3\rho}{m}\right)^{1/2}$$

2. For $T < T_c$, the value of ζ is the order parameter of the 2nd order condensation phase transition

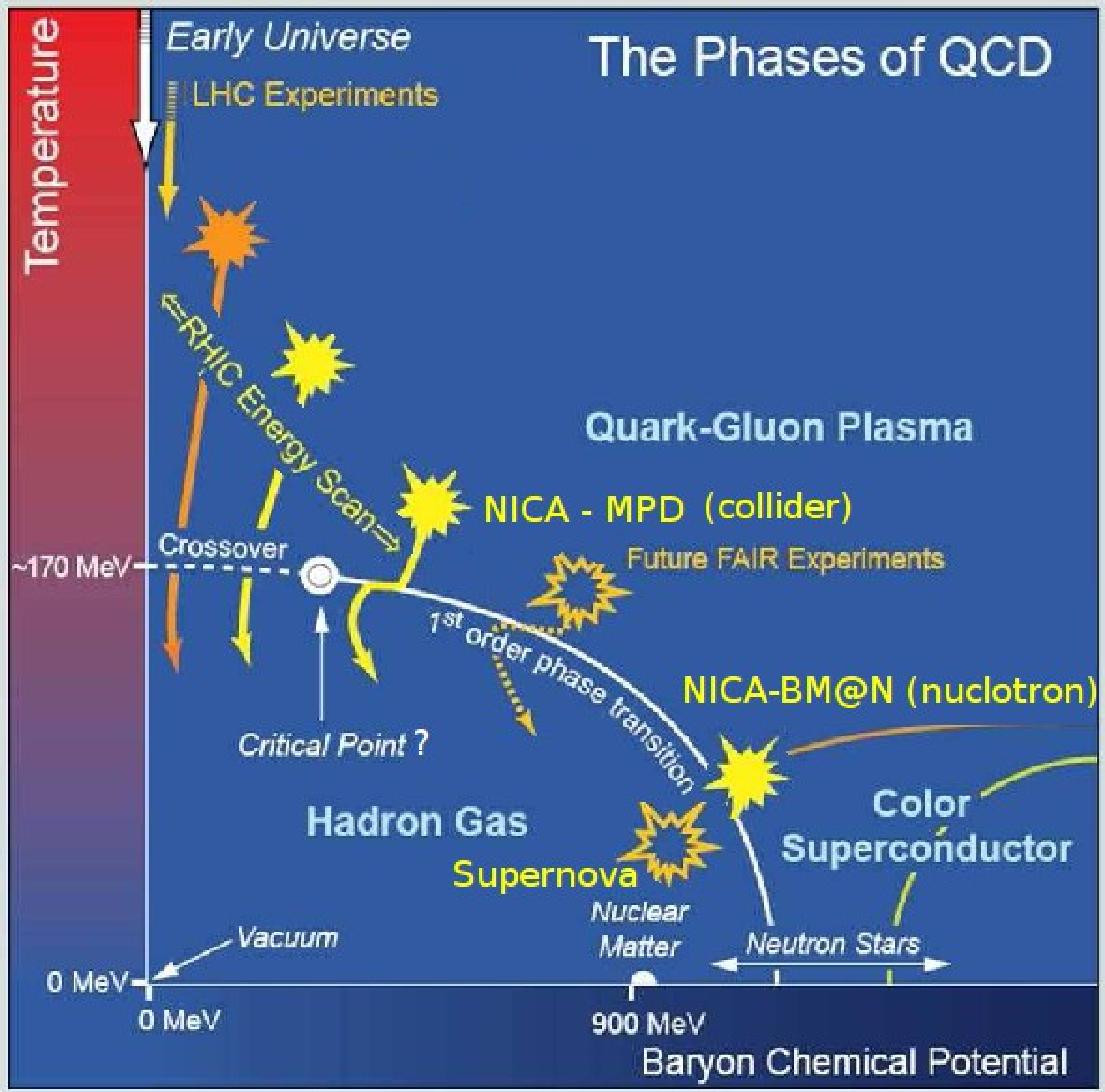
$$\zeta^2 = \left[\rho - \rho^*(\beta, \mu = m)\right]/(2m) \;\; , \;\; T < T_c \, .$$

   At the critical temperature, the expansion $\zeta \sim t^\nu$ for small $t = T - T_c$ determines a critical exponent ν. Find the value of ν in both cases.

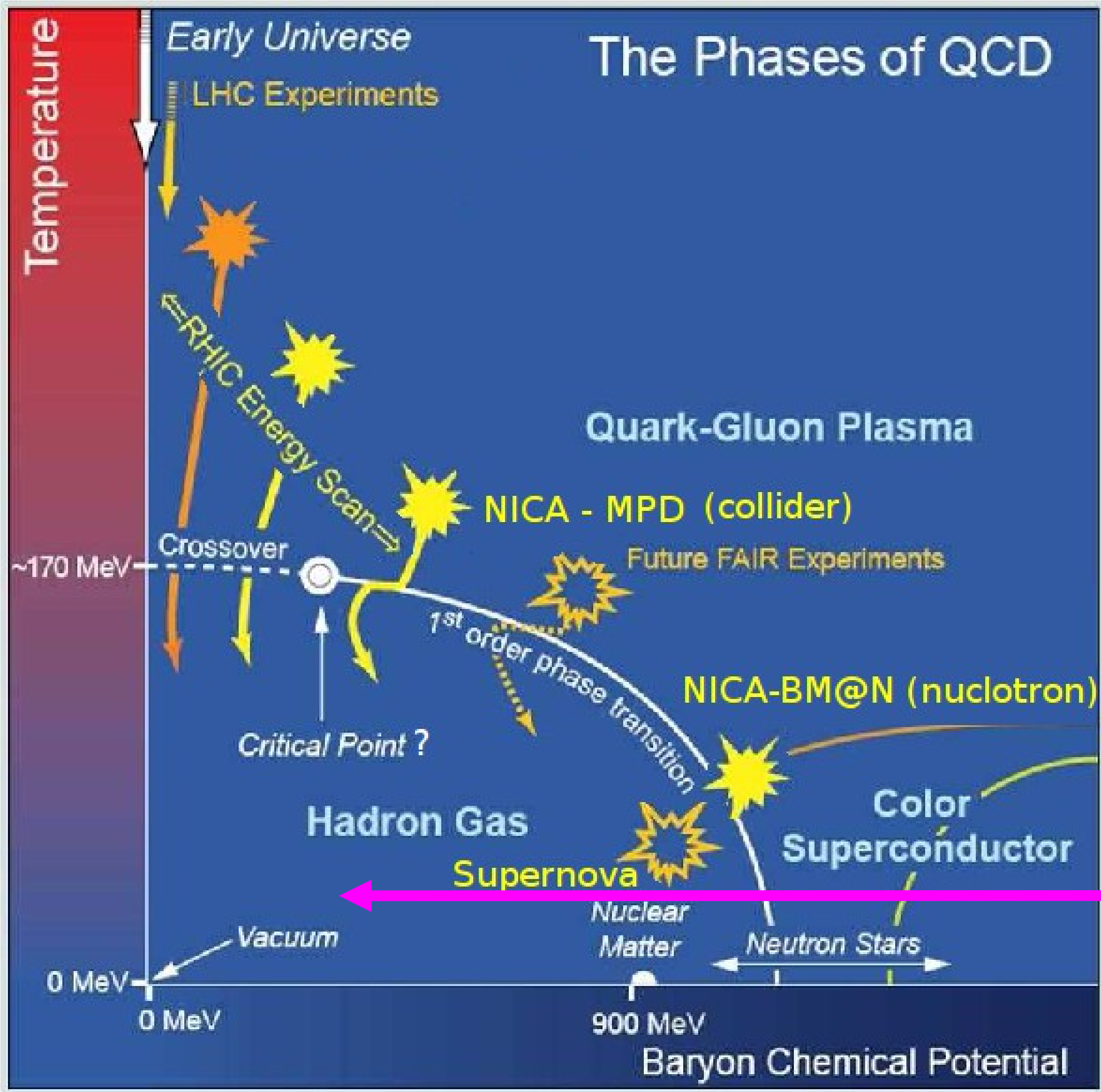
3. Consider the interacting case $\lambda \neq 0$. Perform the Hubbard-Stratonovich transformation by introducing a collective scalar field σ. Discuss the effect of $\lambda \neq 0$ on the thermodynamic potential in the mean-field approximation for σ, i.e. by neglecting the path integral over the σ-fields and determining σ in the thermodynamical equilibrium from a gap equation $\partial \ln Z/\partial\sigma = 0$.

Backup slides

The Goal: Theory of the QCD Phase Diagram



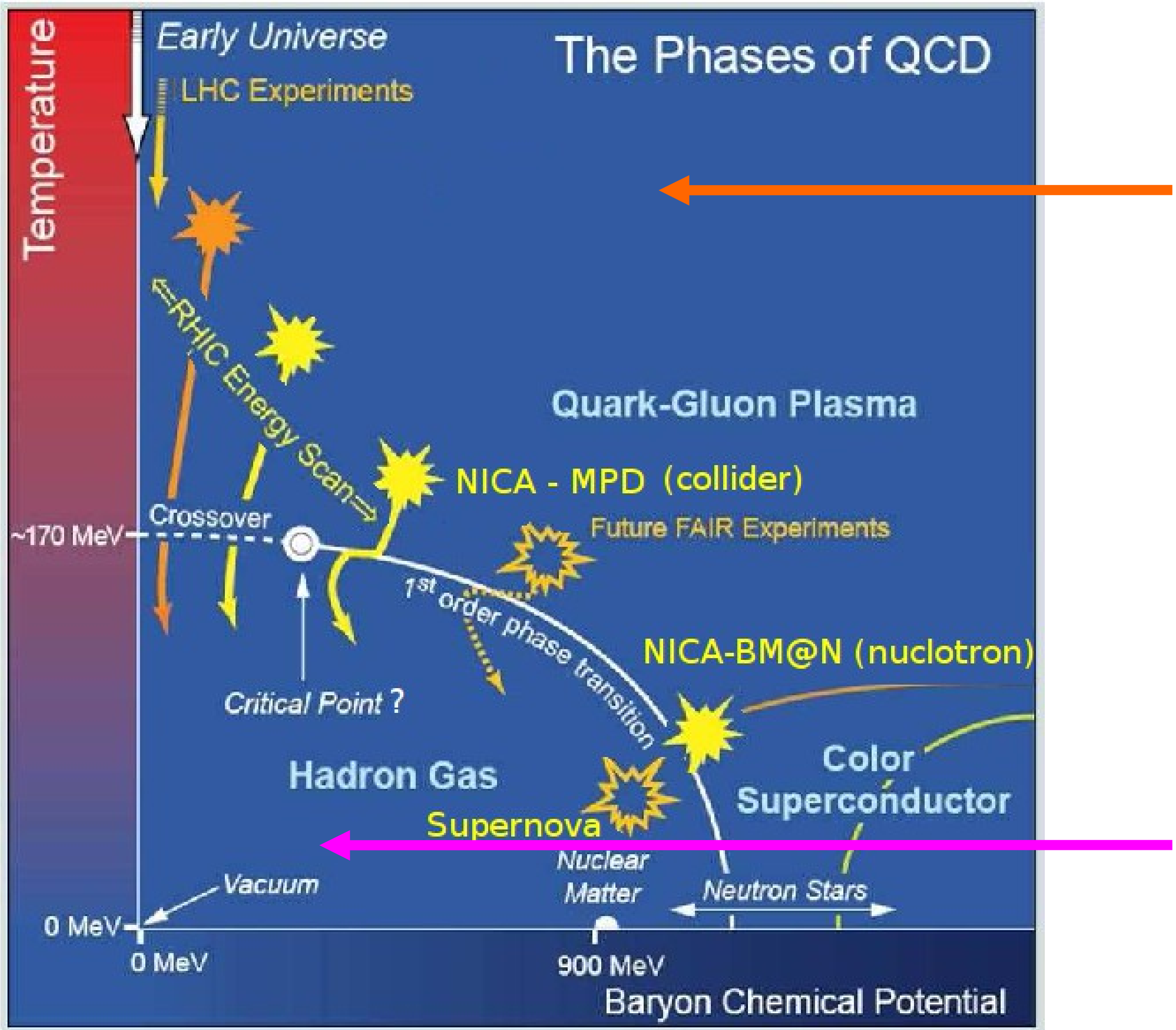
The Goal: Theory of the QCD Phase Diagram



Statistical Model of Hadron Resonance Gas

Well established for Description of chemical freezeout

The Goal: Theory of the QCD Phase Diagram



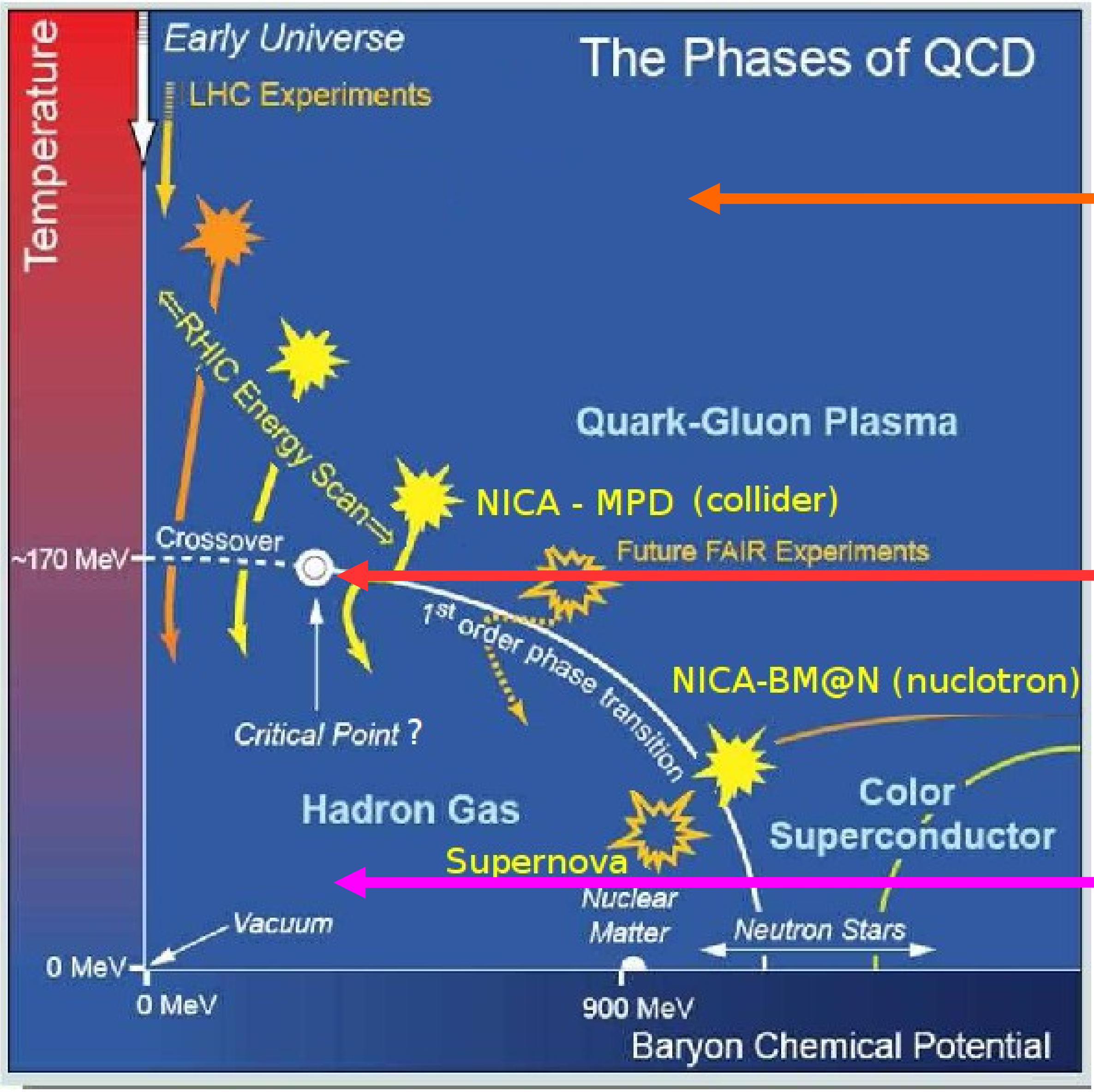
Perturbative QCD

Approximately selfconsistent HTL resummation
($T > 2.5$ Tc , $\mu > 1500$ MeV)

Statistical Model of Hadron Resonance Gas

Well established for Description of chemical freezeout

The Goal: Theory of the QCD Phase Diagram



Perturbative QCD

Approximately selfconsistent HTL resummation
(T > 2.5 Tc , μ > 1500 MeV)

QCD Phase transition(s)

Mott dissociation of hadrons, Deconfinement, χSR

Statistical Model of Hadron Resonance Gas

Well established for Description of chemical freezeout

1. Mott-Anderson localization model for chemical freeze-out

DB, J. Berdermann, J. Cleymans, K. Redlich, Phys. Part. Nucl. Lett. 8 (2011) 811

The basic idea: Localization of (certain) multiquark states ("cluster") = hadronization;
Reverse process = delocalization by quark exchange between hadrons

Freeze-out criterion:

$$H_{\rm exp}(\tau) = \frac{\dot{R}(\tau)}{R(\tau)} = \tau^{-1}_{{\rm coll},i}(T,\mu)\ ,$$

$$\tau^{-1}_{{\rm coll},i}(T,\mu) = \sum_j \sigma_{ij} v n_j(T,\mu)$$

Povh-Huefner law, PRC 46 (1992) 990

$$\sigma_{ij} = \lambda \langle r_i^2 \rangle \langle r_j^2 \rangle$$

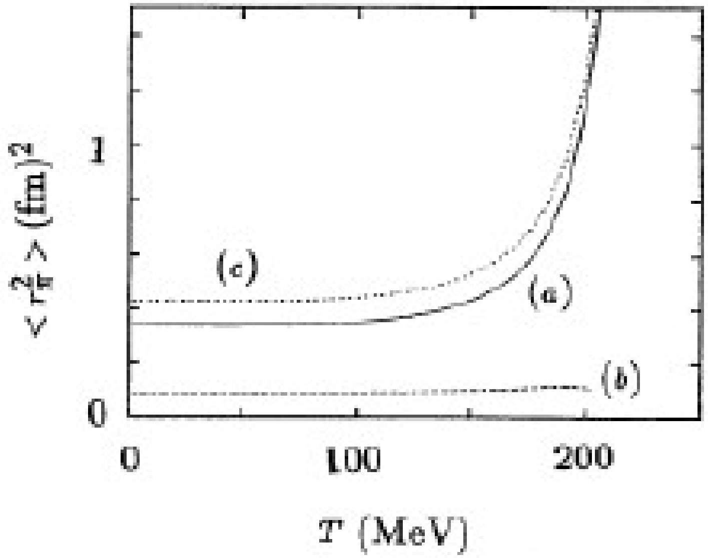
$$r_\pi^2(T,\mu) = \frac{3}{4\pi^2} f_\pi^{-2}(T,\mu)$$

Hippe & Klevansky, PRC 52 (1995) 2172

$$f_\pi^2(T,\mu) = -m_0 \langle \bar{q}q \rangle_{T,\mu} / M_\pi^2$$

$$r_\pi^2(T,\mu) = \frac{3 M_\pi^2}{4\pi^2 m_q} |\langle \bar{q}q \rangle_{T,\mu}|^{-1}$$

$$\langle \bar{q}q \rangle = \langle \bar{q}q \rangle_{\rm MF} \left[1 - \frac{T^2}{8 f_\pi^2(T,\mu)} - \frac{\sigma_N n_{s,N}(T,\mu)}{M_\pi^2 f_\pi^2(T,\mu)} \right]$$



1. Mott-Anderson localization model for chemical freeze-out

DB, J. Berdermann, J. Cleymans, K. Redlich, Phys. Part. Nucl. Lett. 8 (2011) 811

Povh-Huefner law behaviour for quark exchange between hadrons

PHYSICAL REVIEW C VOLUME 51, NUMBER 5 MAY 1995

Quark exchange model for charmonium dissociation in hot hadronic matter

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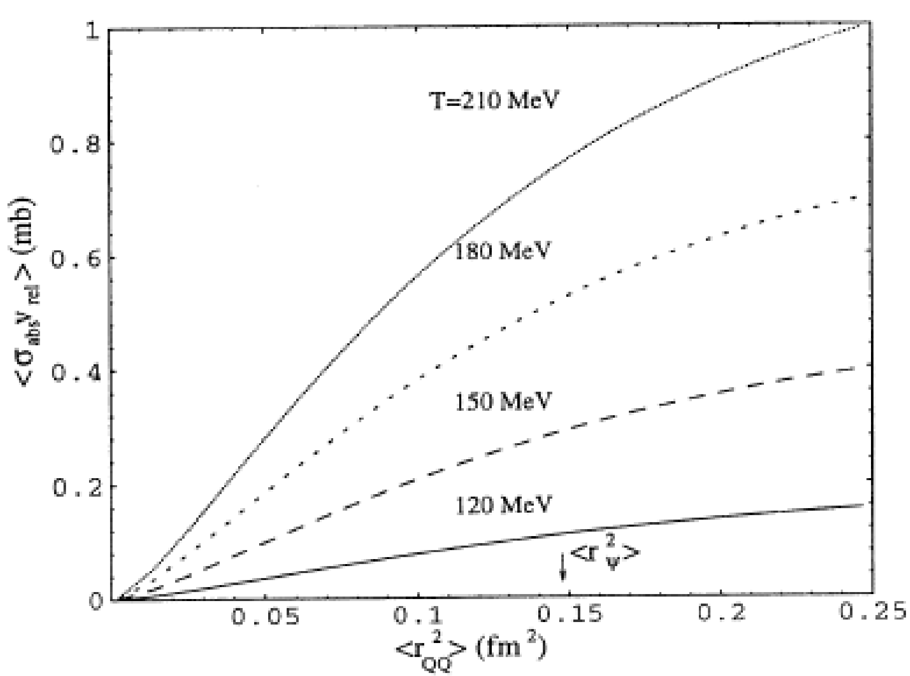
(Received 15 November 1994)



$$\langle \sigma_{\rm abs} v_{\rm rel} \rangle \propto \langle r^2 \rangle_{Q\bar{Q}} \langle r^2 \rangle_{q\bar{q}}$$

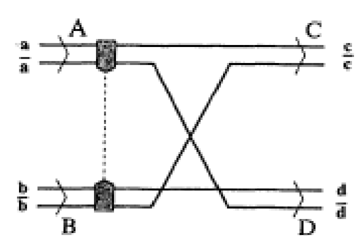
Flavor exchange processes

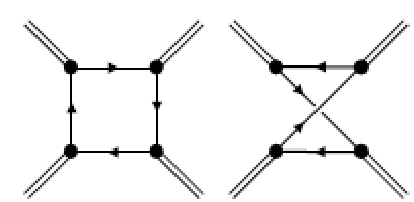
$$\phi + \pi \to K + \bar{K},$$

$$K^- + p \to \Lambda + X,$$
$$K^+ + p \to \Lambda + X,$$

Nonrelativistic → rel. quark loop integrals

$M_{fi} =$





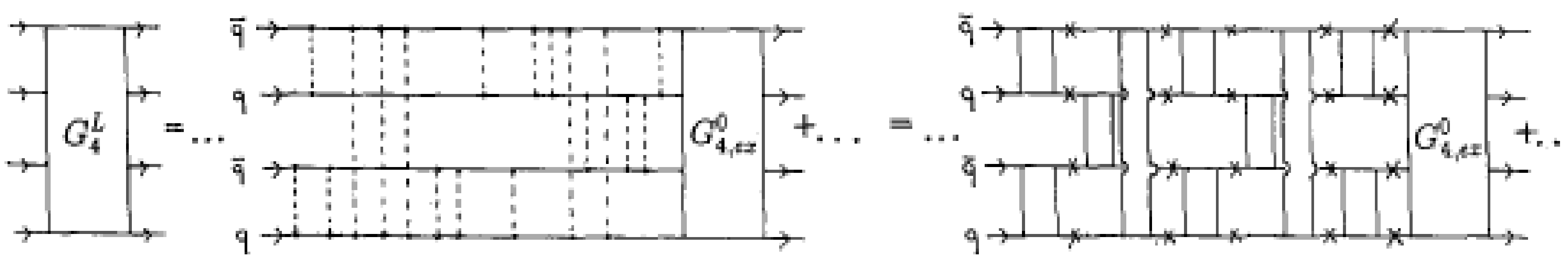
1. Quark exchange in meson-meson scattering

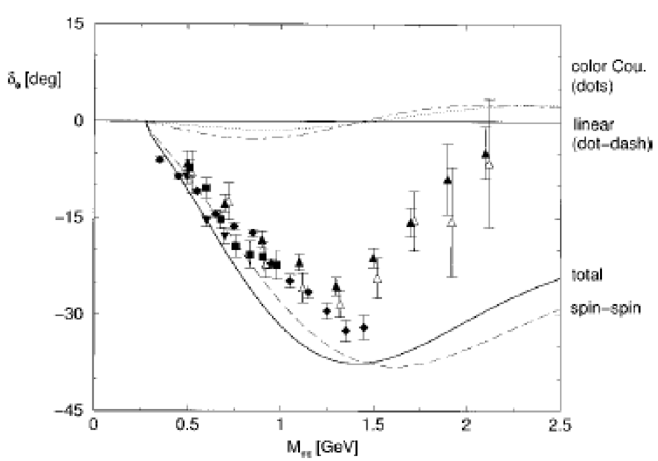
DB, G. Roepke, Phys. Lett. B 299 (1993) 332; T. Barnes et al., PRC 63 (2001) 025204

Povh-Huefner law behaviour for quark exchange between hadrons ?

$$\sigma_{ij} = \lambda \langle r_i^2 \rangle \langle r_j^2 \rangle \qquad r_\pi^2(T,\mu) = \frac{3M_\pi^2}{4\pi^2 m_q} \left| \langle \bar{q}q \rangle_{T,\mu} \right|^{-1}$$

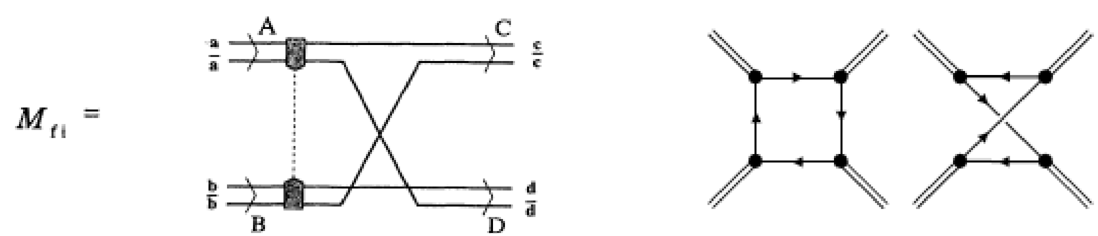
$$\mathcal{V}^{ss}(12,1'2') = \frac{16}{3\sqrt{3}} C_{SFC}(12,1'2') \frac{(2\pi)^3}{\Omega_0} \frac{\alpha_s}{3\pi^2 m_q^2} \exp\left(-\frac{1}{4b^2} (k'^2 + \tfrac{1}{3}k^2) \right) \delta_{K,K'}$$





Quark exchange process in M-M scattering

Nonrelativistic → rel. quark loop integrals



1. Mott-Anderson localization model for chemical freeze-out

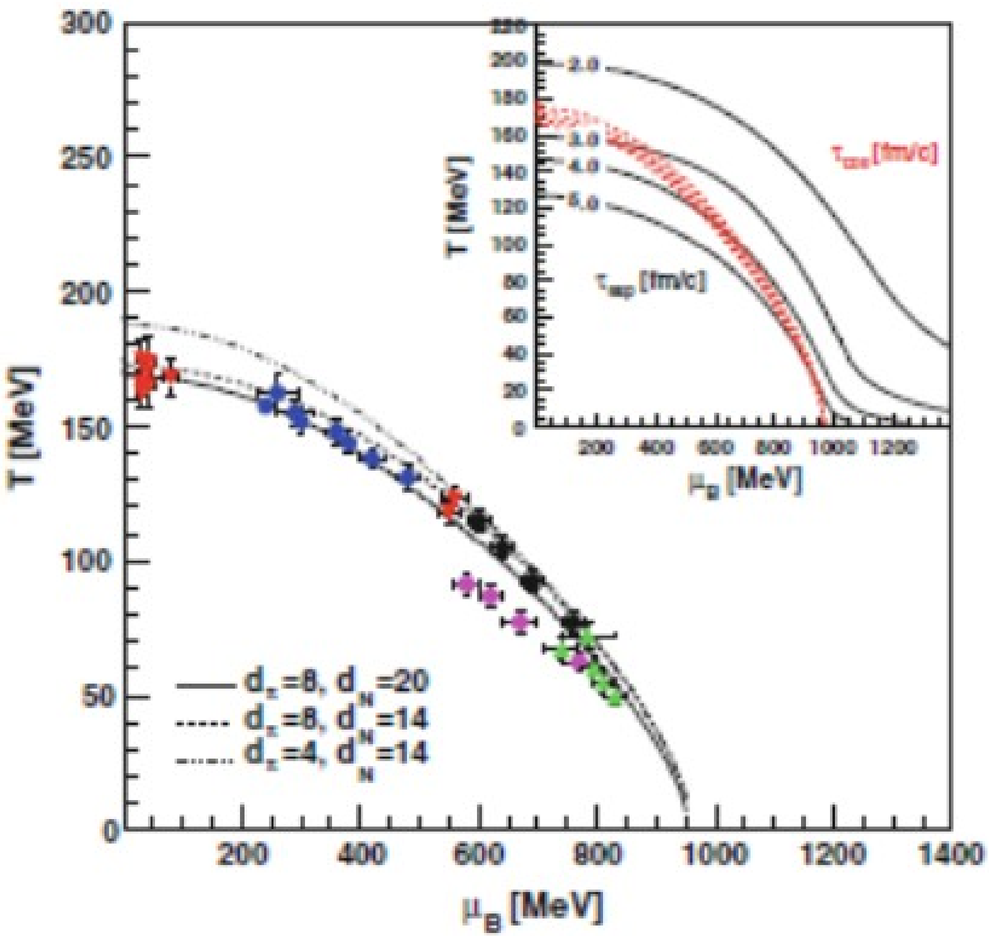
DB, J. Berdermann, J. Cleymans, K. Redlich, Phys. Part. Nucl. Lett. 8 (2011) 811

<u>Model results:</u>

$$\tau_{exp}(T, \mu) = \tau_{coll}(T, \mu)$$

Collision time strongly T, mu dependent !

Schematic resonance gas: dp pions, dN nucleons

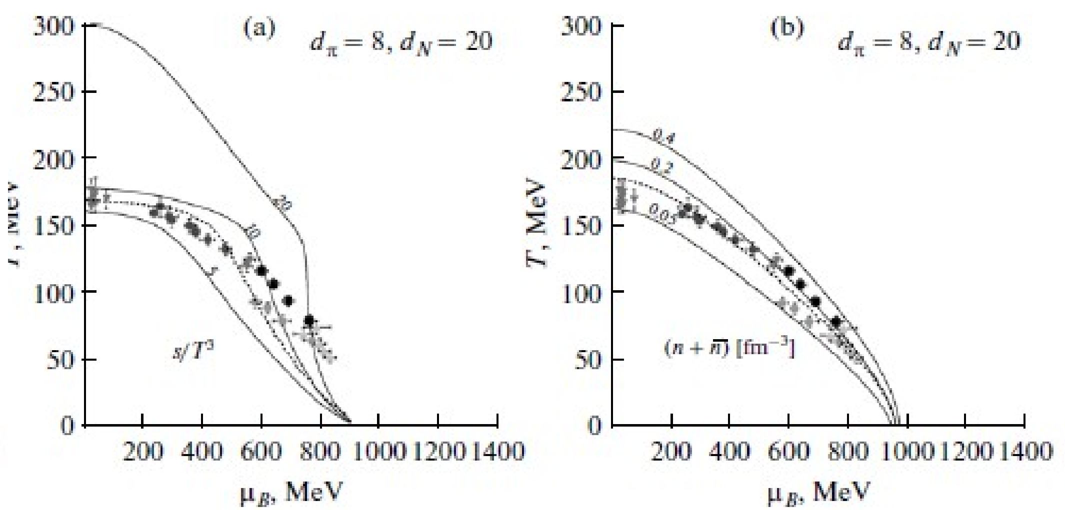


Expansion time scale from entropy conservation:

$$s(T, \mu)\ V(\tau_{exp}) = \text{const}$$

$$\tau_{exp}(T, \mu) = a s^{-1/3}(T, \mu),$$

Thermodynamics consistent with phenomenological Freeze-out rules:



1. Mott-Anderson localization model for chemical freeze-out

DB, J. Berdermann, J. Cleymans, K. Redlich, Few Body Syst. 53 (2012) 99

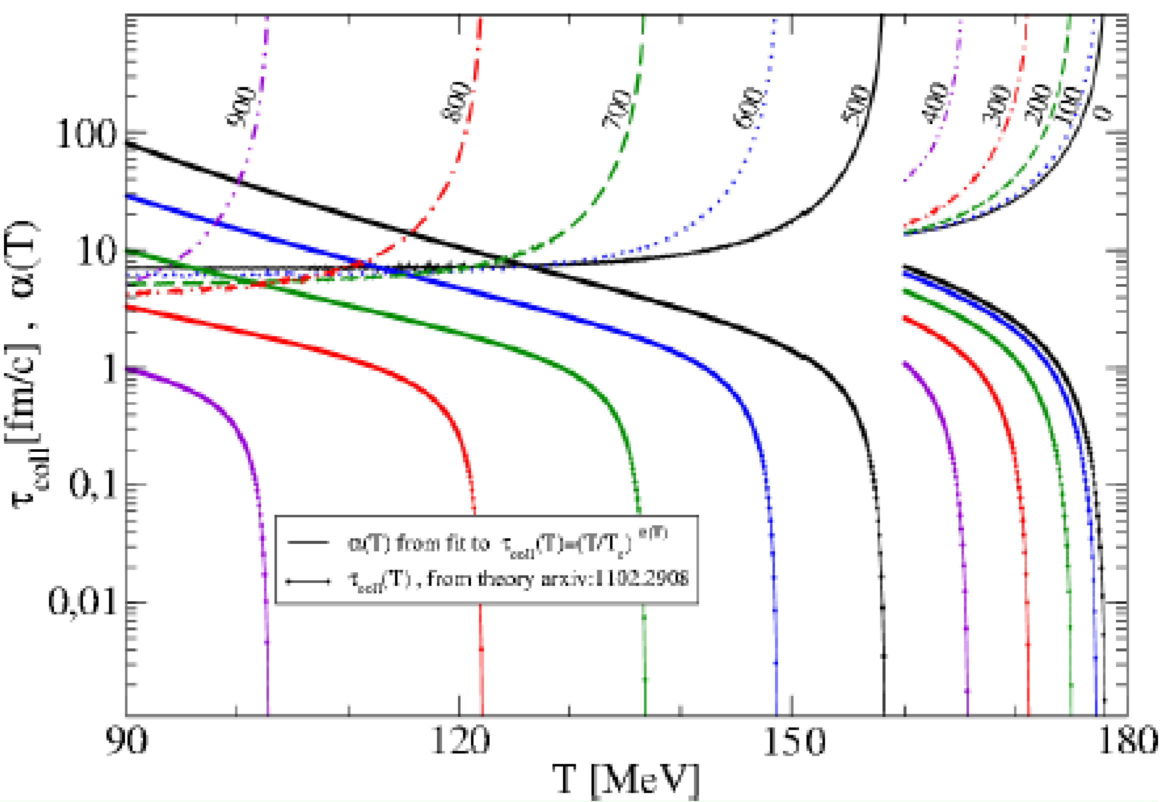
<u>Model results:</u>
Full hadron resonance gas model

$$\sigma_{ij} = \lambda \langle r_i^2 \rangle \langle r_j^2 \rangle \;;$$

$$r_\pi^2(T,\mu) = \frac{3 M_\pi^2}{4\pi^2 m_q} \left| \langle \bar{q} q \rangle_{T,\mu} \right|^{-1}$$

$$r_N^2(T,\mu) = r_0^2 + r_\pi^2(T,\mu)$$

$$\frac{\langle \bar{q}q \rangle}{\langle \bar{q}q \rangle_{\rm vac}} = 1 - \frac{m_0}{F_\pi^2 m_\pi^2} \Bigg\{ 4N_c \int \frac{dp\, p^2}{2\pi^2} \frac{m}{\varepsilon_p} [f_\Phi^+ + f_\Phi^-] + \sum_{M=f_0,\omega,\ldots} d_M (2-N_s) \int \frac{dp\, p^2}{2\pi^2} \frac{m_M}{E_M(p)} f_M(E_M(p)) + \sum_{B=N,\Lambda,\ldots} d_B (3-N_s) \int \frac{dp\, p^2}{2\pi^2} \frac{m_B}{E_B(p)} [f_B^+(E_B(p)) + f_B^-(E_B(p))] \Bigg\} - \sum_{G=\pi,K,\eta,\eta'} \frac{d_G r_G}{4\pi^2 F_G^2} \int dp \frac{p^2}{E_G(p)} f_G(E_G(p)).$$



Collision time follows a power law
t_coll ~ (T/Tc)^a
with a large exponent a ~ 20

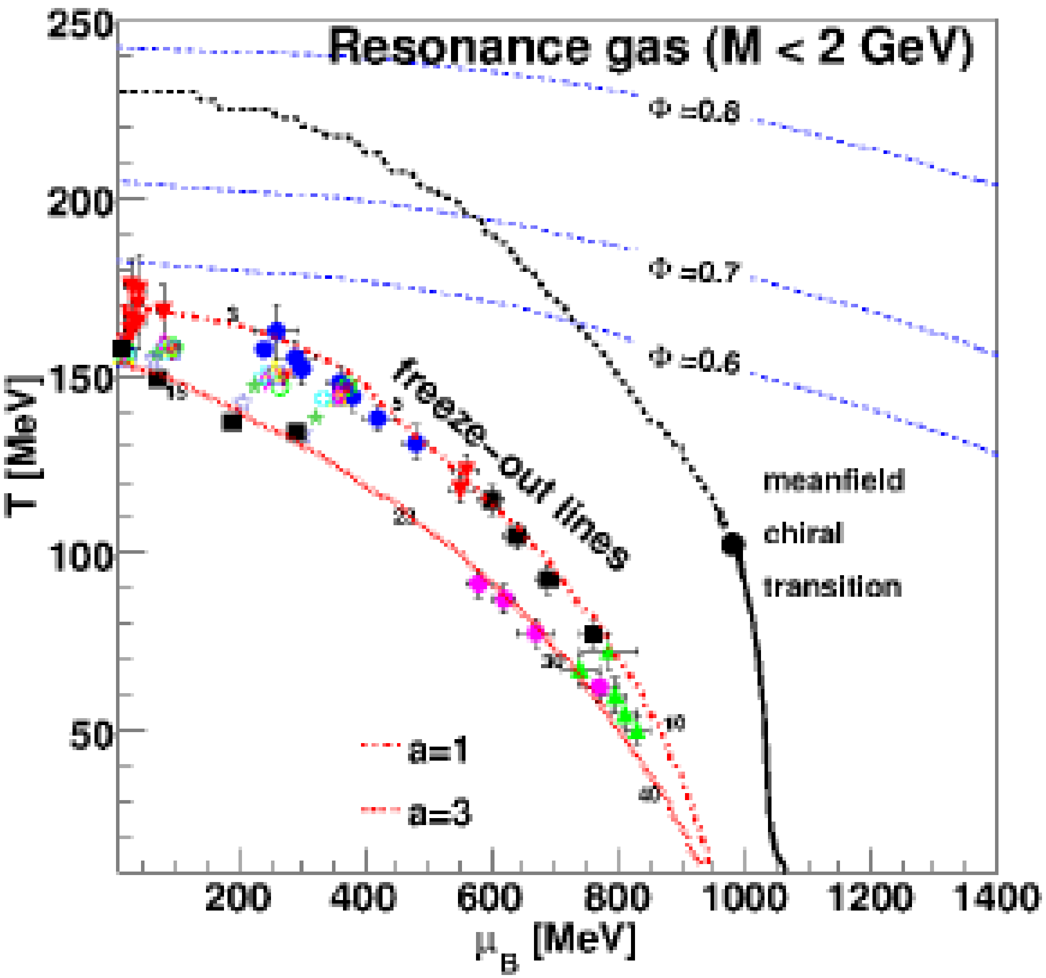
See also: P. Braun-Munzinger, J. Stachel, C. Wetterich, PLB (2004)

1. Mott-Anderson localization model for chemical freeze-out

DB, J. Berdermann, J. Cleymans, K. Redlich, Few Body Syst. 53 (2012) 99

Model results:
Full hadron resonance gas model

See also: S. Leupold, J. Phys. G (2006)



$$\frac{\langle\bar{q}q\rangle}{\langle\bar{q}q\rangle_{vac}} = 1 - \frac{m_0}{F_\pi^2 m_\pi^2}\Big\{4N_c \int \frac{dp\, p^2}{2\pi^2}\frac{m}{\varepsilon_p}[f_\Phi^+ + f_\Phi^-]$$
$$+ \sum_{M=f_0,\omega,\ldots} d_M(2-N_s)\int \frac{dp\, p^2}{2\pi^2}\frac{m_M}{E_M(p)} f_M(E_M(p))$$
$$+ \sum_{B=N,\Lambda,\ldots} d_B(3-N_s)\int \frac{dp\, p^2}{2\pi^2}\frac{m_B}{E_B(p)}[f_B^+(E_B(p)) + f_B^-(E_B(p))]\Big\}$$
$$- \sum_{G=\pi,K,\eta,\eta'} \frac{d_G r_G}{4\pi^2 F_G^2}\int dp\, \frac{p^2}{E_G(p)} f_G(E_G(p)).$$

$$\sigma_{ij} = \lambda \langle r_i^2\rangle\langle r_j^2\rangle \quad ; \quad r_N^2(T,\mu) = r_0^2 + r_\pi^2(T,\mu)$$

$$r_\pi^2(T,\mu) = \frac{3M_\pi^2}{4\pi^2 m_q}\left|\langle\bar{q}q\rangle_{T,\mu}\right|^{-1}$$

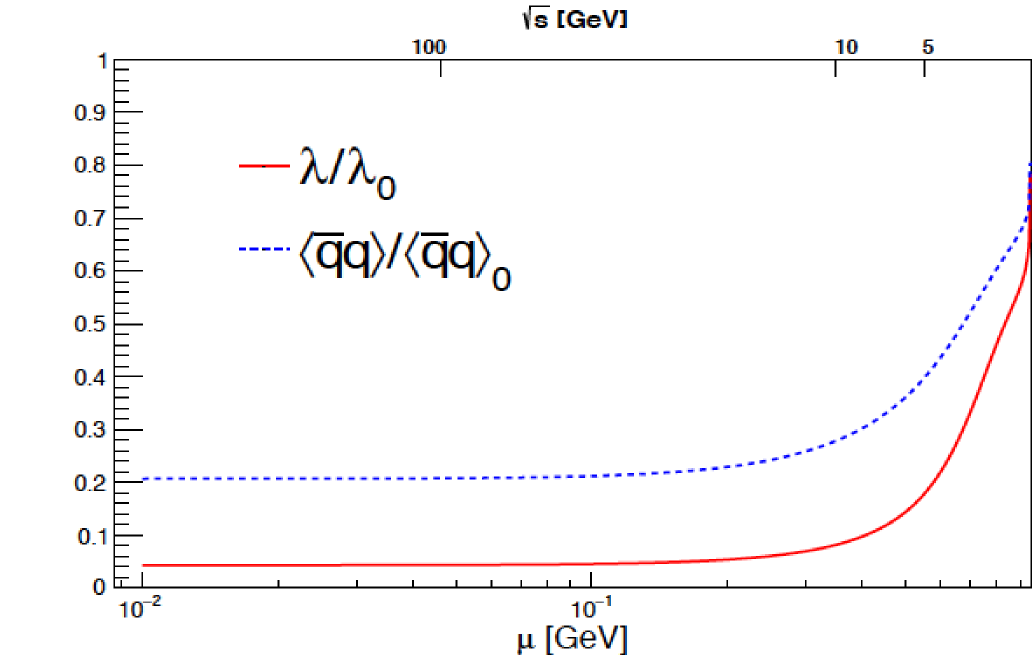
The factor a stands for the inverse system size in the formula

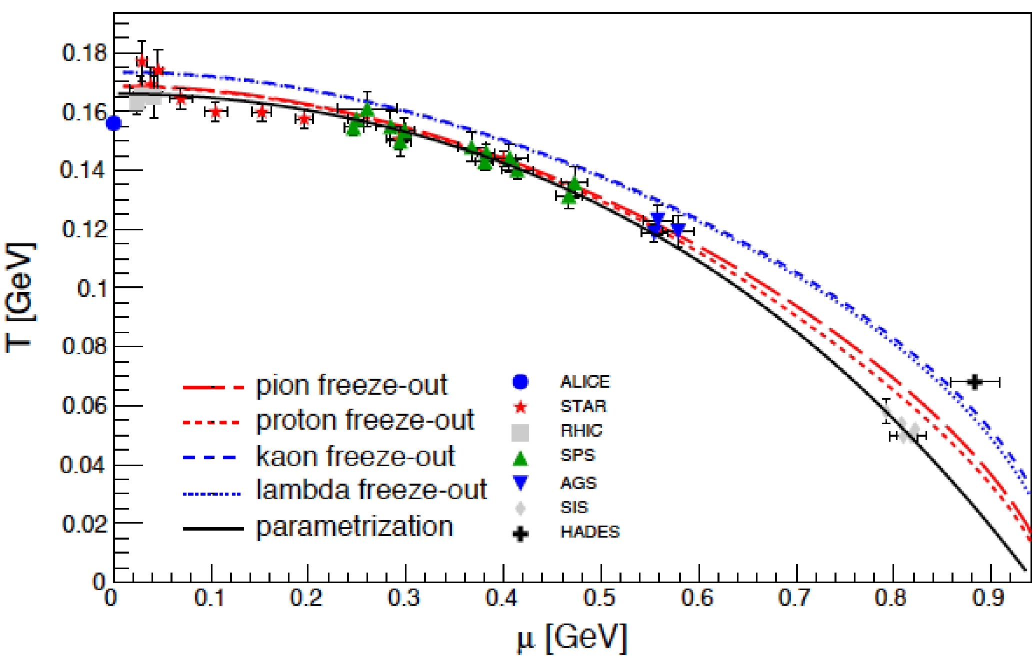
$$\tau_{exp}(T,\mu) = \tau_{coll}(T,\mu)$$

for the 3D expansion time scale assuming entropy conservation

1. Mott-Anderson localization model for K+/π+ "horn"

DB, J. Jankowski, M. Naskret, in prep. (2017)





Full HRG model condensate;
J. Jankowski et al., Phys. Rev. D (2013)

$$\langle \bar{q}q\rangle_{T,\mu} = \langle \bar{q}q\rangle^{MF}_{T,\mu} + \sum_{h=M,B} \frac{\sigma_q^h}{m_q} n_h(T,\mu) ,$$

$$n_h(T,\mu) = \frac{d_h}{2\pi^2}\int_0^\infty dk\, k^2 \frac{m_h}{E_h} \frac{1}{e^{(E_h-\mu_h)/T} \mp 1} .$$

$$\tau^{-1}_{\mathrm{coll},i}(T,\mu) = \sum_j \sigma_{ij} v n_j(T,\mu) \;;\; \sigma_{ij} = \lambda \langle r_i^2\rangle \langle r_j^2\rangle$$

$$\langle r_\pi^2\rangle_{T,\mu} \simeq \frac{3}{4\pi^2} f_\pi^{-2}(T,\mu) = \frac{3M_\pi^2}{4\pi^2 m_q} \left|\langle \bar{q}q\rangle_{T,\mu}\right|^{-1}$$

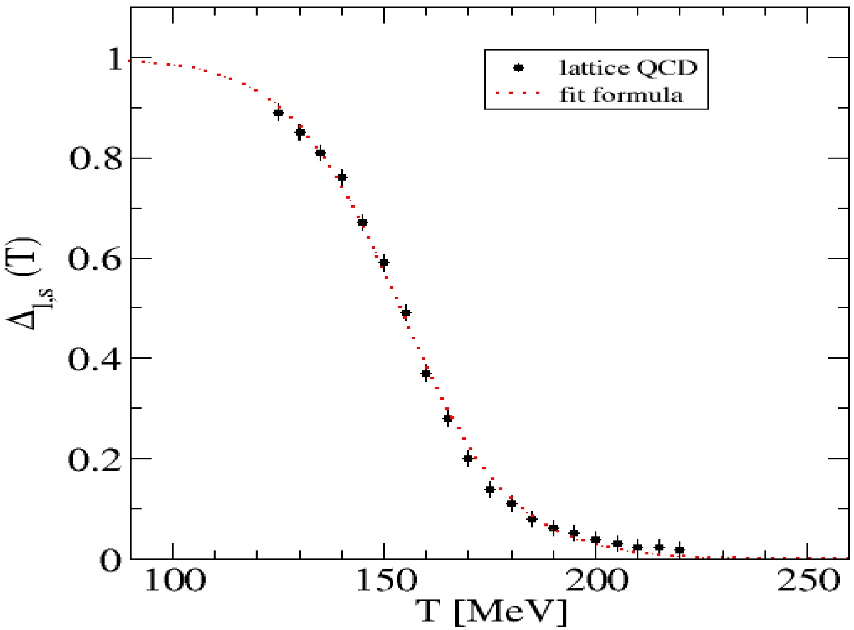
$$\langle r_K^2\rangle_{T,\mu} \simeq \frac{3M_K^2}{\pi^2(m_q+m_s)} \left|\langle \bar{q}q\rangle_{T,\mu} + \langle \bar{s}s\rangle_{T,\mu}\right|^{-1}$$

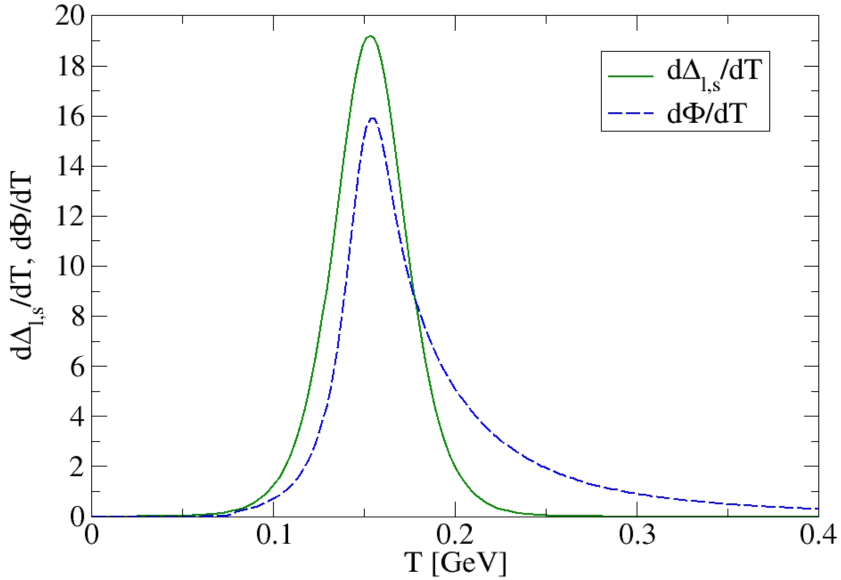
The factor a stands for the inverse system size in the formula

$$\tau_{\mathrm{exp}}(T,\mu) = \tau_{\mathrm{coll}}(T,\mu)$$

for the 3D expansion time scale assuming entropy conservation

3. Mott HRG / PNJL – effective model





$$P_{\rm PNJL}(T) = P_{\rm FG}(T) + \mathcal{U}[\Phi;T] ,$$

$$P_{\rm FG}(T) = 4 \sum_{q=u,d,s} \int \frac{d^3p}{(2\pi)^3} T \ln\left[1 + 3\Phi(Y + Y^2) + Y^3\right]$$

$$Y(E_p) = \exp(-E_p/T)$$

$$\mathcal{U}[\Phi;T] = -\frac{a(T)}{2}\Phi^2 + b(T)\ln(1 - 6\Phi^2 + 8\Phi^3 - 3\Phi^4)$$

T-dependent quark masses from fit to LQCD

$$m(T) = [m(0) - m_0]\Delta_{l,s}(T) + m_0 ,$$

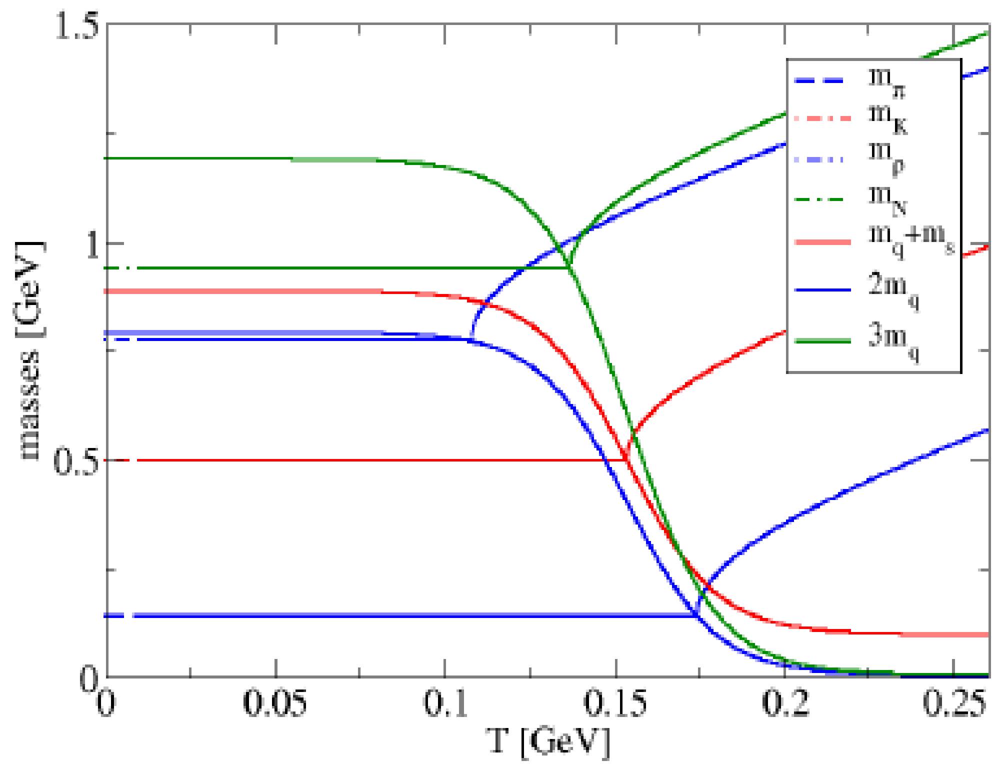
$$m_s(T) = m(T) + m_s - m_0 ,$$

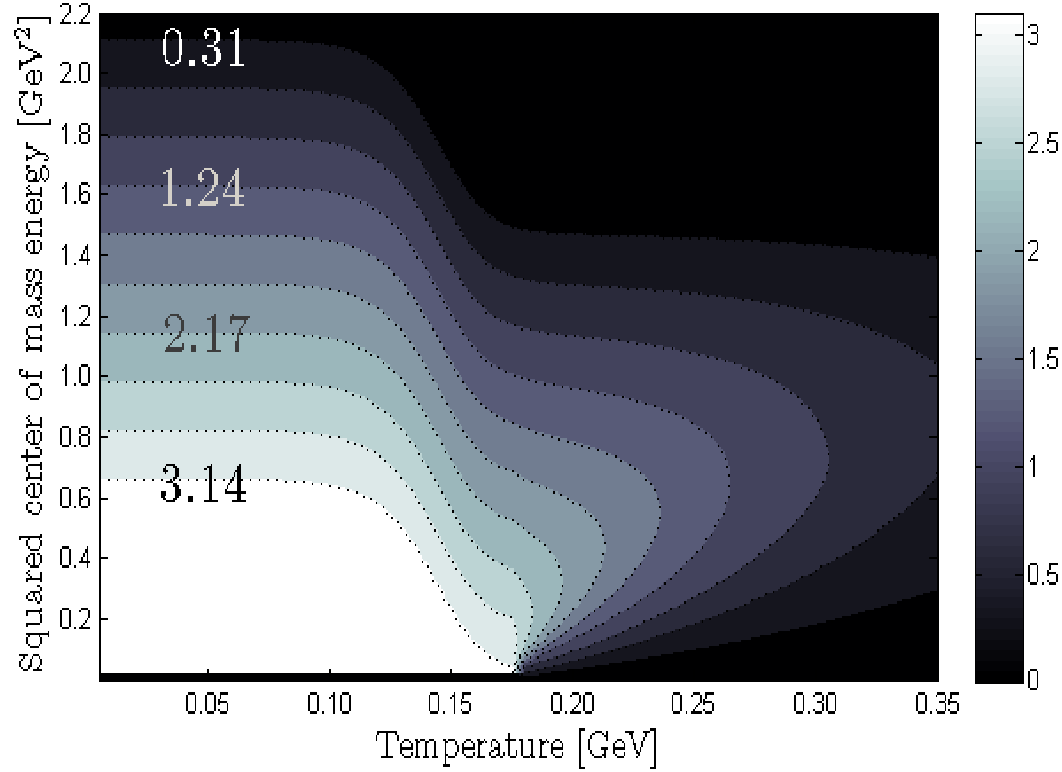
$$\Delta_{l,s}(T) = \frac{1}{2}\left[1 - \tanh\left(\frac{T - T_c}{\delta_T}\right)\right]$$

$$T_c = 154 \text{ MeV} \qquad \delta_T = 26 \text{ MeV}$$

D. Blaschke, A. Dubinin, L. Turko, arxiv:1611.09845

3. Mott HRG / PNJL – effective model





Hadrons + Mott effect

$$P_i(T) = \mp d_i \int_0^\infty \frac{dp\, p^2}{2\pi^2} \int_0^\infty dM\, T \ln\left(1 \mp e^{-\sqrt{p^2+M^2}/T}\right) \frac{2}{\pi} \sin^2 \delta_i(M^2;T) \frac{d\delta_i(M^2;T)}{dM}$$

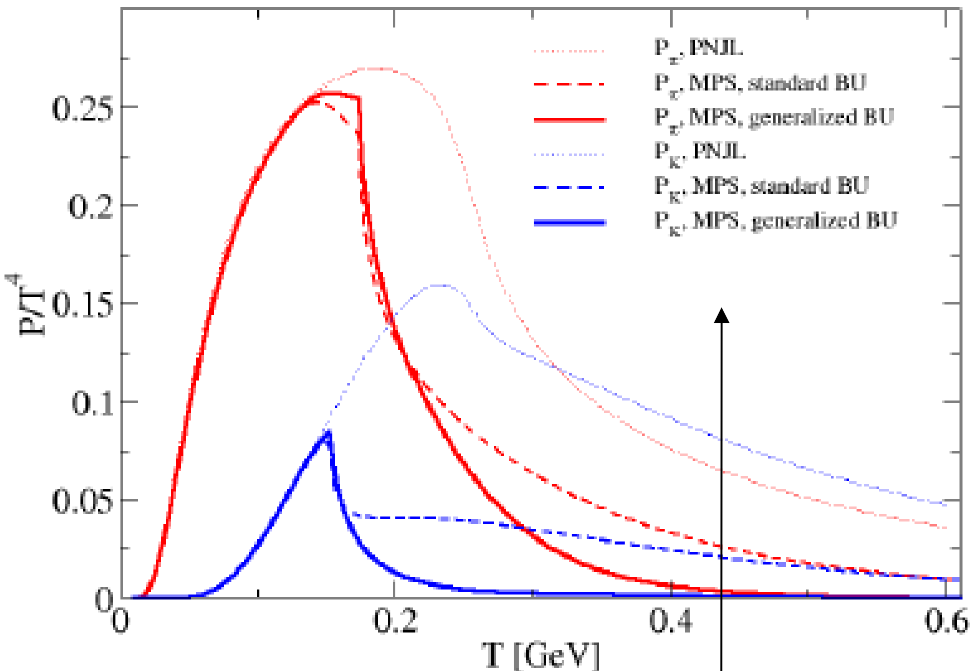
Quarks + rescattering effects

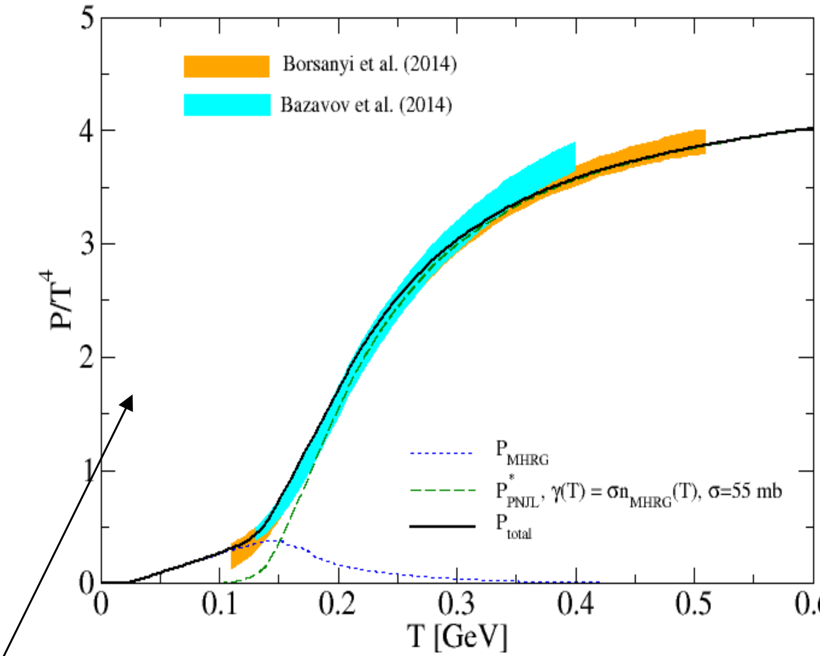
$$f_\Phi(\omega) = \frac{\Phi(1+2Y)Y + Y^3}{1+3\Phi(1+Y)Y+Y^3},$$

$$P^*_{\rm FG}(T) = 4N_c \sum_{q=u,d,s} \int \frac{dp\, p^2}{2\pi^2} \int \frac{d\omega}{\pi} f_\Phi(\omega)\delta_q(\omega;\gamma),$$

$$\delta_q(\omega;\gamma) = \frac{\pi}{2} + \arctan\left[\frac{\omega - \sqrt{p^2+m_q^2}}{\gamma}\right]$$

3. Mott HRG / PNJL – effective model

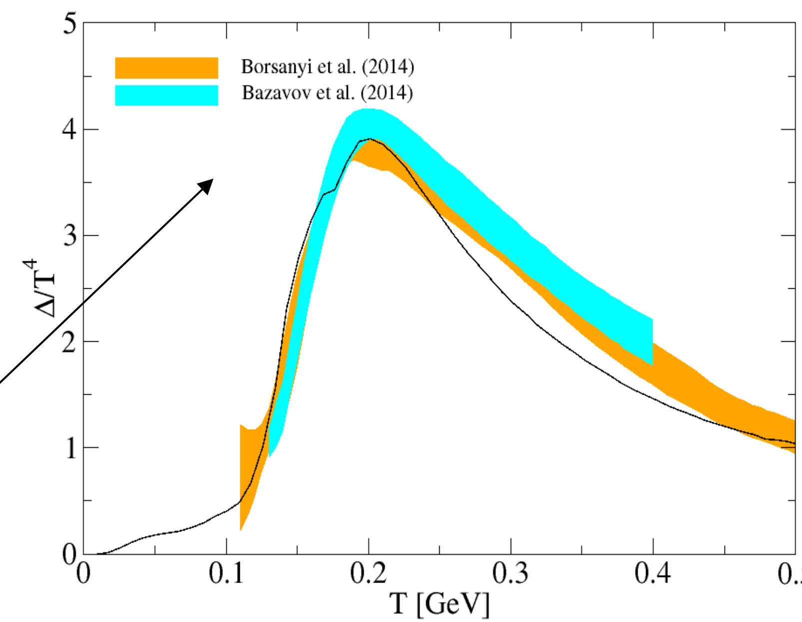




- Mott dissociation of hadrons (here pi, K) at the Chiral restoration temperature T_c = 153 MeV

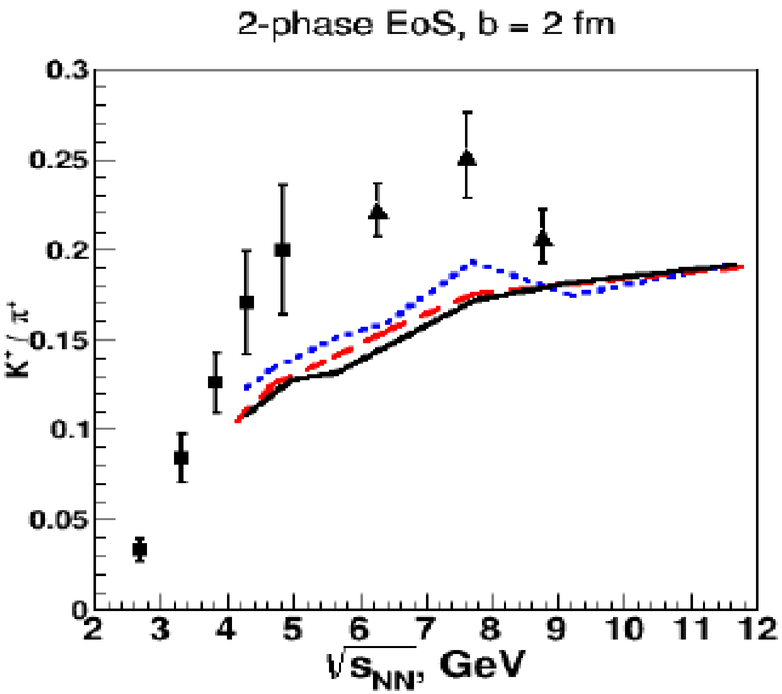
- Asymptotic behaviour of quark-gluon Pressure can be adjusted with rescattering Parameter gamma

- Very good correspondence between lattice QCD Thermodynamics and improved MHRG/PNJL model; Hadronic and partonic contributions quantified



D.B., A. Dubinin, L. Turko, arxiv:1612.09556

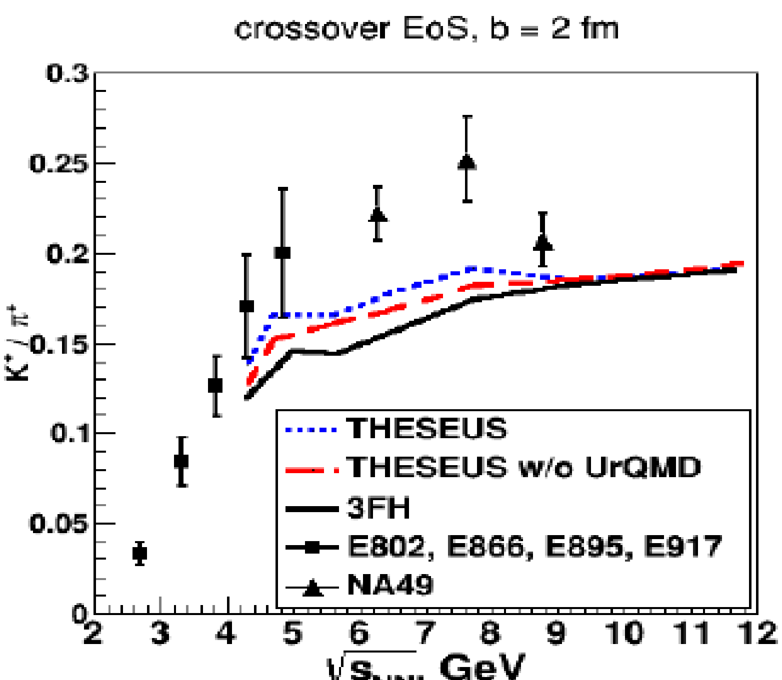
3. What about K+/π+ (Marek's horn) in THESEUS ?



THESEUS simulation reproduces 3FH result,
Thus it has the same discrepancy with experiment

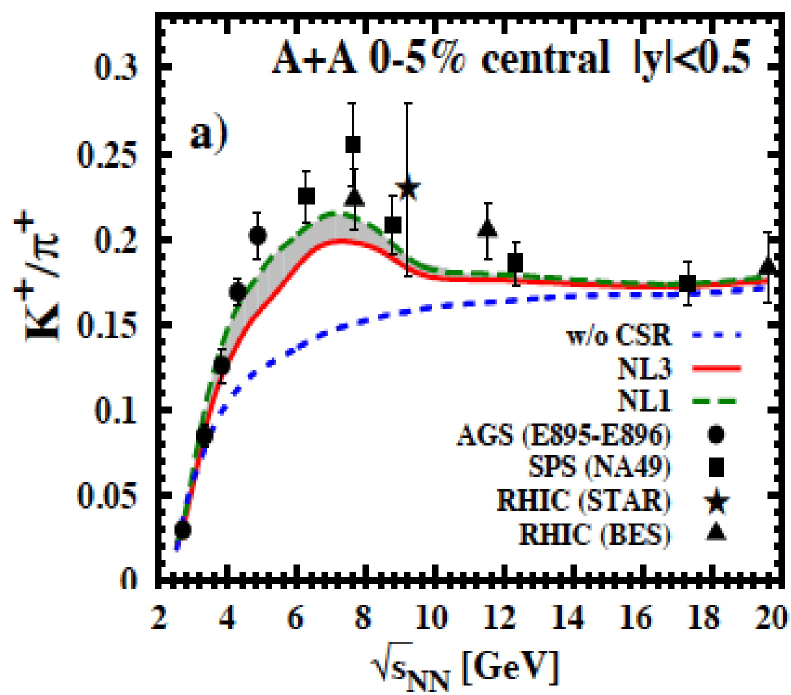
--> some key element still missing in the program

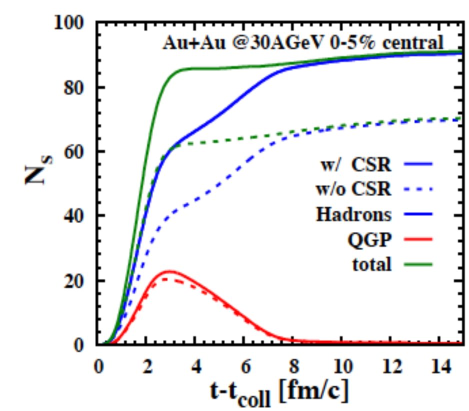
Batyuk, D.B., Bleicher, et al., PRC 94, 044917 (2016)



Recent new development in PHSD

Chiral symmetry restoration in HIC at intermediate ..."
A. Palmese et al., arxiv: 1607.04073; PRC 94, 044912





Strange particle number increase by CSR

3. Mott dissociation of π and K in hot, dense quark matter

D. Blaschke, A. Dubinin, A. Radzhabov, A. Wergieluk, arxiv:1608.05383

Andrey Radzhabov in front of the University of Wroclaw

3. PNJL model for N_f=2+1 quark matter with π and K

$$\mathcal{L} = \bar{q}\left(i\gamma^\mu D_\mu + \hat{m}_0\right)q + G_S \sum_{a=0}^{8}\left[(\bar{q}\lambda^a q)^2 + (\bar{q}i\gamma_5\lambda^a q)^2\right] - \mathcal{U}\left(\Phi[A],\bar{\Phi}[A];T\right)$$

$$\Pi^{M^a}_{ff'}(q_0,\mathbf{q}) = 2N_c T\sum_n \int \frac{d^3p}{(2\pi)^3}\mathrm{tr}_D\left[S_f(p_n,\mathbf{p})\Gamma^{M^a}_{ff'}S_{f'}(p_n+q_0,\mathbf{p}+\mathbf{q})\Gamma^{M^a}_{ff'}\right]$$

$$\Gamma^{P^a}_{ff'} = i\gamma_5\, T^a_{ff'}\ ,\quad \Gamma^{S^a}_{ff'} = T^a_{ff'}\ ,\quad T^a_{ff'} = \begin{cases}(\lambda_3)_{ff'},\\ (\lambda_1 \pm i\lambda_2)_{ff'}/\sqrt{2},\\ (\lambda_4 \pm i\lambda_5)_{ff'}/\sqrt{2},\\ (\lambda_6 \pm i\lambda_7)_{ff'}/\sqrt{2},\end{cases}$$

$$P^a = \pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0$$

$$\Pi^{P^a,S^a}_{ff'}(q_0+i\eta,\mathbf{0}) = 4\{I_1^f(T,\mu_f) + I_1^{f'}(T,\mu_{f'}) \mp \left[(q_0+\mu_{ff'})^2 - (m_f \mp m_{f'})^2\right] I_2^{ff'}(z,T,\mu_{ff'})\}$$

$$I_1^f(T,\mu_f) = \frac{N_c}{4\pi^2}\int_0^\Lambda \frac{dp\,p^2}{E_f}\left(n_f^- - n_f^+\right),$$

$$I_2^{ff'}(z,T,\mu_{ff'}) = \frac{N_c}{4\pi^2}\int_0^\Lambda \frac{dp\,p^2}{E_f E_{f'}}\left[\frac{E_{f'}}{(z-E_f-\mu_{ff'})^2 - E_{f'}^2}\,n_f^- \right.$$

$$\left. - \frac{E_{f'}}{(z+E_f-\mu_{ff'})^2 - E_{f'}^2}\,n_f^+ + \frac{E_f}{(z+E_{f'}-\mu_{ff'})^2 - E_f^2}\,n_{f'}^- - \frac{E_f}{(z-E_{f'}-\mu_{ff'})^2 - E_f^2}\,n_{f'}^+\right]$$

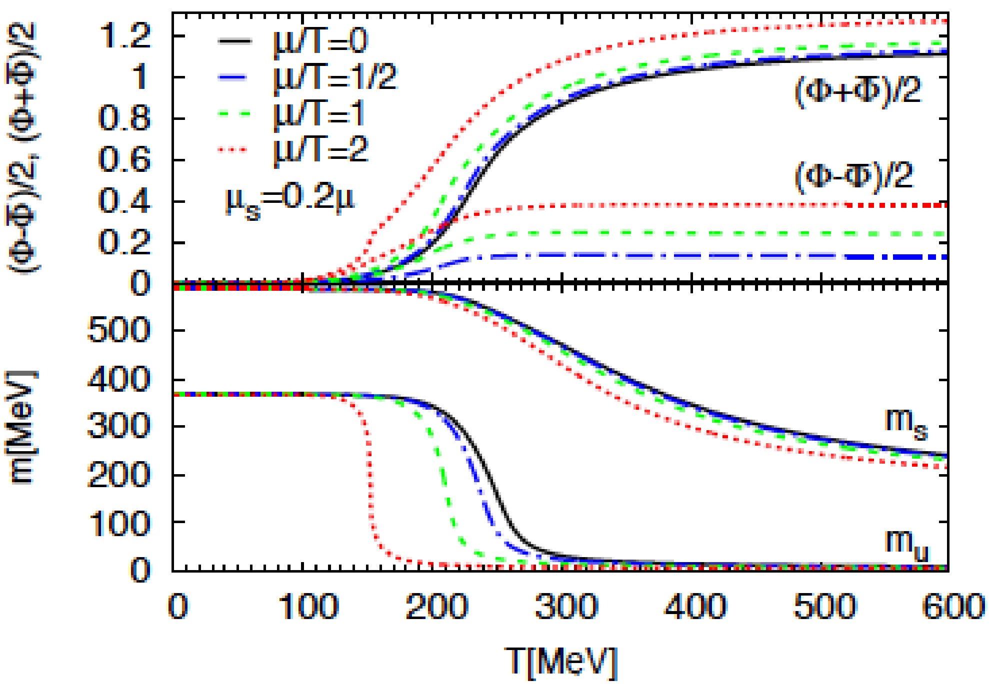
3. PNJL model for N_f=2+1 quark matter with π and K

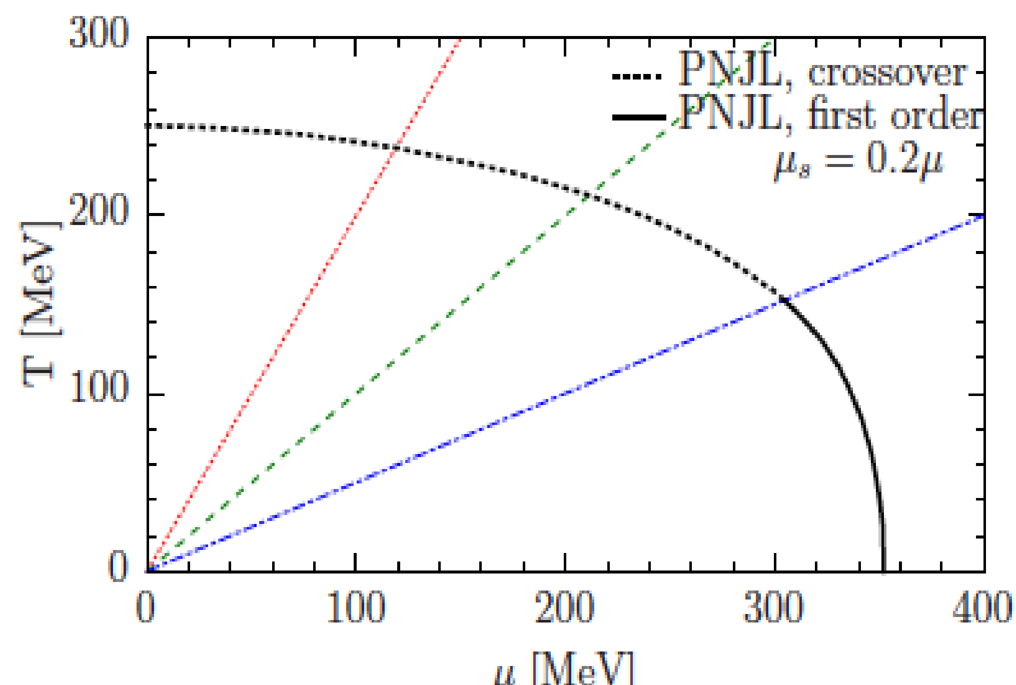
$$m_f = m_{0,f} + 16\, m_f G_S I_1^f(T,\mu), \quad \mathcal{P}^{M^a}_{ff'}(M_{M^a} + i\eta, \mathbf{0}) = 1 - 2G_S \Pi^{M^a}_{ff'}(M_{M^a} + i\eta, \mathbf{0}) = 0.$$

$$P_f = -\frac{(m_f - m_{0,f})^2}{8G} + \frac{N_c}{\pi^2}\int_0^\Lambda dp\, p^2\, E_f + \frac{N_c}{3\pi^2}\int_0^\infty \frac{dp\, p^4}{E_f}\left[f_\Phi^+(E_f) + f_\Phi^-(E_f)\right]$$

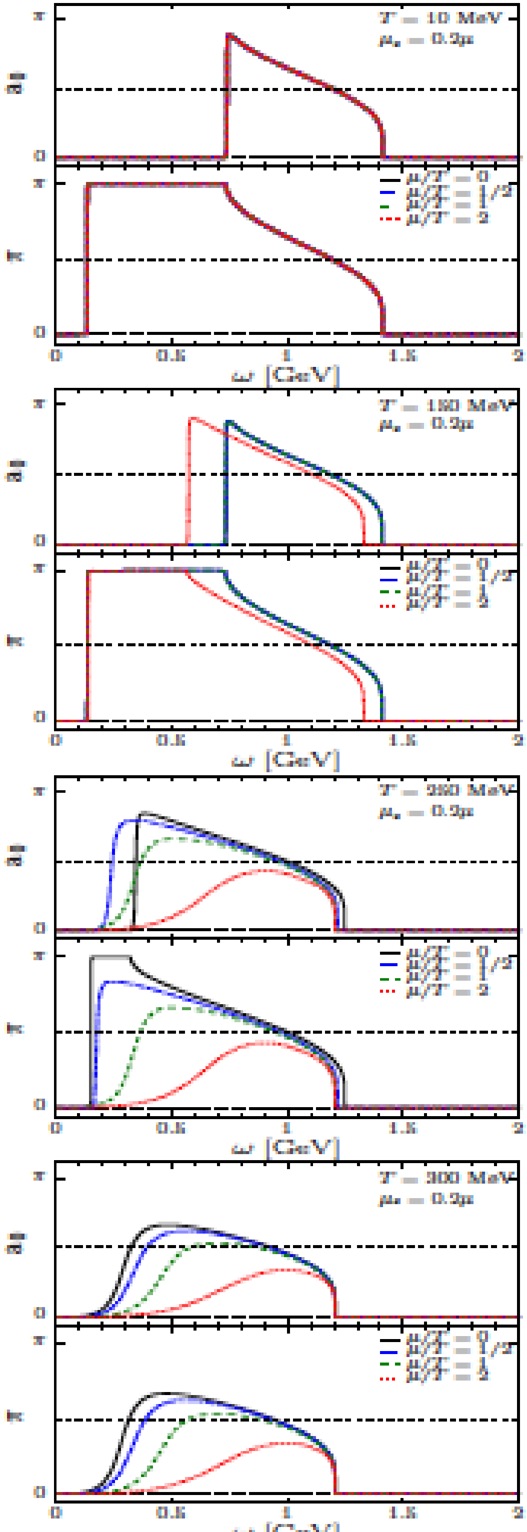
$$P_M = d_M \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi}\left\{ g(\omega - \mu_M) + g(\omega + \mu_M)\right\}\delta_M(\omega, \mathbf{q})$$

$$\delta_M(\omega, \mathbf{q}) = -\arctan\left\{\frac{\mathrm{Im}\left(\mathcal{P}^M_{ff'}(\omega - i\eta, \mathbf{q})\right)}{\mathrm{Re}\left(\mathcal{P}^M_{ff'}(\omega + i\eta, \mathbf{q})\right)}\right\}$$





3. Mott dissociation of pions and kaons in the Beth-Uhlenbeck approach ...



D.B., A. Dubinin, A. Radzhabov, A. Wergieluk, arxiv:1608.05383
D.B., M. Buballa, A. Dubinin, G. Ropke, D. Zablocki, Ann. Phys. (2014)

Thermodynamics of resonances (M) via phase shifts

$$P_M = d_M \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{ds}{4\pi} \frac{1}{\sqrt{s+q^2}} \left\{ g(\sqrt{s+q^2} - \mu_M) \right\} \delta_M(\sqrt{s}; T, \mu)$$

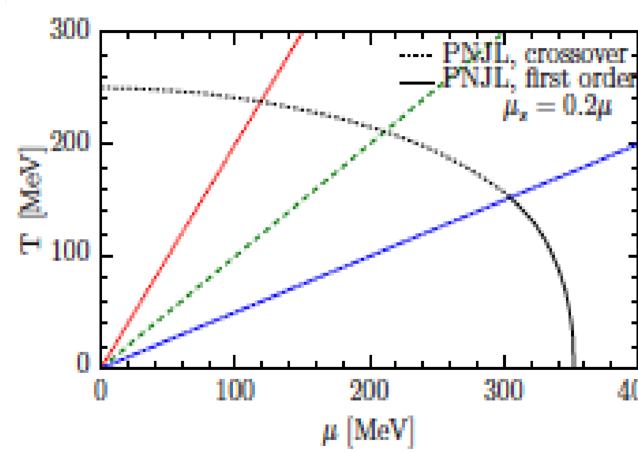
Polyakov-loop Nambu – Jona-Lasinio modell

$$\Pi^{M^a}_{ff'}(q_0, \mathbf{q}) = 2N_c T \sum_n \int \frac{d^3p}{(2\pi)^3} \mathrm{tr}_D \left[S_f(p_n, \mathbf{p}) \Gamma^{M^a}_{ff'} S_{f'}(p_n + q_0, \mathbf{p} + \mathbf{q}) \Gamma^{M^a}_{ff'} \right],$$

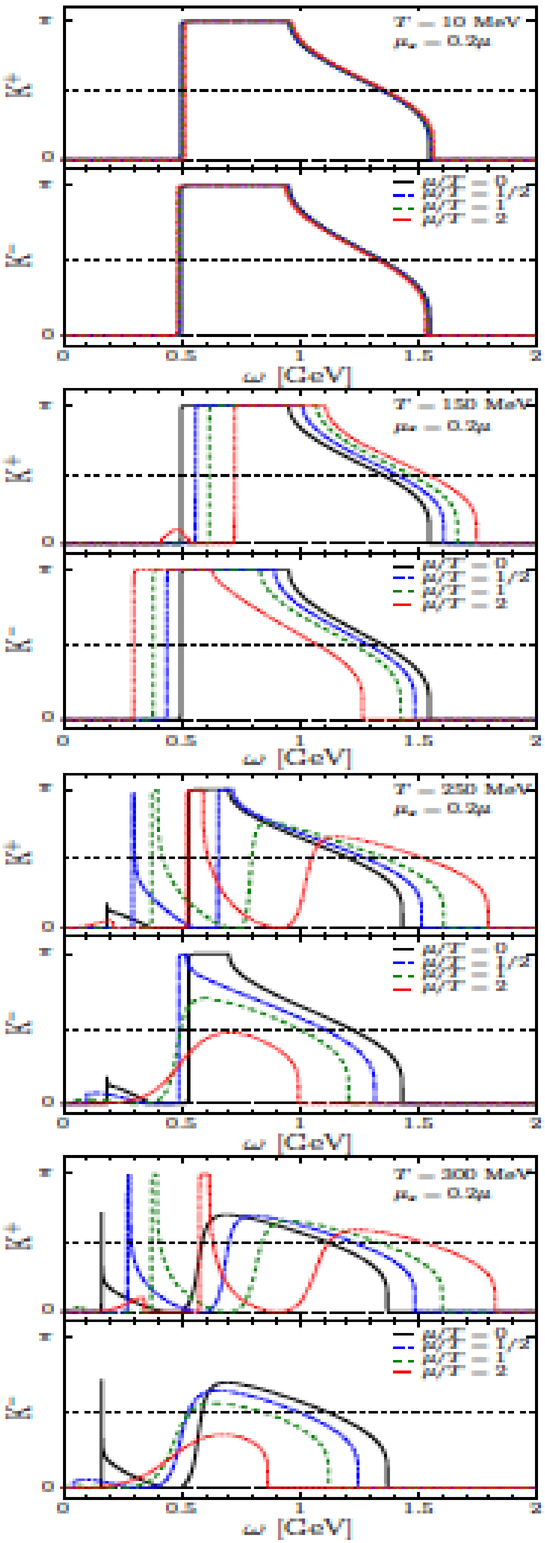
$$\mathcal{P}^{M^a}_{ff'}(M_{M^a} + i\eta, 0) = 1 - 2G_S \Pi^{M^a}_{ff'}(M_{M^a} + i\eta, 0)$$

$$\delta_M(\omega, \mathbf{q}) = -\arctan \left\{ \frac{\mathrm{Im}\left(\mathcal{P}^M_{ff'}(\omega - i\eta, \mathbf{q})\right)}{\mathrm{Re}\left(\mathcal{P}^M_{ff'}(\omega + i\eta, \mathbf{q})\right)} \right\}$$

Evaluation along trajectories
μ/T=const in the phase diagram:



- Pion and a0 as partner states,
- Chiral symmetry restoration,
- Mott dissociation of bound states,
- Levinson theorem



3. Mott dissociation of pions and kaons in the Beth-Uhlenbeck approach ...

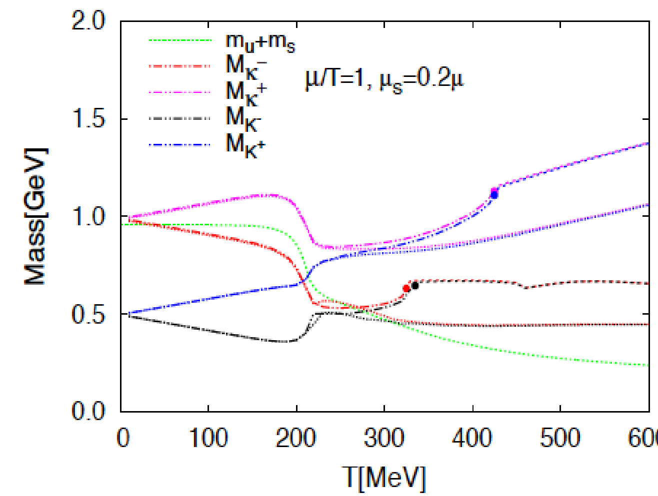
D.B., A. Dubinin, A. Radzhabov, A. Wergieluk, arxiv:1608.05383

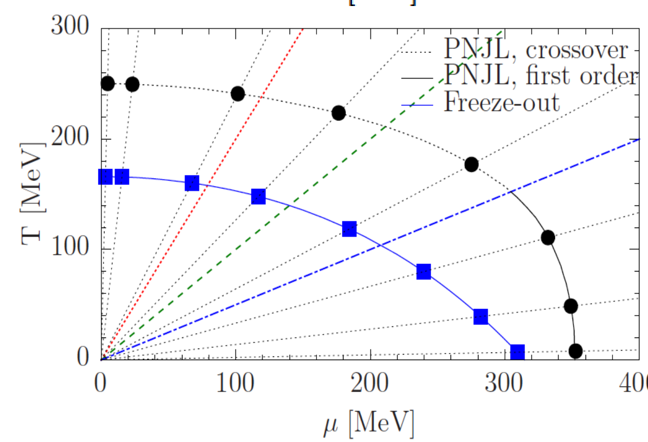
Polarization loop in Polyakov-loop Nambu – Jona-Lasinio model

$$\Pi^{P^a,S^a}_{ff'}(q_0+i\eta,\mathbf{0}) = 4\{I_1^f(T,\mu_f)+I_1^{f'}(T,\mu_{f'}) \mp \left[(q_0+\mu_{ff'})^2-(m_f\mp m_{f'})^2\right] I_2^{ff'}(z,T,\mu_{ff'})\}$$

$$I_1^f(T,\mu_f) = \frac{N_c}{4\pi^2}\int_0^\Lambda \frac{dp\,p^2}{E_f}\left(n_f^- - n_f^+\right),$$

$$I_2^{ff'}(z,T,\mu_{ff'}) = \frac{N_c}{4\pi^2}\int_0^\Lambda \frac{dp\,p^2}{E_f E_{f'}} \left[\frac{E_{f'}}{(z-E_f-\mu_{ff'})^2-E_{f'}^2}\,n_f^- - \frac{E_{f'}}{(z+E_f-\mu_{ff'})^2-E_{f'}^2}\,n_f^+ + \frac{E_f}{(z+E_{f'}-\mu_{ff'})^2-E_f^2}\,n_{f'}^- - \frac{E_f}{(z-E_{f'}-\mu_{ff'})^2-E_f^2}\,n_{f'}^+\right]$$



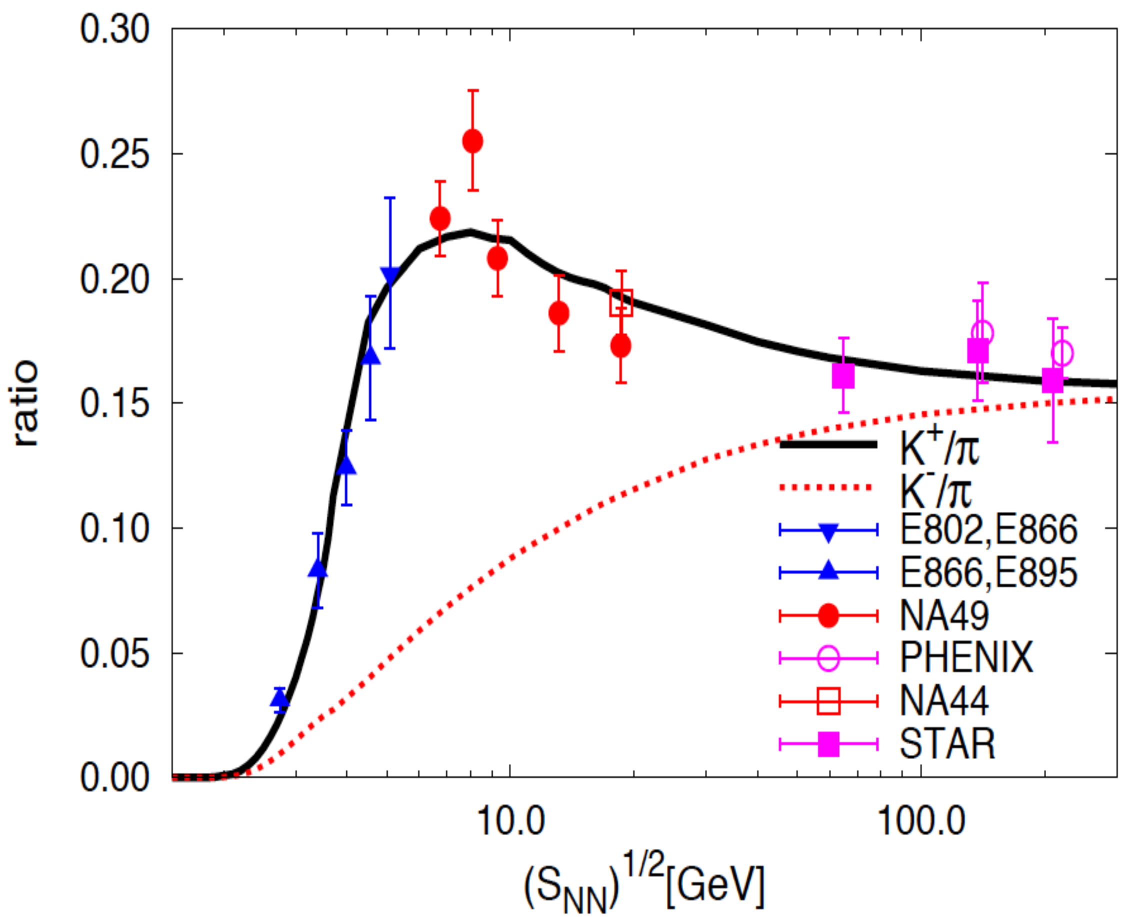


Anomalous low-mass mode for K+ in the dense medium !!

3. Mott dissociation of pions and kaons in Beth-Uhlenbeck: Explanation of the “horn” effect for K+/π+ in HIC?

Ratio of yields in BU approach defined via phase shifts:

$$\frac{n_{K^\pm}}{n_{\pi^\pm}} = \frac{\int dM \int d^3p \; (M/E) g_{K^\pm}(E)[1 + g_{K^\pm}(E)]\delta_{K^\pm}(M)}{\int dM \int d^3p \; (M/E) g_{\pi^\pm}(E)[1 + g_{\pi^\pm}(E)]\delta_{\pi^\pm}(M)}$$



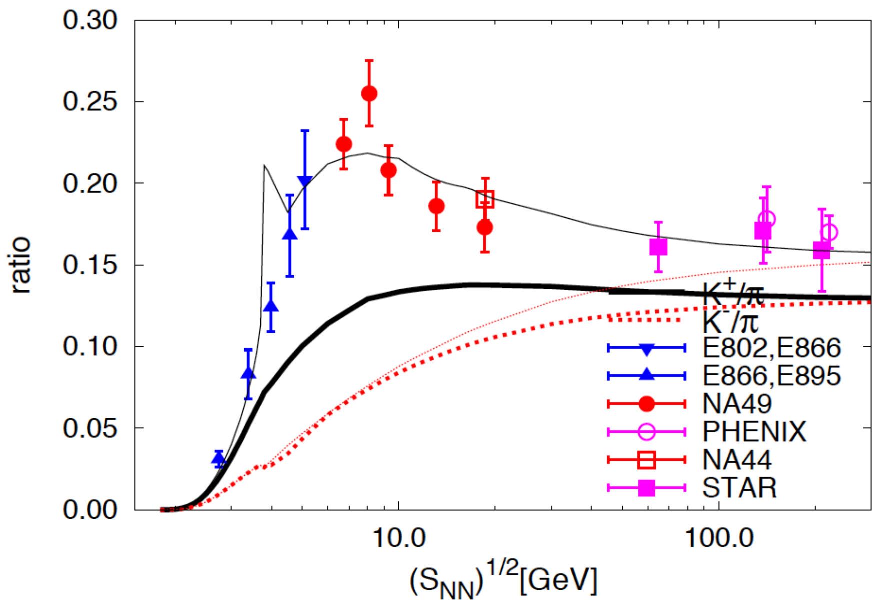
Evaluation along the freeze-out
Curve parametrized by Cleymans et al.

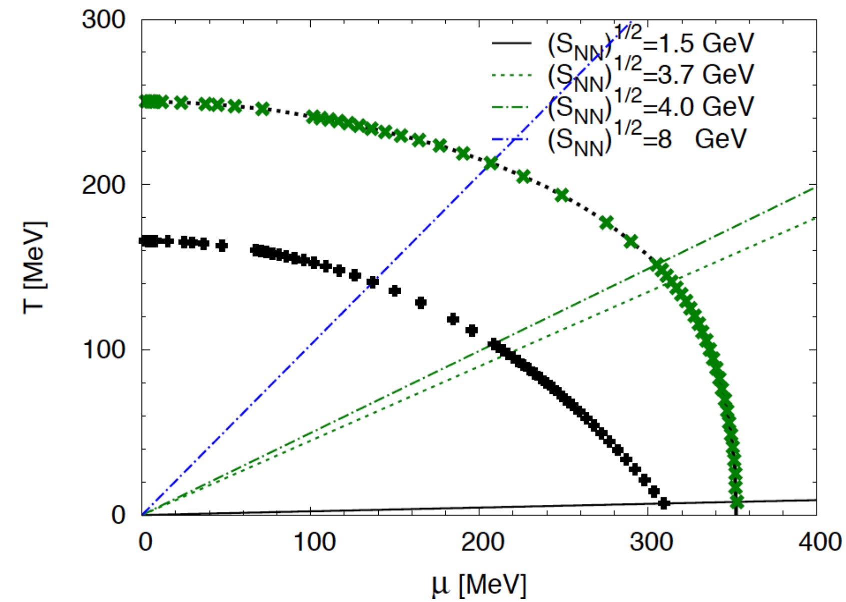
- enhancement for K+ due to anomalous in-medium bound state mode
- no such enhancement for K- or pions

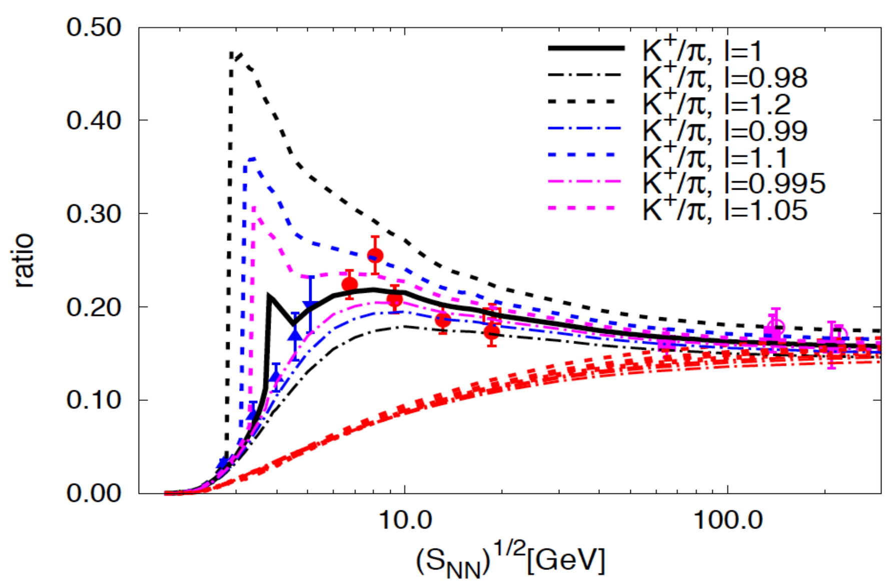
- explore the effect in thermal statistical models and in THESEUS ...

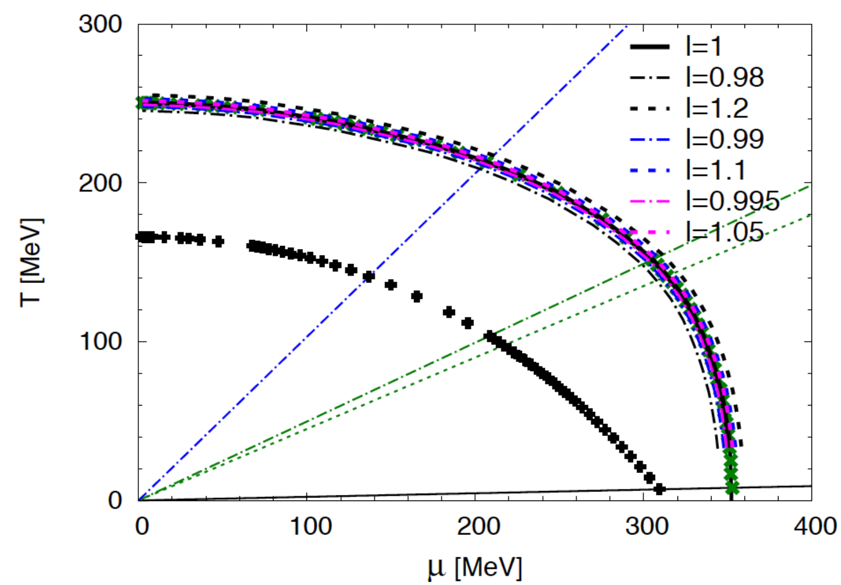
D.B., A. Dubinin, A. Radzhabov, A. Wergieluk, arxiv:1608.05383

3. “Tooth” on the “horn” due to anomalous K+; sign of CEP?





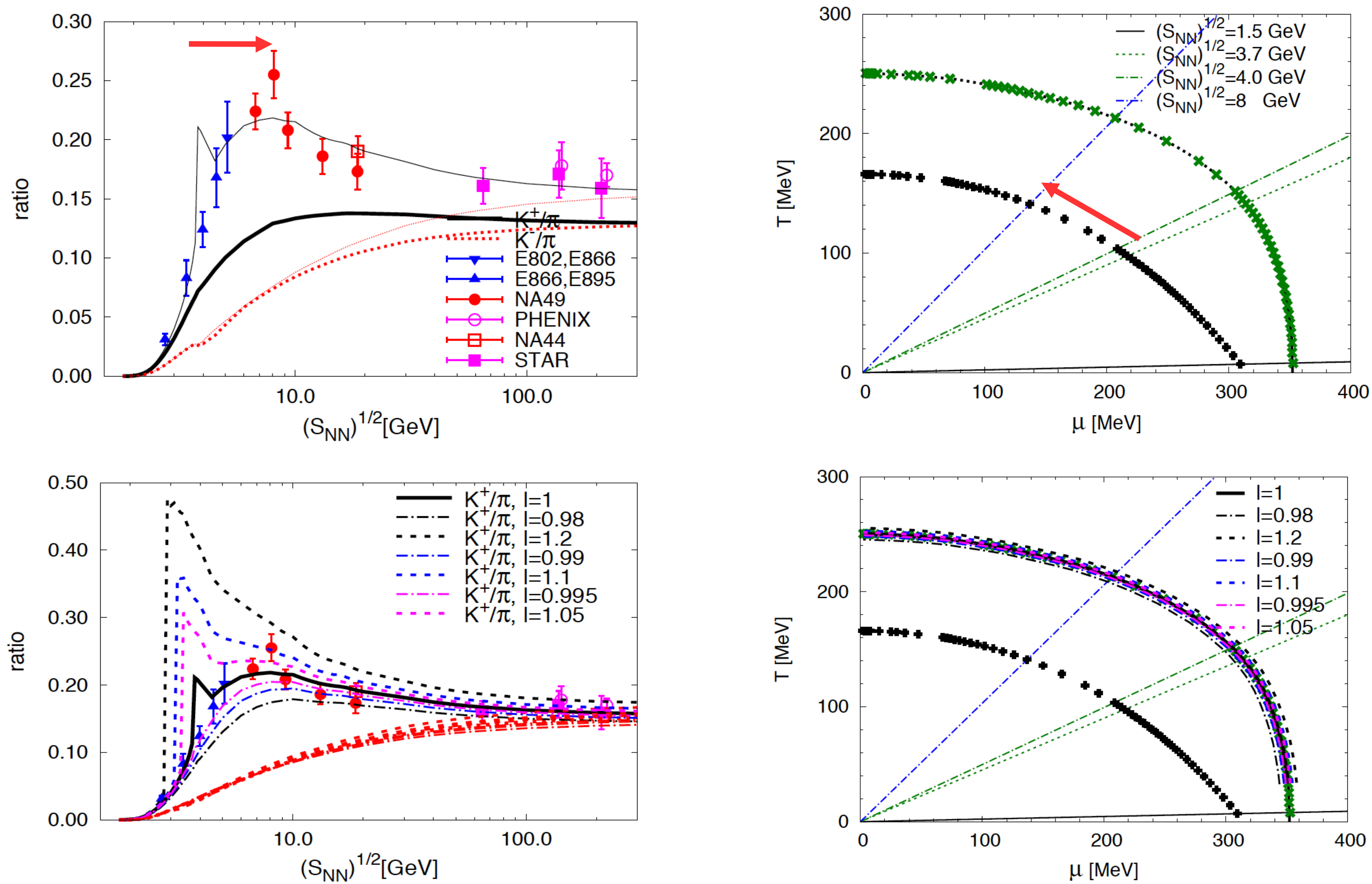




- enhancement for K+ due to anomalous in-medium bound state mode

D.B., A. Radzhabov, in prep. (2017)

3. “Tooth” on the “horn” due to anomalous K+; sign of CEP?



- “tooth” correlated to the CEP → indicator for CEP !!

D.B., A. Radzhabov, in prep. (2017)

Summary:

- GBU accounts consistently for hadron formation and dissociation (Mott effect)
  Inverse: Mott-Anderson Hadronization !!

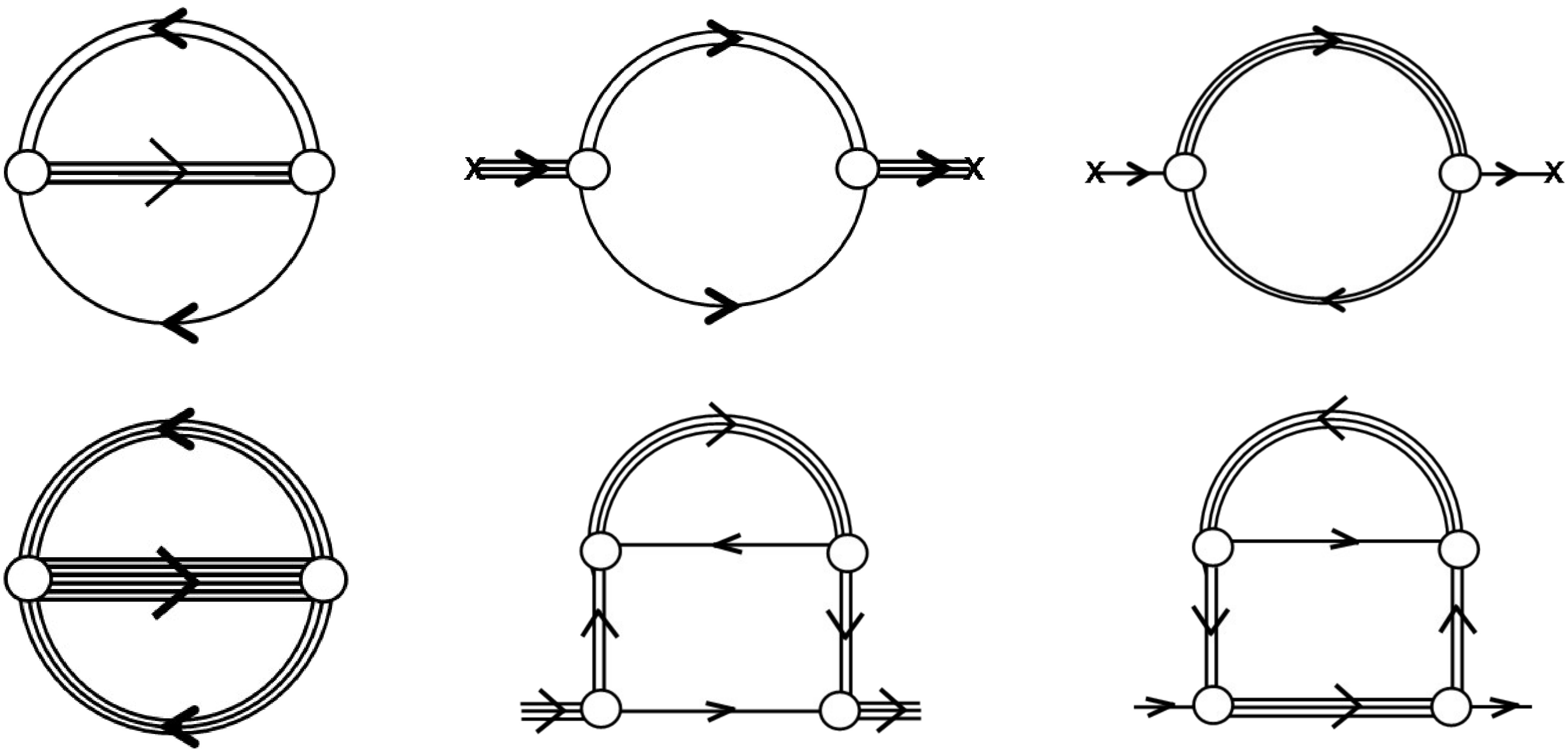
- Quark Pauli blocking leads to stiffening hadronic EoS, precursor of deconfinement

- New modes in medium due to BSE dynamics (e.g., K+)

Additional Slides

Example C: Nucleons in quark matter

Φ-functional | nucleon selfenergy | quark selfenergy



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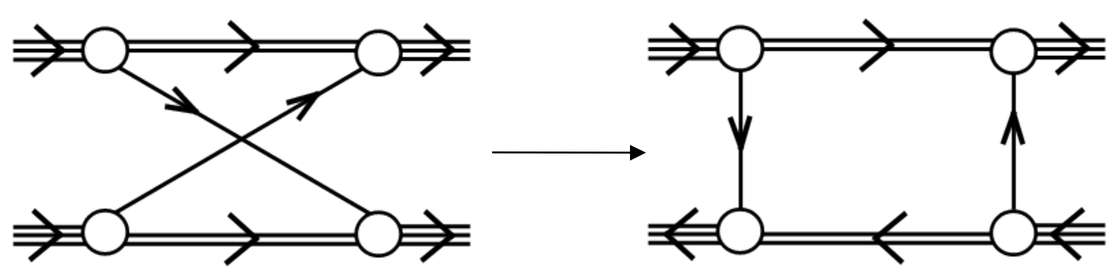
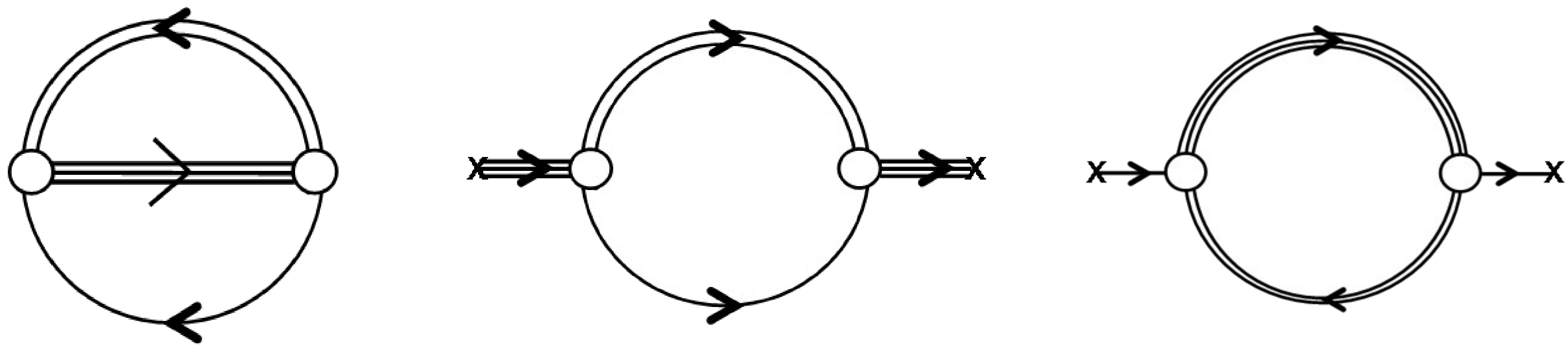
quark exchange interaction
between nucleons:

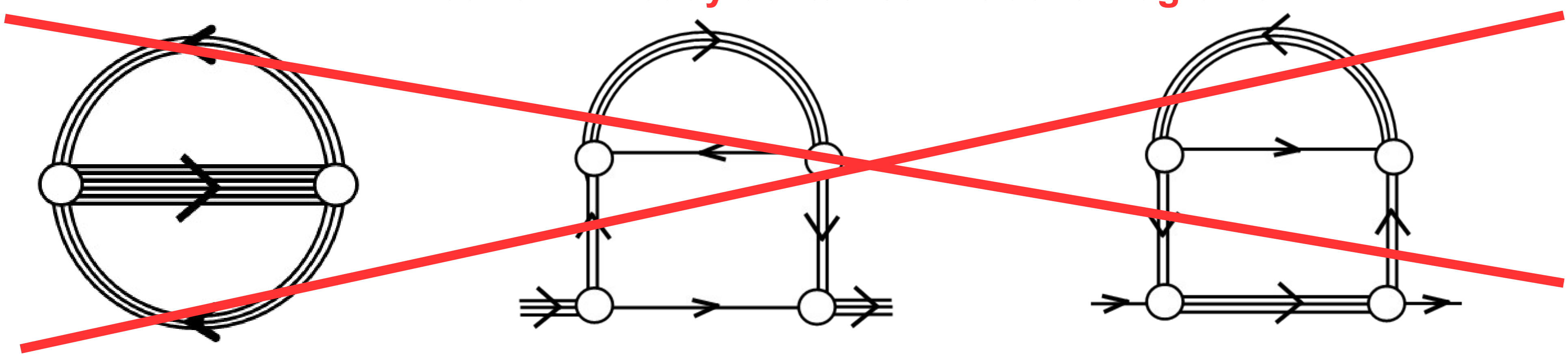


Example C: Nucleons in quark matter

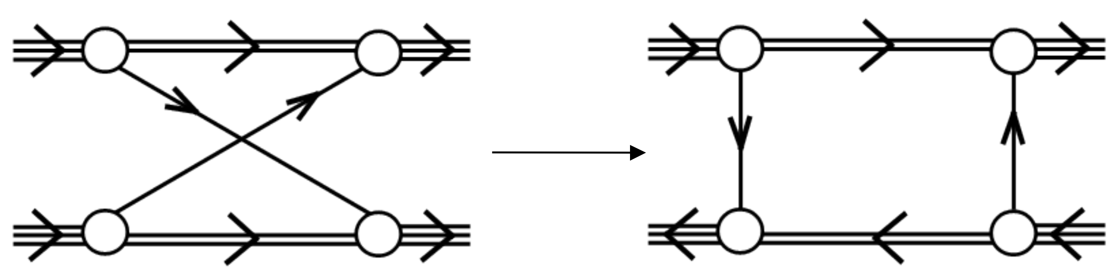
Φ-functional | nucleon selfenergy | quark selfenergy



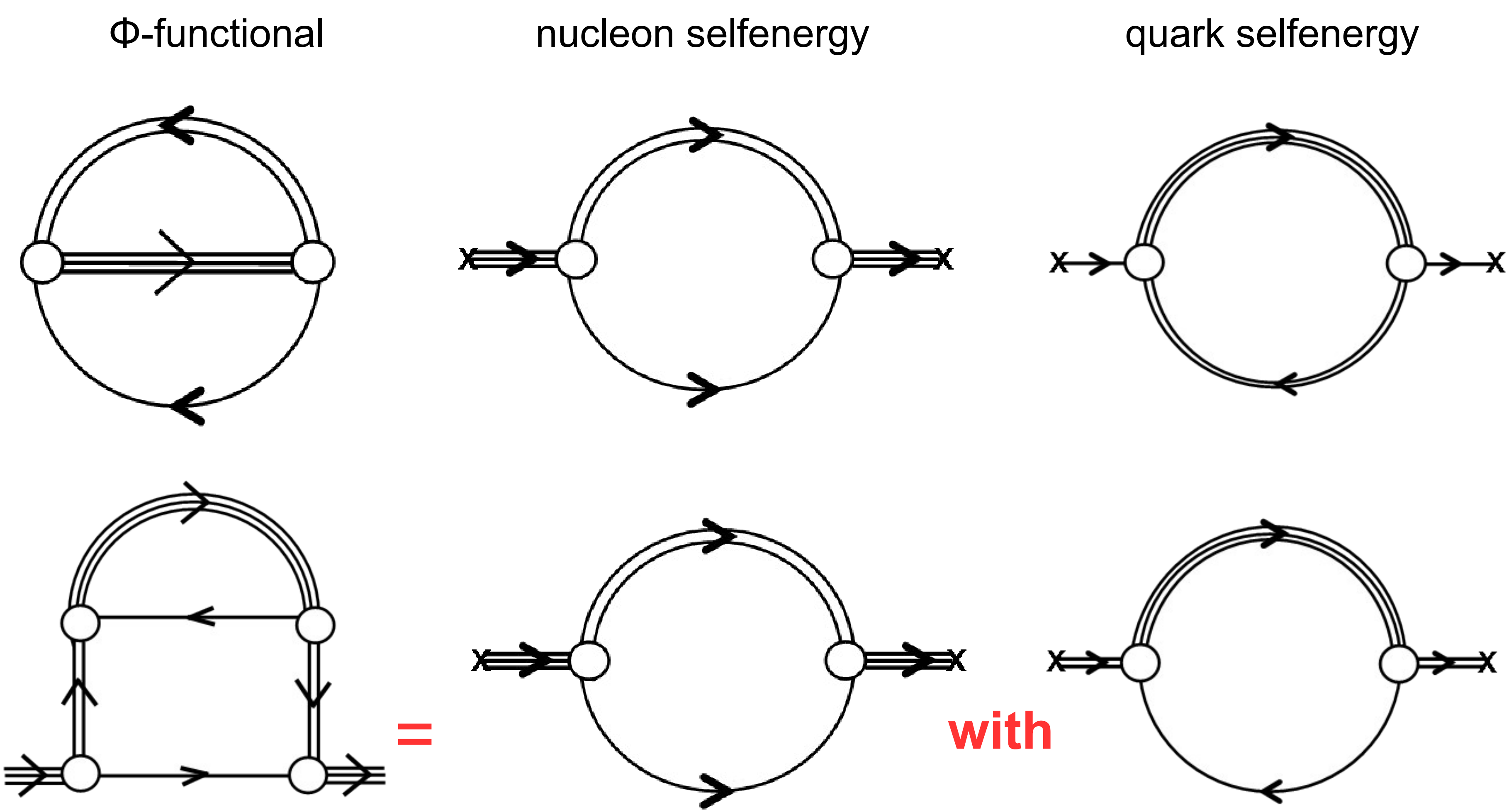
Not new! Already contained in above diagrams!



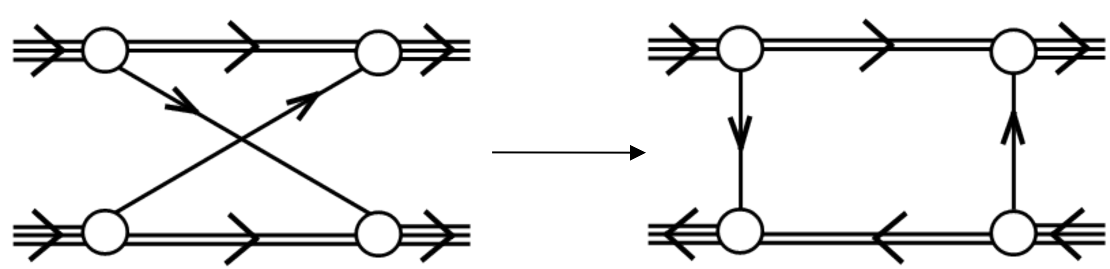
quark exchange interaction
between nucleons:



Example C: Nucleons in quark matter



quark exchange interaction
between nucleons:



Intermezzo: Structure of the baryon?

12-Apostle Church, Kars

Intermezzo: Structure of the baryon?

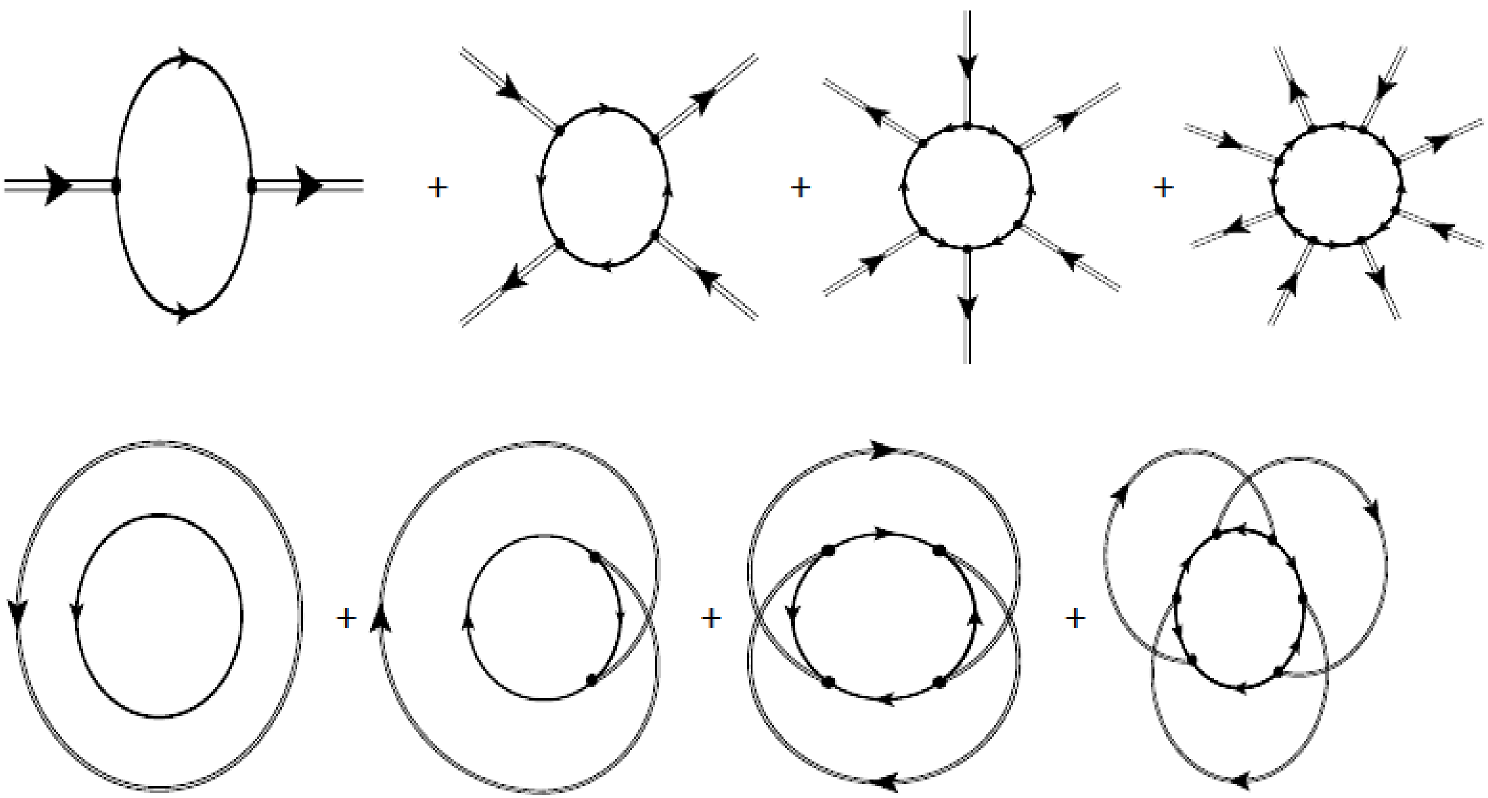
12-Apostle
Church,
Kars

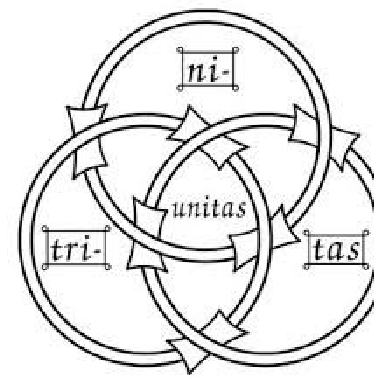
Intermezzo: Structure of the baryon?

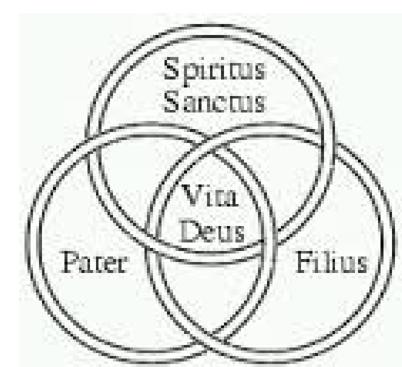
Borromean ? !!

$$Z_{\text{fluct}} = \int D\Delta^\dagger D\Delta D\phi \exp\{-\frac{|\Delta|^2}{4G_D} - \frac{\phi^2}{4G} - Tr\, lnS^{-1}[\Delta, \Delta^\dagger, \phi]\}$$

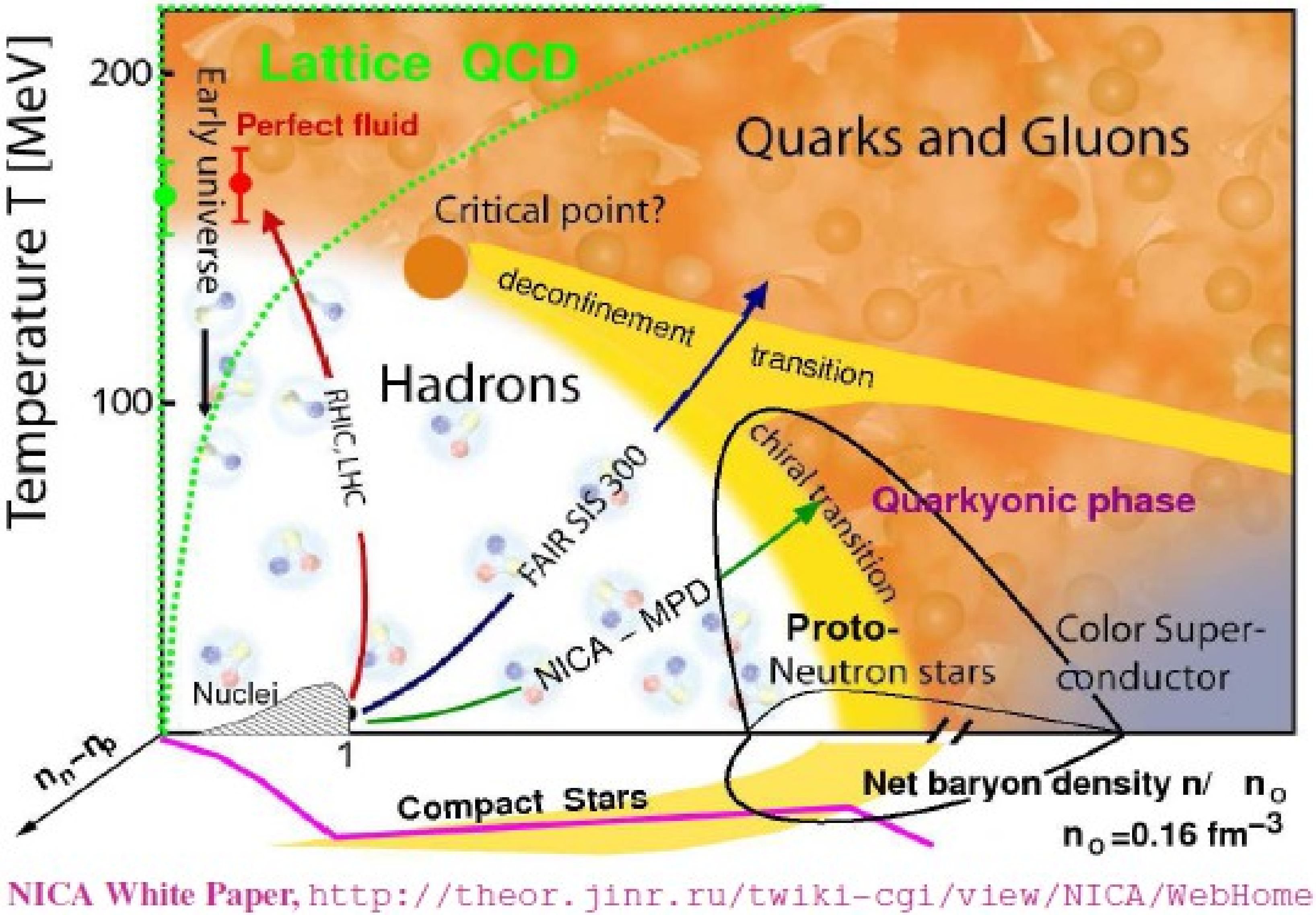
Cahill, Roberts, Prashifka: Aust. J. Phys. 42 (1989) 129, 161
Cahill, ibid, 171; Reinhardt: PLB 244 (1990) 316; Buck, Alkofer, Reinhardt: PLB 286 (1992) 29







Support a CEP in QCD phase diagram with Astrophysics?



S. Benic et al., A&A 577, A40 (2015)

Crossover at finite T (Lattice QCD) + First order at zero T (Astrophysics) = Critical endpoint exists!

Introduction: Beth-Uhlenbeck vs. Generalized BU

Beth-Uhlenbeck: 2nd virial coefficient B(T)

$$pV = NkT\left(1 + \frac{B(T)}{V} + \frac{C(T)}{V^2} + \dots\right)$$

BU for virial expansion of density:

$$n(\mu, T) = n_{\text{free}}(\mu, T) + 2n_{\text{corr}}(\mu, T)$$

$$n_{\text{free}}(\mu, T) = 4\int \frac{d^3p}{h^3} e^{-(p^2/2m-\mu)/T} = \frac{4}{\lambda^3} e^{\mu/T}$$

$$n_{\text{corr}}(\mu, T) = \int \frac{d^3\mathbf{P}}{h^3} e^{-(P^2/4m-2\mu)/T} \int_{-\infty}^{\infty} \frac{dE}{\pi} e^{-E/T} D(E)$$

$$= \frac{2^{3/2}}{\lambda^3} e^{2\mu/T} \int_{-\infty}^{\infty} \frac{dE}{\pi} e^{-E/T} D(E).$$

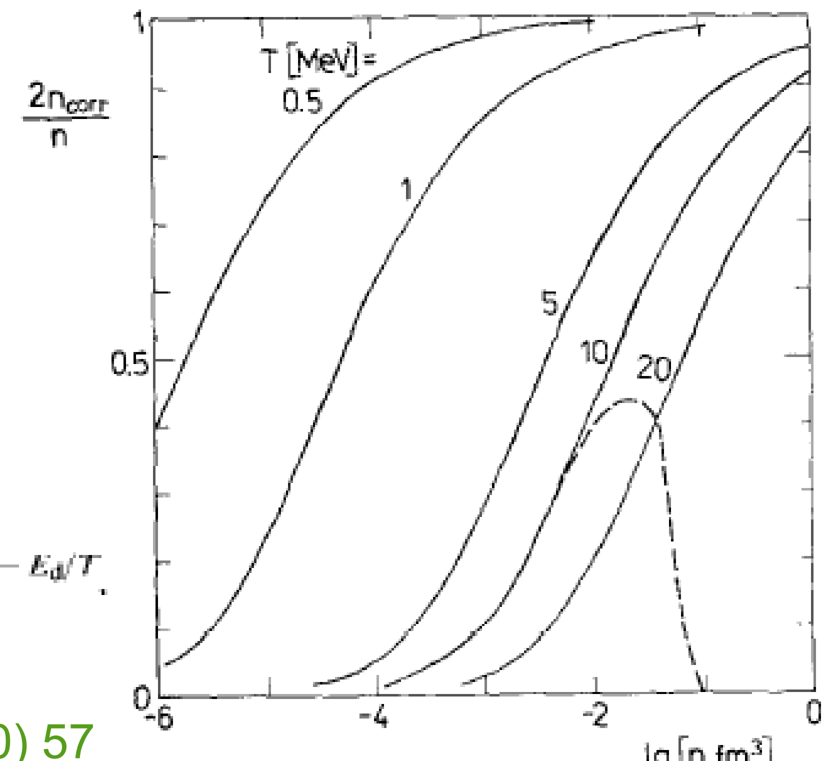
Density of states: bound and scattering part

$$D(E) = \sum_\alpha c_\alpha \left[\pi\delta(E - E_\alpha) + \frac{d}{dE}\delta_\alpha(E)\right],$$

Example: Deuterons in nuclear matter

$$n = n_{\text{free}} + 2n_{\text{free}}^2 I(T)$$

$$I(T) = \lambda^3 \frac{2^{1/2}}{8}\left[3(e^{-E_d/T} - 1) + \int_0^{\infty} \frac{dE}{\pi T} e^{-E/T} \sum_\alpha c_\alpha \delta_\alpha(E)\right].$$



For T<<E$_d$: $n = n_{\text{free}} + 2n_{\text{deut}}\,,\quad n_{\text{deut}} = n_{\text{free}}^2 \lambda^3 3\,\frac{2^{1/2}}{8} e^{-E_d/T}.$

E. Beth and G.E. Uhlenbeck, Physica IV (1937) 915;
S. Schmidt, G. Roepke, H. Schulz, Ann. Phys. 202 (1990) 57

Introduction: Beth-Uhlenbeck vs. Generalized BU

Thermodynamic Greens function approach:

$$n(1,\mu_1,T)=\int\frac{dE}{2\pi}f_1(E)\,A(1,E)$$

$$A(1,E)=\frac{2\Sigma_I(1,E-i0)}{(E-E(1)-\Sigma_R(1,E))^2+\Sigma_I(1,E-i0)^2}\approx\frac{2\pi\,\delta(E-e(1))}{1-((d/dz)\,\Sigma_R(1,z))|_{z=e(1)+i0}}-2\Sigma_I(1,E+i0)\frac{d}{dE}\frac{\mathrm{P}}{E-e(1)}$$

Density formula
(free and corr. Quasiparticles):

$$n(\mu,T)=n_{\text{free}}(\mu,T)+2n_{\text{corr}}(\mu,T),$$

$$\Sigma(1,z_\nu)=T\sum_2\sum_{z_{\nu'}}[T(1212,z_\nu+z_{\nu'})-\text{ex}]\,G(2,z_{\nu'}).$$

$$n_{\text{corr}}(\mu,T)=\int\frac{dE}{2\pi}g(E)\,F(E)$$

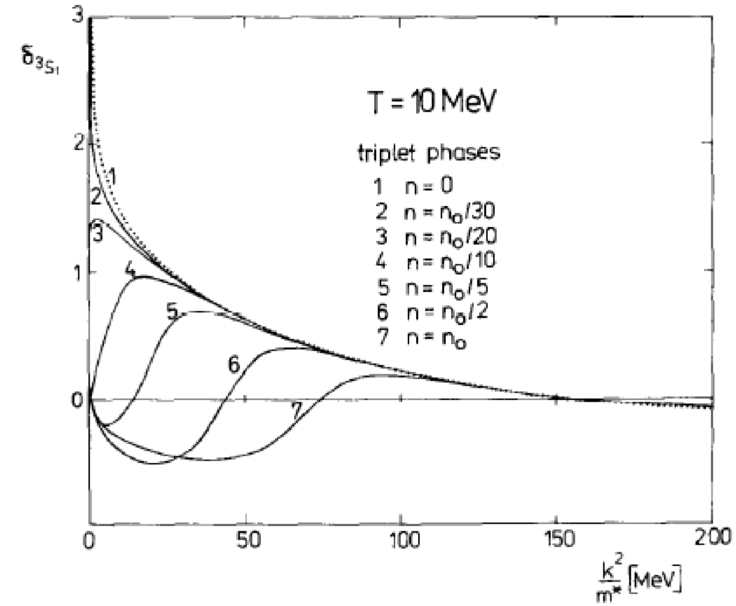
$$F(E)=F_{\text{deut}}(E)+\frac{2}{4\pi}\sum_\alpha c_\alpha F_\alpha(E),$$

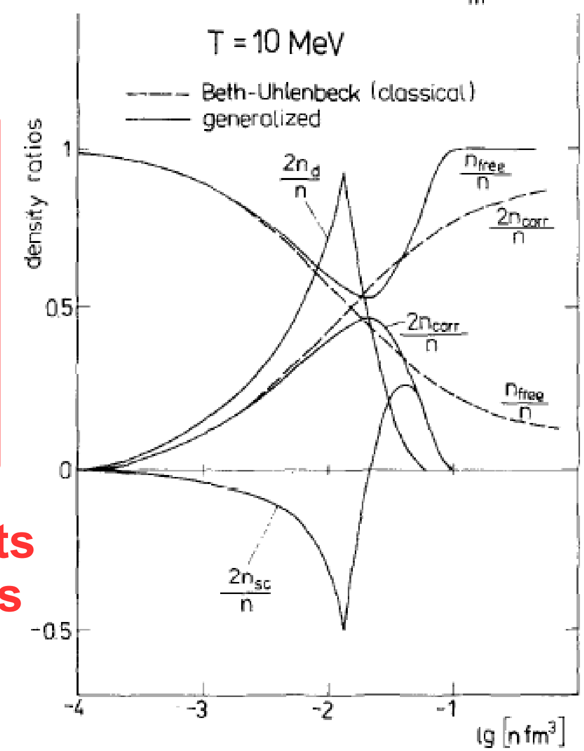
$$F_{\text{deut}}(E)=6\sum_{K>K^{\text{Mott}}}\pi\delta(E-E_b(K,\mu,T)).$$

$$F(E)=\sum_{12}[1-f(e(1))-f(e(2))]\cdot\left[(T_I(1212,E+i0)-\text{ex})\frac{d}{dE}\frac{\mathrm{P}}{e(1)+e(2)-E}-\pi\delta(E-e(1)-e(2))\frac{d}{dE}(T_R(1212,E+i0)-\text{ex})\right].$$

$$F_\alpha(E)=8\pi\sum_K\sin^2\delta_\alpha(E,K,\mu,T)\frac{d}{dE}\delta_\alpha(E,K,\mu,T).$$

The $\sin^2\delta$ term accounts for quasiparticle effects





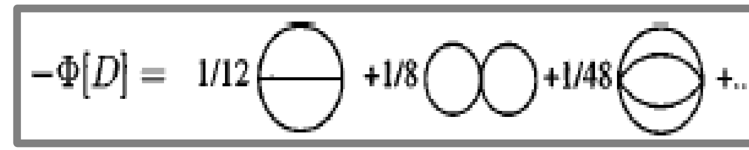
S. Schmidt, G. Röpke, H. Schulz, Ann. Phys. 202 (1990) 57

Φ-derivable approach, 2-loop approximation

J.-P. Blaizot, E. Iancu, A. Rebhan, Phys. Rev. D 63 (2001) 065003

Skeleton expansion for thermodynamic potential and entropy

$$\beta\Omega[D] = -\log Z = \frac{1}{2}\mathrm{Tr}\log D^{-1} - \frac{1}{2}\mathrm{Tr}\,\Pi D + \Phi[D]$$



Inv. Temp: 1/T trace in conf. Space self-energy related to D

Dyson equation: $D^{-1} = D_0^{-1} + \Pi$ Free propagator Do is known

Essential property of Ω[D] is Stationarity under variation of D: δ Ω[D] / δ D = 0

This implies δ Φ[D] / δ D = 1/2 Π

Physical propagator and selfenergy are defined self-consistently !

Self-consistent approximations are defined by the choice of Φ

→ **Φ – derivable theories**

G. Baym, Phys. Rev. 127 (1962) 1391; Vanderheyden & Baym; J. Stat. Phys. 93, 843 (1998)

Approximately selfconsistent thermodynamics

Matsubara summation:

$$\Omega/V = \int \frac{d^4k}{(2\pi)^4} n(\omega)[\mathrm{Im}\log(-\omega^2+k^2+\Pi) - \mathrm{Im}\,\Pi D] \;\; + T\Phi[D]/V$$

Analytic properties:

$$D(\omega,k) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\rho(k_0,k)}{k_0-\omega}, \qquad \mathrm{Im}\,D(\omega,k) \equiv \mathrm{Im}\,D(\omega+i\epsilon,k) = \frac{\rho(\omega,k)}{2}.$$

Thermodynamics from entropy density: $\mathcal{S} = -\partial(\Omega/V)/\partial T$

$$\mathcal{S} = -\int \frac{d^4k}{(2\pi)^4} \frac{\partial n(\omega)}{\partial T} \mathrm{Im}\log D^{-1}(\omega,k) \; + \int \frac{d^4k}{(2\pi)^4} \frac{\partial n(\omega)}{\partial T} \mathrm{Im}\,\Pi(\omega,k)\mathrm{Re}\,D(\omega,k) + \mathcal{S}'$$

$$\mathcal{S}' \equiv -\left.\frac{\partial(T\Phi/V)}{\partial T}\right|_D + \int \frac{d^4k}{(2\pi)^4} \frac{\partial n(\omega)}{\partial T} \mathrm{Re}\,\Pi\,\mathrm{Im}\,D \longrightarrow 0$$

for two-loop skeleton diagrams

Loosely speaking: S' accounts for residual interactions of "independent quasiparticles"

$d/d\omega$ [Im log D^{-1} + ImΠ ReD] = 2 Im [D ImΠ ($d/d\omega$ D^*) ImΠ] = 2 $\sin^2\delta$ $d\delta/d\omega$, for $D = |D|e^{i\delta}$

D. B., in preparation (2017)

Proof of cancellations resulting in S'=0 (I)

$$S' \equiv -\frac{\partial(T\Phi/V)}{\partial T}\bigg|_D + \int \frac{d^4k}{(2\pi)^4}\left\{\frac{\partial n(\omega)}{\partial T}\mathrm{Re}\Pi\,\mathrm{Im}D\right\}$$

First term

$$-\frac{T}{V}\Phi = \frac{g^2}{12}T^2\sum_{\omega_1,\omega_2}\int\frac{d^3k_1d^3k_2}{(2\pi)^6}D(\omega_1,|k_1|)D(\omega_2,|k_2|)D(-\omega_1-\omega_2,|-k_1-k_2|)$$

Spectral representation

$$D(\omega,k) = \int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\rho(k_0,k)}{k_0-\omega}$$

Matsubara sums

$$-\frac{T}{V}\Phi = \frac{g^2}{12}T^2\sum_{\omega_1,\omega_2}\int\frac{d^4kd^4k'd^4k"}{(2\pi)^9}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\rho(k)\rho(k')\rho(k'')\frac{-1}{\omega_1-k_0}\frac{-1}{\omega_2-k_0'}\frac{1}{\omega_1+\omega_2+k_0''}$$

Partial fraction decomposition of the three energy denominators and Matsubara summation over ω_1, ω_2 yields:

$$\frac{1}{k_0+k_0'+k_0''}\{[n(k_0'')+1][n(k_0)+n(k_0')+1]+n(k_0)n(k_0'')\}$$

Temperature derivative and renaming variables under the integrals

$$\partial_T\left[n(k_0+n(k_0')+n(k_0'')+n(k_0')n(k_0)+n(k_0')n(k_0'')+n(k_0)n(k_0'')\right]\to 3\partial_T n(k_0)\left[1+n(k_0')+n(k_0'')\right]$$

Proof of cancellations resulting in S'=0 (II)

Second term:

$$\begin{aligned}\mathrm{Re}\Pi(\omega,q) &= -\frac{g^2}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{dk_0}{2\pi}\int\frac{dk_0'}{2\pi}\rho(k_0,|k|)\rho(k_0',|\mathbf{k}+\mathbf{q}|)\sum_{\omega_1}\frac{1}{\omega_1-k_0}\,\frac{1}{\omega_1+\omega-k_0'}\\ &= -\frac{g^2}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{dk_0}{2\pi}\int\frac{dk_0'}{2\pi}\rho(k_0,|k|)\rho(k_0',|\mathbf{k}+\mathbf{q}|)\frac{1+n(k_0)+n(k_0')}{\omega+k_0+k_0'}\end{aligned} \tag{7}$$

$$\int\frac{d^4q}{(2\pi)^4}\frac{\partial n(k_0)}{\partial T}\mathrm{Re}\Pi(\omega,q)\mathrm{Im}D(\omega,q) =$$

$$= -\frac{g^2}{2\cdot 2}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^k}{(2\pi)^4}\int\frac{d^4k'}{2\pi}\delta^{(3)}(\mathbf{q}+\mathbf{k}+\mathbf{k}')\rho(q)\rho(k)\rho(k')\partial_T n(q_0)\left[1+n(k_0)+n(k_0')\right]\frac{1}{q_0+k_0+k_0'} \tag{8}$$

This proves the cancellation of $\mathcal{S}'$ for the scalar theory with cubic selfinteraction in the 2-loop approximation (sunset diagram) for the $\Phi-$ functional.

This cancellation holds as well for the pressure and the density!

For the pressure we obtain

$$p(T) = -\int\frac{d^4q}{(2\pi)^4}n(q_0)\left[\delta(q)-\sin\delta(q)\cos\delta(q)\right] = -\int\frac{d^4q}{(2\pi)^4}T\ln\left(1-\mathrm{e}^{-q_0/T}\right)\frac{\partial\delta(q)}{\partial q_0}\,2\sin^2\delta(q) \tag{9}$$

Note that in the approximation $\delta(q_0,q) = -\arctan[\omega\gamma/(q_0^2-\omega^2)]$ the "spectral distribution" does not correspond to a Lorentzian (Breit-Wigner) function as naïvely expected, but to a "squared Lorentzian"

$$\frac{q_0(\omega\gamma)^3}{[(q_0^2-\omega^2)^2+(\omega\gamma)^2]^2} \tag{10}$$

See, e.g., Vanderheyden & Baym (1998); Morozov & Röpke, Ann. Phys. 324 (2009) 1261

Approximately selfconsistent HTL resumm. QCD

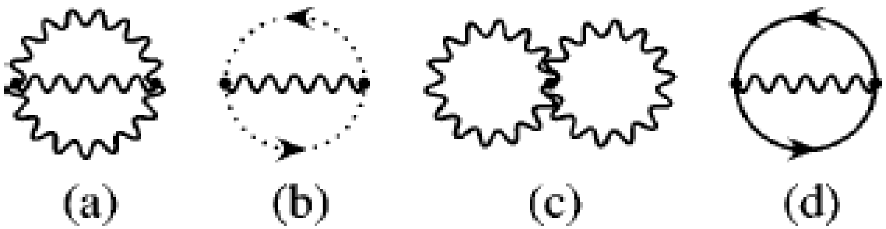
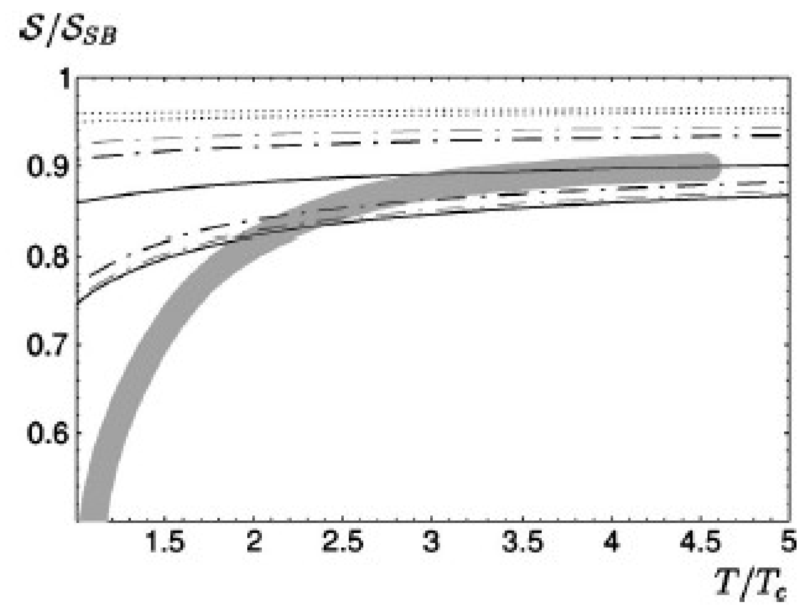


FIG. 3. Diagrams for Φ at 2-loop order in QCD. Wiggly, plain, and dotted lines refer respectively to gluons, quarks, and ghosts.



In ghost-free gauge, HTL resummed QCD thermodyn.

$$\mathcal{S}_2 = -\frac{g^2 N_g T}{48}\left\{\frac{4N+5N_f}{3}T^2 + \frac{3N_f}{\pi^2}\mu^2\right\},$$

$$\mathcal{N}_2 = -\frac{g^2 \mu N_g N_f}{16\pi^2}\left(T^2 + \frac{\mu^2}{\pi^2}\right),$$

$$P_2 = -\frac{g^2 N_g}{32}\left\{\frac{4N+5N_f}{18}T^4 + \frac{N_f}{\pi^2}\mu^2 T^2 + \frac{N_f}{2\pi^4}\mu^4\right\}$$

J.-P. Blaizot, E. Iancu, A. Rebhan, Phys. Rev. D 63 (2001) 065003

Generalized Optical Theorems

See derivations for T-matrices by R. Zimmermann & H. Stolz, pss (b) 131, 151 (1985)
Here we consider the analogue of $T^{-1} = V^{-1} - G_2^0$, the propagator $S^{-1} = G^{-1} - \Pi$, G real, static

Assuming the inverse exists we have two identities: $S = S^* S^{*-1} S$ and $S^* = S^* S^{-1} S$

$$\begin{aligned} S_R + \mathrm{i}S_I &= S^*(S_R^{-1} - \mathrm{i}S_I^{-1})S \,, \\ S_R - \mathrm{i}S_I &= S^*(S_R^{-1} + \mathrm{i}S_I^{-1})S \,. \end{aligned} \longrightarrow \begin{aligned} S_R &= S^* S_R^{-1} S \,, \\ S_I &= -S^* S_I^{-1} S \,, \end{aligned}$$

With definition $S^{-1} = G^{-1} - \Pi$ follows off-shell optical theorem: $S_I = S^* \Pi_I S = S \Pi_I S^*$

Using the fact that G is a real constant, we have: $(S_R^{-1})' = -\Pi_R'$ and $S_I^{-1} = -\Pi_I$

$$\begin{aligned} S_R' &= S^{*\prime} S_R^{-1} S + S^*(S_R^{-1})' S + S^* S_R^{-1} S' \\ &= S^{*\prime}(\underbrace{S_R^{-1} + \mathrm{i}S_I^{-1}}_{S^{-1}} - \mathrm{i}S_I^{-1})S + S^*(S_R^{-1})' S + S^*(\underbrace{S_R^{-1} - \mathrm{i}S_I^{-1}}_{S^{*-1}} + \mathrm{i}S_I^{-1})S' \\ &= \underbrace{S^{*\prime} + S'}_{2S_R'} - \mathrm{i}S^{*\prime} S_I^{-1} S + \mathrm{i}S^* S_I^{-1} S' + S^*(S_R^{-1})' S \\ &= S^* \Pi_R' S - \mathrm{i}S^{*\prime} \Pi_I S + \mathrm{i}S^* \Pi_I S' \,, \end{aligned}$$

Derivative optical theorem:

$$S_R' \Pi_I = \underbrace{S^* \Pi_R' S \Pi_I}_{\Pi_R' S_I} + \underbrace{\mathrm{i}S^* \Pi_I S' \Pi_I - \mathrm{i}S^{*\prime} \Pi_I S \Pi_I}_{2\,\mathrm{Im}[\Pi_I S \Pi_I S^{*\prime}]} \,, \longrightarrow S_R' \Pi_I - \Pi_R' S_I = 2\,\mathrm{Im}\left[\Pi_I S \Pi_I S^{*\prime}\right]$$


Wir fahren aufs Land

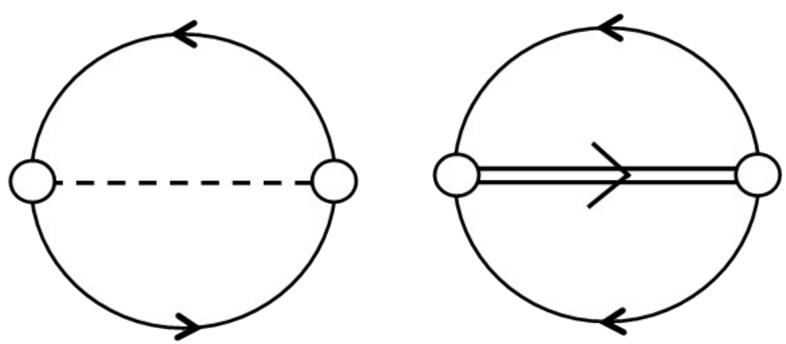
Φ-derivable Q-M-D PNJL model, 2-loop approximation

$$\Omega = \frac{1}{2}\frac{T}{V}\sum_{i=Q,M,D} c_i \,\mathrm{Tr}\left\{\ln\left[S_i^{-1}\right] + \left[S_i\Pi_i\right]\right\} + \Phi\left[S_Q, S_M, S_D\right] \ ,$$

$$S_i^{-1}(\mathrm{i}z_n,\mathbf{q}) = S_{i,0}^{-1}(\mathrm{i}z_n,\mathbf{q}) - \Pi_i(\mathrm{i}z_n,\mathbf{q}) \ , \qquad \frac{\delta\Omega}{\delta S_i} = 0 \ , \quad \text{if} \quad \Pi_i = \frac{\delta\Omega}{\delta S_i} \ .$$

$$\Omega = \frac{1}{2}T\sum_{i=Q,M,D}\int\frac{d^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f_i(\omega)\,\mathrm{Tr}\left\{\mathrm{Im}\ln\left[S_i^{-1}\right] + \left[\mathrm{Re}\,S_i\,\mathrm{Im}\,\Pi_i\right]\right\} + \tilde{\Omega}$$

$$\tilde{\Omega} = \Phi\left[S_Q, S_M, S_D\right] - \frac{1}{2}T\sum_{i=Q,M,D}\int\frac{d^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f_i(\omega)\,\mathrm{Tr}\left\{\left[\mathrm{Im}\,S_i\,\mathrm{Re}\,\Pi_i\right]\right\} \ ,$$

$\Phi =$ 

$$\mathcal{S} = -\frac{\partial\Omega}{\partial T} = \sum_i \mathcal{S}_i + \tilde{\mathcal{S}}$$

$$\mathcal{N} = -\frac{\partial\Omega}{\partial\mu} = \sum_i \mathcal{N}_i + \tilde{\mathcal{N}} \ .$$

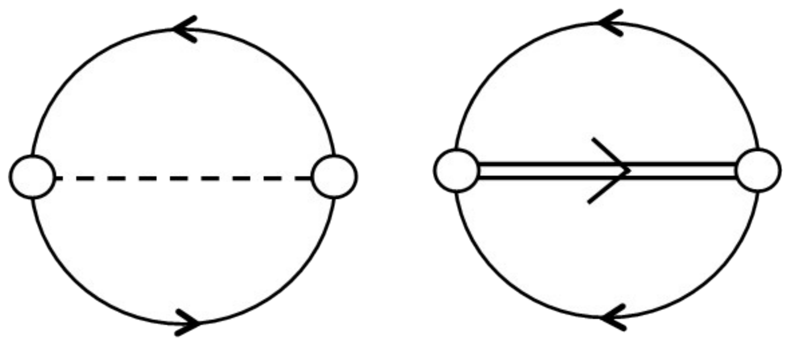
Φ-derivable Q-M-D PNJL model, 2-loop approximation

$$\Omega = \frac{1}{2}\frac{T}{V}\sum_{i=Q,M,D} c_i \,\mathrm{Tr}\left\{\ln\left[S_i^{-1}\right] + \left[S_i \Pi_i\right]\right\} + \Phi\left[S_Q, S_M, S_D\right] \; ,$$

$$S_i^{-1}(iz_n, \mathbf{q}) = S_{i,0}^{-1}(iz_n, \mathbf{q}) - \Pi_i(iz_n, \mathbf{q}) \; , \qquad \frac{\delta\Omega}{\delta S_i} = 0 \; , \quad \text{if} \quad \Pi_i = \frac{\delta\Omega}{\delta S_i} \; .$$

$$\Omega \; = \; \frac{1}{2}T \sum_{i=Q,M,D} \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_i(\omega)\, \mathrm{Tr}\left\{\mathrm{Im}\ln\left[S_i^{-1}\right] + \left[\mathrm{Re}\, S_i \,\mathrm{Im}\, \Pi_i\right]\right\} + \tilde{\Omega}$$

$$\tilde{\Omega} \; = \; \Phi\left[S_Q, S_M, S_D\right] - \frac{1}{2}T \sum_{i=Q,M,D} \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_i(\omega)\, \mathrm{Tr}\left\{\left[\mathrm{Im}\, S_i \,\mathrm{Re}\, \Pi_i\right]\right\} \; ,$$

$\Phi =$ 

$$\mathcal{S} = -\frac{\partial\Omega}{\partial T} = \sum_i \mathcal{S}_i + \tilde{\mathcal{S}}$$

$$\mathcal{N} = -\frac{\partial\Omega}{\partial \mu} = \sum_i \mathcal{N}_i + \tilde{\mathcal{N}} .$$

Φ-derivable Q-M-D PNJL model, 2-loop approximation

$$\left(\mathrm{Im}\ln S^{-1}\right)' = -\,\mathrm{Im}\,(S\Pi') = \underbrace{S'_R\Pi_I - S_I\Pi'_R}_{2\,\mathrm{Im}(S\Pi_I S^{*\prime}\Pi_I)} - \underbrace{(\Pi_I S'_R + S_R\Pi'_I)}_{(\Pi_I S_R)'} ,$$

Use optical theorems ...

$$S\Pi_I \;=\; \sin\delta\,\mathrm{e}^{\mathrm{i}\delta} \;,\quad S^{*\prime}\Pi_I = -\mathrm{i}\delta'\sin\delta\,\mathrm{e}^{-\mathrm{i}\delta} \;,\quad 2\,\mathrm{Im}(S\Pi_I S^{*\prime}\Pi_I) = -2\delta'\sin^2\delta \;.$$

Generalized Beth-Uhlenbeck EoS

$$\Omega \;=\; -\sum_{i=Q,M,D} d_i \int \frac{\mathrm{d}^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, T\ln[1-\mathrm{e}^{-(\omega-\mu_i)/T}]\,\sin^2\delta_i(\omega,\mathbf{q})\frac{\partial\delta_i(\omega,\mathbf{q})}{\partial\omega}$$

Effect of the sin^2 term ... example: Breit-Wigner ...

$$\delta_i(\omega) = -\arctan\left[\frac{\omega_i\Gamma_i}{\omega^2-\omega_i^2}\right] \;,\quad \frac{\partial\delta_i(\omega)}{\partial\omega} = \frac{2\omega\omega_i\Gamma_i}{(\omega^2-\omega_i^2)^2+\omega_i^2\Gamma_i^2} \;,$$

$$\sin^2\delta_i(\omega)\frac{\partial\delta_i(\omega)}{\partial\omega} \;=\; \frac{2\omega(\omega_i\Gamma_i)^3}{[(\omega^2-\omega_i^2)^2+\omega_i^2\Gamma_i^2]^2} \;.$$

"Squared Lorentzian" ...
Vanderheyden & Baym (1998)
Morozov & Roepke (2009)

1. Cluster expansion in the 2PI formalism

- $\Phi-$ derivable approach to the grand canonical thermodynamic potential
  [Baym, Phys. Rev. 127 (1962) 139]

$$J = -\mathrm{Tr}\,\{\ln(-G_1)\} - \mathrm{Tr}\{\Sigma_1 G_1\} + \mathrm{Tr}\,\{\ln(-G_2)\} + \mathrm{Tr}\{\Sigma_2 G_2\} + \Phi[G_1, G_2]$$

  with full propagators:
  $G_1^{-1}(1,z) = z - E_1(p_1) - \Sigma_1(1,z)$; $G_2^{-1}(12,1'2',z) = z - E_1(p_1) - E_2(p_2) - \Sigma_2(12,1'2',z)$
  and selfenergies

$$\Sigma_1(1,1') = \frac{\delta\Phi}{\delta G_1(1,1')}\,;\Sigma_2(12,1'2',z) = \frac{\delta\Phi}{\delta G_2^{-1}(12,1'2',z)}.$$

  Because of stationarity equivalent to

$$n = -\frac{1}{\Omega}\frac{\partial J}{\partial \mu} = \frac{1}{\Omega}\sum_1 \int_{-\infty}^{\infty}\frac{d\omega}{\pi} f_1(\omega) S_1(1,\omega)\,,$$

  (baryon number conservation)

- Generalization to A-nucleon clusters in nuclear matter

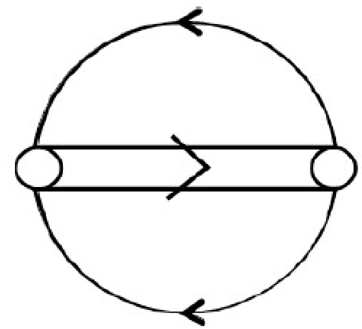
$$\Omega \;=\; \sum_A (-1)^A \left[\mathrm{Tr}\ln\left(-G_A^{-1}\right) + \mathrm{Tr}\left(\Sigma_A\, G_A\right)\right] + \Phi\,,$$

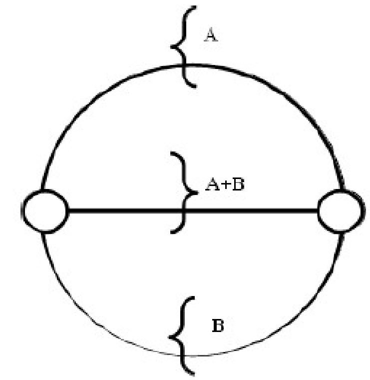
$$G_A^{-1} \;=\; G_A^{(0)-1} - \Sigma_A\,,\;\; \Sigma_A(1\ldots A, 1'\ldots A', z_A) = \frac{\delta\Phi}{\delta G_A(1\ldots A, 1'\ldots A', z_A)}\,.$$

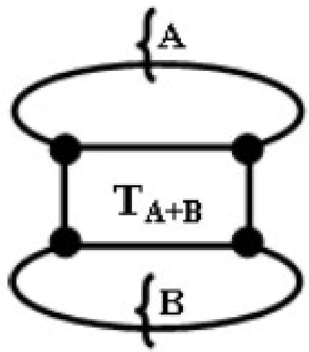
1. Cluster expansion in the 2PI formalism

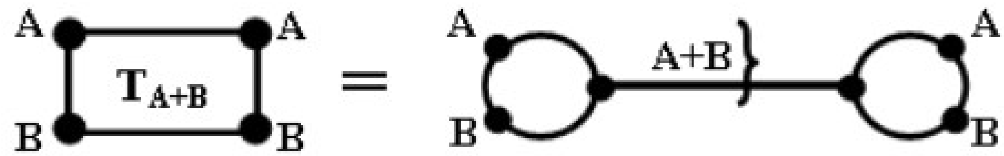
A) Choice of the Φ-functional:

- 2-particle irreducible diagrams
- closed 2-loop diagram involving 3 cluster propagators (A, B, A+B) and 2 vertices
- equivalent to 1 T-matrix + 2 propagators









B) Ansatz for thermodynamic potential:

$$\Omega = \sum_A (-1)^A \left[\mathrm{Tr}\ln\left(-G_A^{-1}\right) + \mathrm{Tr}\left(\Sigma_A\, G_A\right)\right] + \sum_{A,B} \Phi[G_A, G_B, G_{A+B}]\,,$$

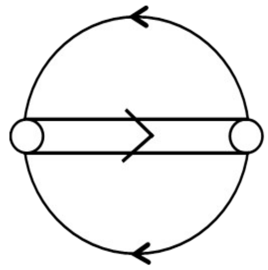
$$G_A^{-1} = G_A^{(0)^{-1}} - \Sigma_A\,,\quad \Sigma_A(1\ldots A, 1'\ldots A', z_A) = \frac{\delta\Phi}{\delta G_A(1\ldots A, 1'\ldots A', z_A)}\,.$$

C) Check: conservation laws, e.g.:
(correspondence to GF formalism)

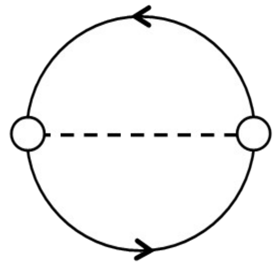
$$n = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu} = \frac{1}{V}\sum_1 \int_{-\infty}^{\infty}\frac{d\omega}{\pi} f_1(\omega) A_1(1,\omega)$$

Cluster virial expansion in the 2PI formalism, Examples:

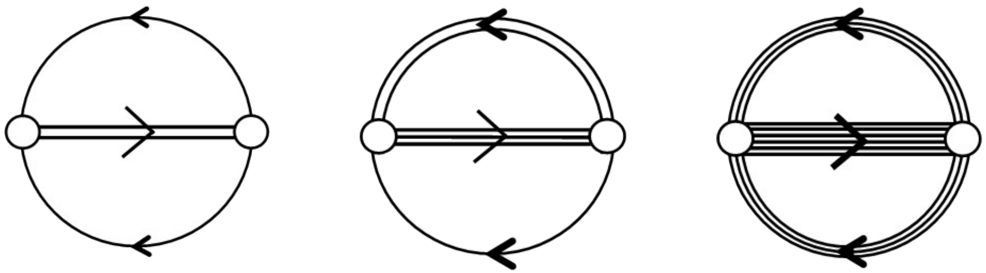
A) Deuterons in nuclear matter:



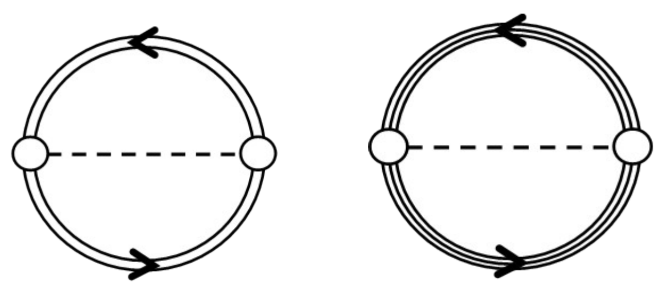
B) Mesons in quark matter:



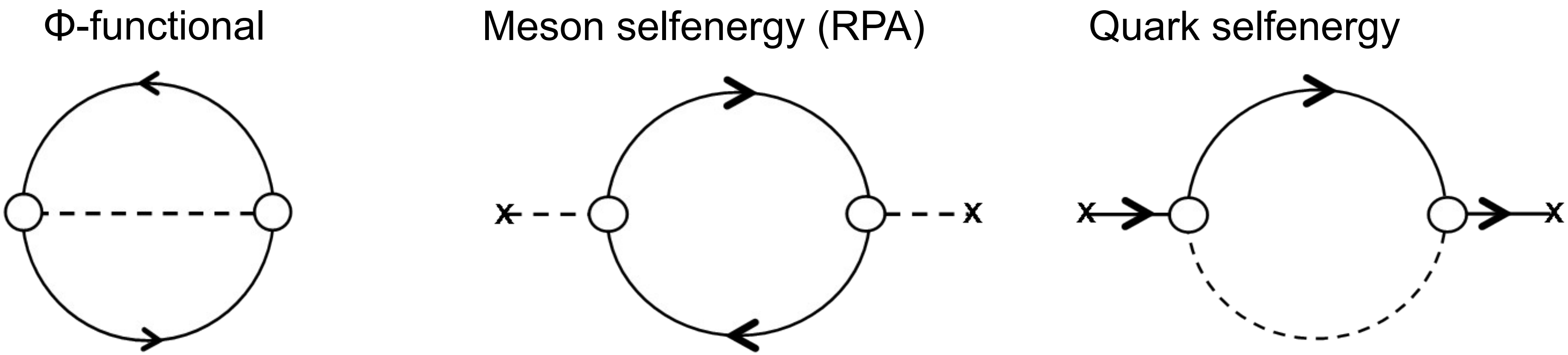
C) Nucleons in quark matter:



D) Nucleons and mesons (hadron resonance gas) in quark matter:

B) + C) + 

Example B: Mesons in quark matter



$$T_M^{-1}(q,\omega+i\eta) = G_S^{-1} - \Pi_M(q,\omega+i\eta) = |T_M(q,\omega)|^{-1} e^{-i\delta_M(q,\omega)} , \quad \delta_M(q,\omega) = \arctan(\Im T_M / \Re T_M)$$

$$\Omega = \Omega_{\rm MF} + \Omega_M , \qquad \sigma_{\rm MF} = 2N_f N_c G_S \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} [1 - f_-(E_p) - f_+(E_p)] ,$$

$$\Omega_{\rm MF} = \frac{\sigma_{\rm MF}^2}{4G_S} - 2N_c N_f \int \frac{d^3p}{(2\pi)^3} \left[E_p + T \ln\left(1 + e^{-(E_p - \Sigma_+ - \mu)/T}\right) + T \ln\left(1 + e^{-(E_p + \Sigma_- + \mu)/T}\right)\right] ,$$

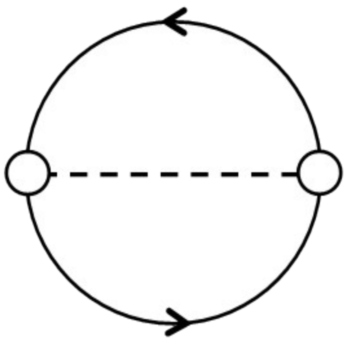
$$\Omega_M = d_M \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left\{ \omega + 2T \ln\left[1 - e^{-\omega/T}\right] 2\sin^2\delta_M(k,\omega) \, \frac{\delta_M(k,\omega)}{d\omega} \right\} ,$$

$$\Sigma_M(0,p_0) = d_M \int \frac{d^4q}{(2\pi)^4} \pi \rho_M(\mathbf{q},q_0) \left\{ \frac{(\gamma_0 + m/E_q)[1 + g(q_0) - f_-(E_q)]}{q_0 - p_0 + E_q - \mu - i\eta} + \frac{(\gamma_0 - m/E_q)[g(q_0) + f_+(E_q)]}{q_0 - p_0 - E_q - \mu - i\eta} \right\} ,$$

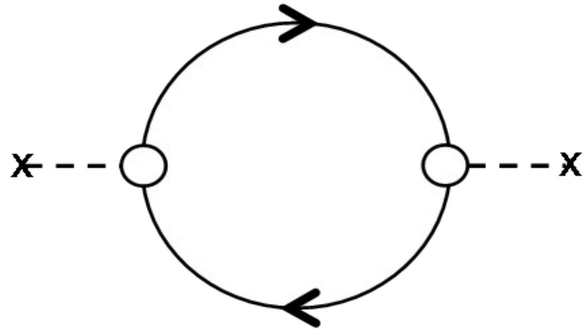
D. Blaschke, M. Buballa, A. Dubinin, G. Roepke, D. Zablocki: Ann. Phys. 348 (2014) 228
D. Blaschke, PoS Baldin ISHEPXXII (2015) 113; arxiv:1502.06279

Example B: Mesons in quark matter

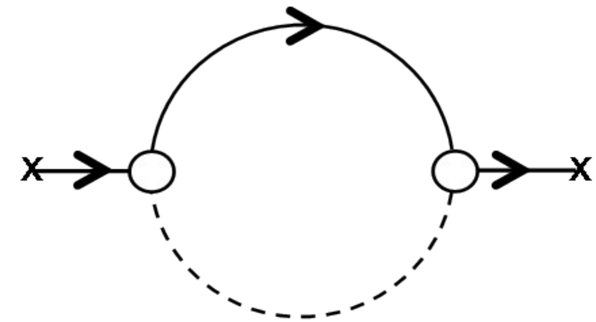
Φ-functional



Meson selfenergy (RPA)



Quark selfenergy



$$T_M^{-1}(q,\omega+i\eta) = G_S^{-1} - \Pi_M(q,\omega+i\eta) = |T_M(q,\omega)|^{-1} \mathrm{e}^{-i\delta_M(q,\omega)} \,, \quad \delta_M(q,\omega) = \arctan(\Im T_M/\Re T_M)$$

$$\Omega = \Omega_{\mathrm{MF}} + \Omega_M \,, \qquad \sigma_{\mathrm{MF}} = 2N_f N_c G_S \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} [1 - f_-(E_p) - f_+(E_p)] \,,$$

$$\Omega_{\mathrm{MF}} = \frac{\sigma_{\mathrm{MF}}^2}{4G_S} - 2N_c N_f \int \frac{d^3p}{(2\pi)^3} \left[E_p + T \ln\left(1 + \mathrm{e}^{-(E_p - \Sigma_+ - \mu)/T}\right) + T \ln\left(1 + \mathrm{e}^{-(E_p + \Sigma_- + \mu)/T}\right) \right] \,,$$

$$\Omega_M = d_M \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left\{ \omega + 2T \ln\left[1 - \mathrm{e}^{-\omega/T}\right] \boxed{2\sin^2 \delta_M(k,\omega)} \frac{\delta_M(k,\omega)}{d\omega} \right\} \,,$$

new !

$$\Sigma_M(0,p_0) = d_M \int \frac{d^4q}{(2\pi)^4} \pi \rho_M(\mathbf{q},q_0) \left\{ \frac{(\gamma_0 + m/E_q)[1 + g(q_0) - f_-(E_q)]}{q_0 - p_0 + E_q - \mu - i\eta} + \frac{(\gamma_0 - m/E_q)[g(q_0) + f_+(E_q)]}{q_0 - p_0 - E_q - \mu - i\eta} \right\} \,,$$

D. Blaschke, M. Buballa, A. Dubinin, G. Roepke, D. Zablocki: Ann. Phys. 348 (2014) 228
D. Blaschke, PoS Baldin ISHEPXXII (2015) 113; arxiv:1502.06279

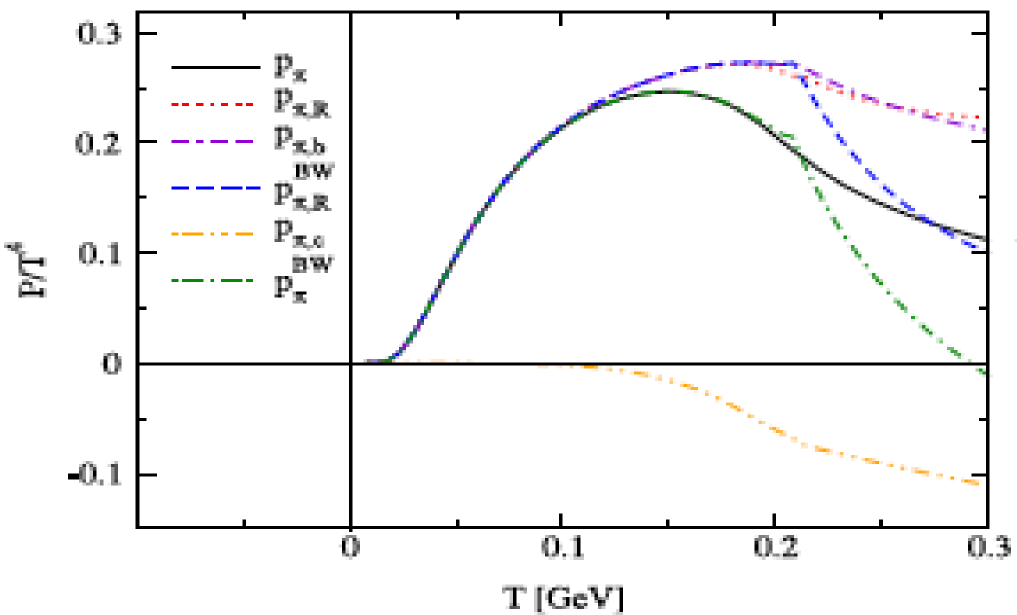
Example B: Mesons in quark matter

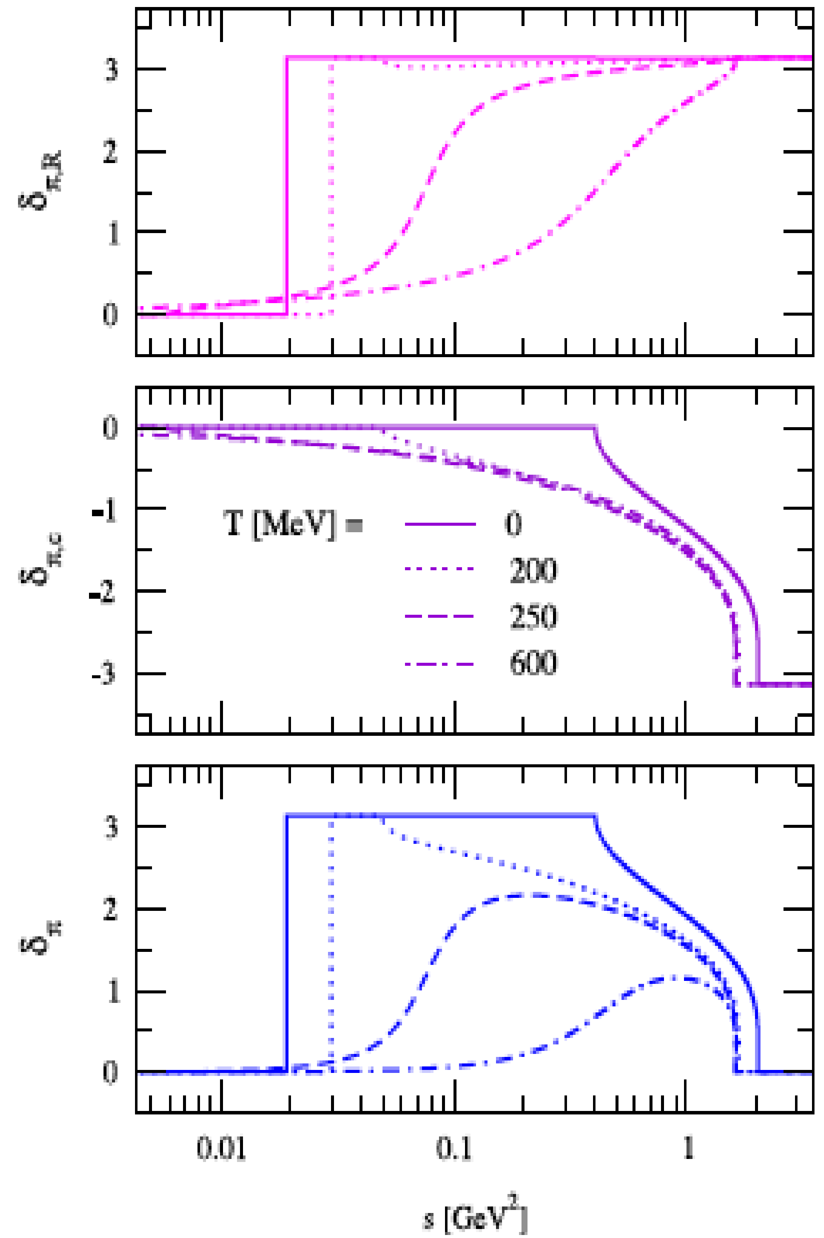
$$\Omega_X(T,\mu) = -d_X \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n_X^-(\omega)\delta_X(\omega,\mathbf{q}),$$

$$\int_0^\infty d\omega \frac{1}{\pi}\frac{d\delta_X(\omega;T)}{d\omega} = 0 = \underbrace{\int_0^{\omega_{\rm thr}(T)} d\omega \frac{1}{\pi}\frac{d\delta_X(\omega;T)}{d\omega}}_{n_{B,X}(T)} + \underbrace{\frac{1}{\pi}\int_{\omega_{\rm thr}(T)}^\infty d\omega \frac{d\delta_X(\omega;T)}{d\omega}}_{\frac{1}{\pi}[\delta_X(\infty;T)-\delta_X(\omega_{\rm thr};T)]},$$

$$p_\pi(T) = -d_\pi T \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{d\omega}{\pi} \ln\left(1 - e^{-\omega/T}\right) \frac{d\delta_\pi(\omega,\mathbf{q})}{d\omega}$$

$$\delta_\pi \doteq \delta_{\pi,c} + \delta_{\pi,R}$$





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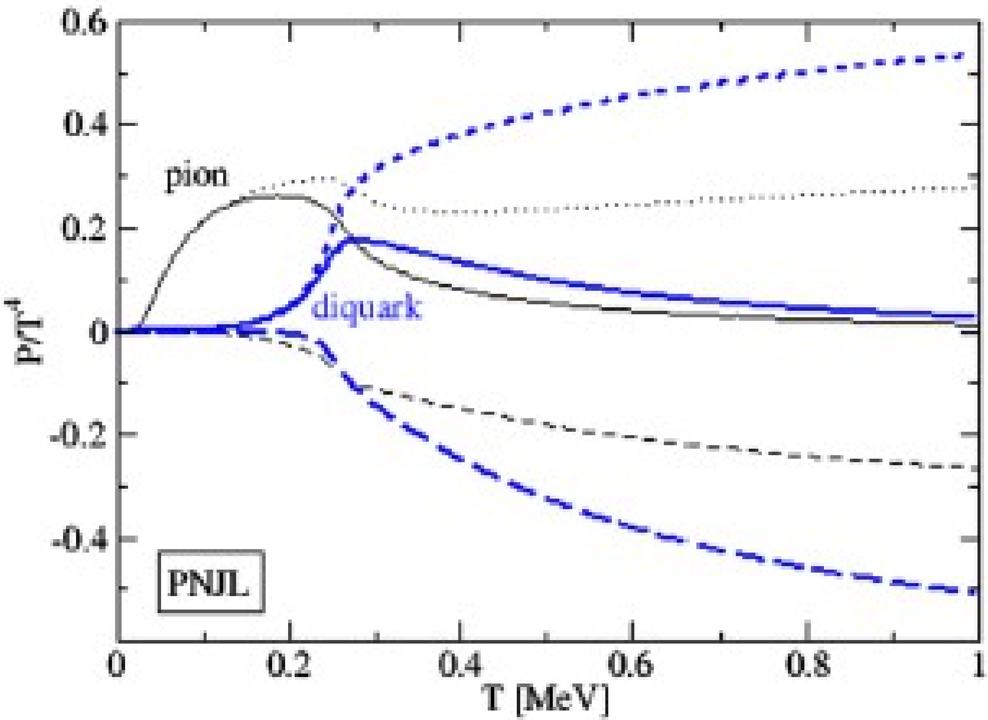
Example B*: Mesons+diquarks in quark matter

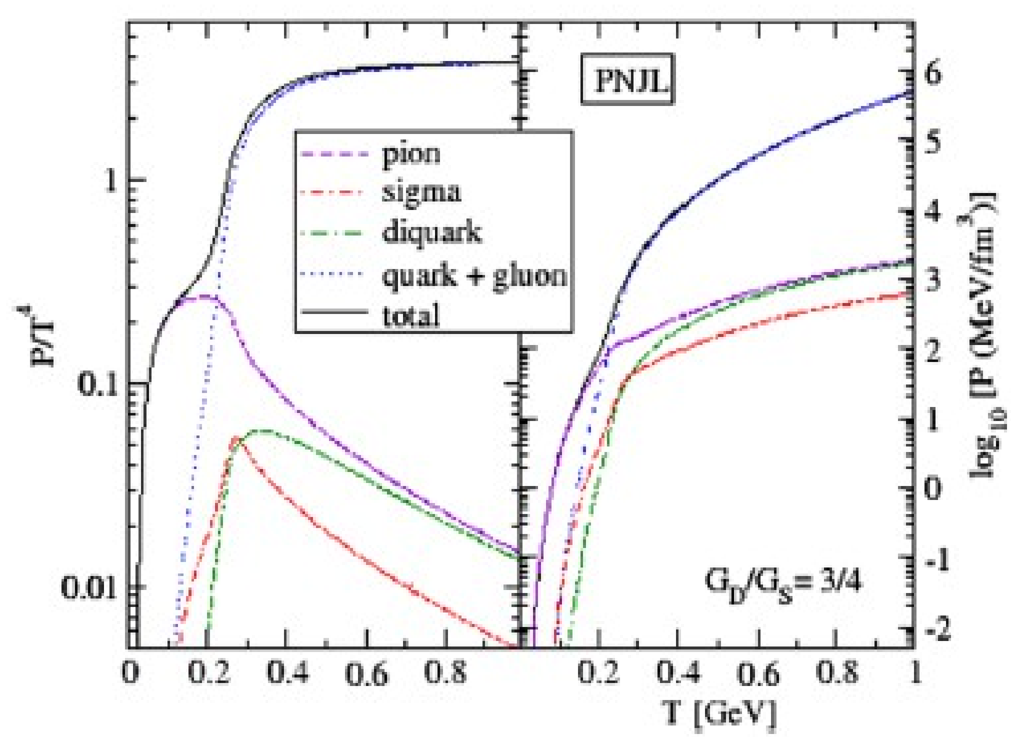
$$\Omega_Q = -\frac{2N_c N_f}{3}\int \frac{dp}{2\pi^2}\frac{p^4}{E_p}[f_\Phi^+(E_p) + f_{\bar\Phi}^-(E_p)],\ \ f_\Phi^+(E_p) = \frac{(\bar\Phi + 2\Phi Y)Y + Y^3}{1 + 3(\bar\Phi + \Phi Y)Y + Y^3},\ \ Y = \mathrm{e}^{-(E_p-\mu)/T}$$

$$\Omega_D = -3\int \frac{d^3p}{(2\pi)^3}\int \frac{d\omega}{2\pi}[g_\Phi^+(\omega) + g_{\bar\Phi}^-(\omega)]\delta_D(\omega),\ \ g_\Phi^+(\omega) = \frac{(\Phi - 2\bar\Phi X)X + X^3}{1 - 3(\Phi - \bar\Phi X)X - X^3},\ \ X = \mathrm{e}^{-(\omega-2\mu)/T}$$

Suppression of colored states by Polyakov-loop Φ

Confinement: Φ=0





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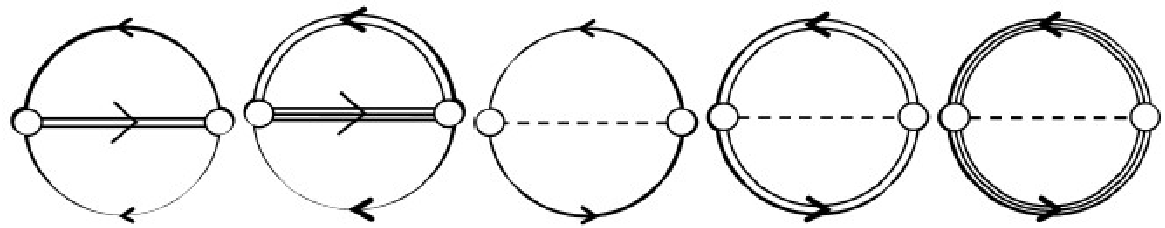
Example D: Hadron resonance gas – effect. model

Φ-functional:



Selfenergies: